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Calculus I

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Unit 7. Derivatives

Solutions

D.7 Derivatives

Problem 7.1

(i)

$$h'(x) = \frac{f(x)f'(x) + g(x)g'(x)}{\sqrt{f(x)^2 + g(x)^2}}.$$

(ii)

$$h'(x) = \frac{1}{1 + \left(\frac{f(x)}{g(x)}\right)^2} \cdot \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} = \frac{f(x)g'(x) - f'(x)g(x)}{f(x)^2 + g(x)^2}.$$

(iii)

$$h'(x) = f'\big(g(x)\big)g'(x)e^{f(x)} + f\big(g(x)\big)f'(x)e^{f(x)} = \left[f'\big(g(x)\big)g'(x) + f\big(g(x)\big)f'(x)\right]e^{f(x)}.$$

(iv) First of all $h(x) = \log\big(g(x)\big) + \log\big(\sin f(x)\big)$, so

$$h'(x) = \frac{g'(x)}{g(x)} + \frac{f'(x)\cos f(x)}{\sin f(x)} = \frac{g'(x)}{g(x)} + f'(x)\cot f(x).$$

(v) We first write $f(x)^{g(x)} = \exp\{g(x)\log f(x)\}$. Then

$$\begin{aligned} h'(x) &= \left[g'(x)\log f(x) + \frac{g(x)f'(x)}{f(x)}\right]\exp\{g(x)\log f(x)\} \\ &= \left[g'(x)\log f(x) + \frac{g(x)f'(x)}{f(x)}\right]f(x)^{g(x)} \\ &= f(x)^{g(x)}g'(x)\log f(x) + g(x)f'(x)f(x)^{g(x)-1}. \end{aligned}$$

(vi)

$$h'(x) = -\frac{1}{\left[\log\big(f(x) + g(x)^2\big)\right]^2} \cdot \frac{f'(x) + 2g(x)g'(x)}{f(x) + g(x)^2}.$$

Problem 7.2 In both items we are asked to figure out a function $g(x)$ such that

$$f(x) = \begin{cases} 1, & |x| \leqslant 1, \\ 0, & |x| \geqslant 2, \\ g(x), & 1 < x < 2, \\ g(-x), & -2 < x < -1, \end{cases}$$

is the requested function.

(a) For $f(x)$ to be continuous we need $g(x)$ to be continuous and fulfill the two conditions $g(1) = 1$, $g(2) = 0$. The simplest such function is the straight line $g(x) = ax + b$, for which these two conditions imply

$$\begin{cases} a + b = 1, \\ 2a + b = 0, \end{cases} \qquad \Leftrightarrow \qquad a = -1, \quad b = 2.$$

Thus $g(x) = -x + 2$.

(b) Since the derivative of f for $|x| < 1$ and $|x| > 2$ is 0, now we need $g(x)$ to satisfy also $g'(1) = g'(2) = 0$. These are four equations, so the simplests function would be a polynomial with four unknown coefficients, namely $g(x) = ax^3 + bx^2 + cx + d$. But we can determine the polynomial more precisely given the information we have. For instance, the fact that $g(1) = 0$ means that $g(x) = (x-1)q(x)$, with $q(x)$ is a second degree polynomial. Given this expression, $g'(x) = q(x) + (x-1)q'(x)$, so $0 = g'(1) = q(1)$, and this implies $q(x) = (x-1)r(x)$, with $r(x)$ a lineal polynomial. In other words,

$$g(x) = (x-1)^2(ax+b), \qquad g'(x) = 2(x-1)(ax+b) + a(x-1)^2.$$

We can now impose the constraints $g(2) = 1$, $g'(2) = 0$, and this leads to

$$\begin{cases} 2a+b=1, \\ 2(2a+b)+a=0, \end{cases} \Leftrightarrow \begin{cases} 2a+b=1, \\ 5a+2b=0, \end{cases} \Leftrightarrow \begin{cases} a=-2, \\ b=5. \end{cases}$$

Thus $g(x) = (x-1)^2(5-2x)$.

There is a simpler way to achieve the same result though. It amounts to finding a continuous and differentiable function with a local maximum and a local minimum. One such function is $\cos^2 a(x-b)$. This function reaches a maximum at $x = b$, where it is 1, and a minimum at $a(x-b) = \pi/2$, where it is 0. If we want the maximum to be at $x = 1$ then we must choose $b = 1$. If we want the minimum to be at $x = 2$ we must choose $a(2-1) = \pi/2$, i.e., $a = \pi/2$. Thus $g(x) = \cos^2 \frac{\pi}{2}(x-1)$.

Problem 7.3

(i) $f'(x) = -\dfrac{c}{x^2}$, therefore

$$xf' + f = -\frac{c}{x} + \frac{c}{x} = 0.$$

(ii) $f'(x) = \tan x + x(1+\tan^2 x)$, therefore

$$xf' - f - f^2 = x\tan x + x^2 - x^2\tan^2 x - x\tan x - x^2\tan^2 x = x^2.$$

(iii) $f'(x) = 3c_1\cos 3x - 3c_2\sin 3x$ and $f''(x) = -9c_1\sin 3x - 9c_2\cos 3x$, therefore

$$f'' + 9f = -9c_1\sin 3x - 9c_2\cos 3x + 9(c_1\sin 3x + c_2\cos 3x) = 0.$$

(iv) $f'(x) = 3c_1e^{3x} - 3c_2e^{-3x}$ and $f''(x) = 9c_1e^{3x} + 9c_2e^{-3x}$, therefore

$$f'' - 9f = 9c_1e^{3x} + 9c_2e^{-3x} - 9(c_1e^{3x} + c_2e^{-3x}) = 0.$$

(v) $f'(x) = 2c_1e^{2x} + 5c_2e^{5x}$ and $f''(x) = 4c_1e^{2x} + 25c_2e^{5x}$, therefore

$$\begin{aligned} f'' - 7f' + 10f &= 4c_1e^{2x} + 25c_2e^{5x} - 7(2c_1e^{2x} + 5c_2e^{5x}) + 10(c_1e^{2x} + c_2e^{5x}) \\ &= (4-14+10)e^{2x} + (25-35+10)e^{5x} = 0. \end{aligned}$$

(vi) $f'(x) = \dfrac{c_1e^x - e^{-x}}{c_1e^x + e^{-x}}$ and

$$f''(x) = \frac{(c_1e^x + e^{-x})^2 - (c_1e^x - e^{-x})^2}{(c_1e^x + e^{-x})^2} = 1 - \left(\frac{c_1e^x - e^{-x}}{c_1e^x + e^{-x}}\right)^2,$$

therefore

$$f'' - (f')^2 = 1 - \left(\frac{c_1e^x - e^{-x}}{c_1e^x + e^{-x}}\right)^2 + \left(\frac{c_1e^x - e^{-x}}{c_1e^x + e^{-x}}\right)^2 = 0.$$

Problem 7.4

(i) Differentiating $f(x) = \arctan x + \arctan \frac{1}{x}$,

$$\frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}}\left(-\frac{1}{x^2}\right) = \frac{1}{1+x^2} - \frac{1}{x^2+1} = 0.$$

Therefore $f(x) = c$, a constant. To find out which constant we must evaluate $f(x)$ at any point $x > 0$, say $x = 1$. Then $f(1) = c = \arctan 1 + \arctan 1 = 2\pi/4 = \pi/2$.

(ii) Differentiating $f(x) = \arctan \frac{1+x}{1-x} - \arctan x$,

$$f'(x) = \frac{1}{1+\left(\frac{1+x}{1-x}\right)^2}\frac{1-x+1+x}{(1-x)^2} - \frac{1}{1+x^2} = \frac{2}{(1-x)^2+(1+x)^2} - \frac{1}{1+x^2}$$
$$= \frac{2}{1-2x+x^2+1+2x+x^2} - \frac{1}{1+x^2} = \frac{2}{2+2x^2} - \frac{1}{1+x^2} = 0.$$

Therefore $f(x) = c$, a constant. To find out which constant we must evaluate $f(x)$ at any point $x < 1$, say $x = 0$. Then $f(0) = c = \arctan 1 + \arctan 0 = \pi/4$.

(iii) Differentiating $f(x) = 2\arctan x + \arcsin \frac{2x}{1+x^2}$,

$$f'(x) = \frac{2}{1+x^2} + \frac{1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}}\frac{2(1+x^2)-2x\cdot 2x}{(1+x^2)^2}$$
$$= \frac{2}{1+x^2} + \frac{1+x^2}{\sqrt{(1+x^2)^2-4x^2}}\frac{2(1-x^2)}{(1+x^2)^2} = \frac{2}{1+x^2} + \frac{2(1-x^2)}{(1+x^2)\sqrt{(1-x^2)^2}}$$
$$\underset{(*)}{=} \frac{2}{1+x^2} + \frac{2(1-x^2)}{(1+x^2)(x^2-1)} = \frac{2}{1+x^2} - \frac{2}{1+x^2} = 0,$$

where in (*) we have used the fact that $x \geqslant 1$ implies that $\sqrt{(1-x^2)^2} = x^2 - 1 \geqslant 0$. Therefore $f(x) = c$, a constant. To find out which constant we must evaluate $f(x)$ at any point $x \geqslant 1$, say $x = 1$. Then $f(1) = c = 2\arctan 1 + \arcsin 1 = 2\pi/4 + \pi/2 = \pi$.

Problem 7.5 If we calculate $f'(x) = 1 + \frac{1}{3}(\sin x)^{-2/3}\cos x$ we observe that this function diverges whenever $\sin x = 0$, i.e., for $x = n\pi$ with $n \in \mathbb{Z}$. Those are the points where the tangent straight line is vertical.

Problem 7.6 Let us calculate the derivative on the left, $f'(0^-)$ and on the right, $f'(0^+)$. Since $f(0) = 0$,

$$f'(0^-) = \lim_{x\to 0^-}\frac{f(x)-f(0)}{x} = \lim_{x\to 0^-}\frac{1}{1+e^{1/x}} = \lim_{t\to -\infty}\frac{1}{1+e^t} = 1,$$
$$f'(0^+) = \lim_{x\to 0^+}\frac{f(x)-f(0)}{x} = \lim_{x\to 0^+}\frac{1}{1+e^{1/x}} = \lim_{t\to \infty}\frac{1}{1+e^t} = 0.$$

So the slope of the tangent on the left is 1 —hence it forms an angle $\pi/4$ with the X axis— and that on the right is 0 —hence it is parallel to the X axis. Thus the angle between both tangents is $\pi/4$.

Problem 7.7 The domain of this function requires that $x+2 \geqslant 0$ and $-1 \leqslant x+2 \leqslant 1$ be satisfied simultaneously. This happens for x such that $0 \leqslant x+2 \leqslant 1$, in other words, for $x \in [-2,-1]$. Within this domain the function is continuous because so are $x+2$, $\sqrt{x}$, and $\cos x$ —hence its inverse— in their respective domains.

About differentiability,

$$f'(x) = \frac{\arccos(x+2)}{2\sqrt{x+2}} - \frac{\sqrt{x+2}}{\sqrt{1-(x+2)^2}} = \frac{\arccos(x+2)}{2\sqrt{x+2}} - \sqrt{\frac{x+2}{-3-4x-x^2}},$$

which diverges when $x=-2$ and is defined only if $x^2+4x+3=(x+1)(x+3)<0$. This happens for $x\in(-3,-1)$, an interval that overlaps with the domain excluding the point $x=-1$. Thus the derivative exits only for $x\in(-2,-1)$.

Problem 7.8 Function $f(x)$ will be differentiable if and only if $\alpha x^2-x+3\geqslant 0$ for all $x\in\mathbb{R}$ or $\alpha x^2-x+3\leqslant 0$ for all $x\in\mathbb{R}$. The reason is that in either of these two cases the parabola does not cross the X axis or it just touches the axis at one point (it is only if the parabola crosses the axis that its absolute value generates points with no derivative). The condition for this to happen is that the discriminant of the parabola be $\leqslant 0$, i.e., $1-12\alpha\leqslant 0$. Thus $\alpha\geqslant 1/12$.

Problem 7.9 Function $f(x)$ is even, so it is enough to make sure that it is continuous and differentiable at $x=c$. The function will be continuous at $x=c$ if

$$a+bc^2=\frac{1}{c}.$$

On the other hand, for $x\geqslant 0$ the function is

$$f(x)=\begin{cases} a+bx^2, & 0\leqslant x\leqslant c,\\ \dfrac{1}{x}, & x>c,\end{cases}$$

so its derivative will be

$$f'(x)=\begin{cases} 2bx, & 0\leqslant x<c,\\ -\dfrac{1}{x^2}, & x>c,\end{cases}$$

and therefore $f(x)$ will be differentiable at $x=c$ if

$$2bc=-\frac{1}{c^2} \qquad \Leftrightarrow \qquad b=-\frac{1}{2c^3}.$$

And from the previous equation we obtain

$$a=\frac{1}{c}-bc^2=\frac{1}{c}+\frac{1}{2c}=\frac{3}{2c}.$$

So for $|x|<c$ the function is defined as

$$f(x)=\frac{1}{2c}\left(3-\frac{x^2}{c^2}\right).$$

Problem 7.10

(a) The two pieces defining this function are continuous and differentiable within their respective sets, so the only critical point is $x=1$. Let us first check the continuity at this point. So

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\frac{1}{x}=1, \qquad \lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\frac{3-x^2}{2}=1,$$

hence

$$\lim_{x\to 1} f(x) = 1 = f(1),$$

which proves that the function is continuous also at this point. As for differentiability,

$$f'(1^+) = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{\frac{1}{x}-1}{x-1} = \lim_{x\to 1^+} \frac{1-x}{x(x-1)} = -1,$$

$$f'(1^-) = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{\frac{3-x^2}{2}-1}{x-1} = \lim_{x\to 1^-} \frac{1-x^2}{2(x-1)} = \lim_{x\to 1^-} \frac{(1-x)(1+x)}{2(x-1)}$$
$$= \lim_{x\to 1^-} \frac{-(1+x)}{2} = -1,$$

so f is differentiable at this point and $f'(1) = -1$. Summarising, f is continuous and differentiable in $\mathbb{R}$.

(b) Given that f is differentiable in $\mathbb{R}$, there must exist $c \in (0,2)$ such that

$$f(2) - f(0) = f'(c)(2-0) \quad \Leftrightarrow \quad \frac{1}{2} - \frac{3}{2} = 2f'(c) \quad \Leftrightarrow \quad -\frac{1}{2} = f'(c).$$

We do not know whether c is in $(0,1)$ or in $[1,2)$, so we have to check both. We have

$$f'(x) = \begin{cases} -x, & x < 1, \\ -\dfrac{1}{x^2}, & x \geqslant 1. \end{cases}$$

Assuming $0 < c < 1$, the equation becomes

$$-\frac{1}{2} = -c \quad \Rightarrow \quad c = \frac{1}{2}.$$

Assuming $1 \leqslant c < 2$, the equation becomes

$$-\frac{1}{2} = -\frac{1}{c^2} \quad \Rightarrow \quad c = \sqrt{2}.$$

Problem 7.11 The derivative is

$$f'(x) = -\frac{2}{3x^{1/3}},$$

so f is not differentiable at $x = 0$. This is the hypothesis that is not met.

Problem 7.12

(i) Assume that $a \leqslant x_1 < x_2 < \cdots < x_{k-1} < x_k \leqslant b$ are the k points where f vanishes in $[a,b]$. In any of the $k-1$ intervals $[x_j, x_{j+1}]$, with $j = 1, 2, \ldots, k-1$, we can apply Rolle's theorem and conclude that there must be at least a point in each of them where f' vanishes. This means that f' vanishes at least $k-1$ times in (a,b) —hence in $[a,b]$.

(ii) We can recursively apply the previous result and obtain

$$\begin{aligned} f \text{ vanishes } n+1 \text{ times in } [a,b] \quad &\Rightarrow \quad f' \text{ vanishes } n \text{ times in } [a,b] \\ f' \text{ vanishes } n \text{ times in } [a,b] \quad &\Rightarrow \quad f'' \text{ vanishes } n-1 \text{ times in } [a,b] \\ f'' \text{ vanishes } n-1 \text{ times in } [a,b] \quad &\Rightarrow \quad f''' \text{ vanishes } n-2 \text{ times in } [a,b] \\ &\vdots \\ f^{(n-1)} \text{ vanishes 2 times in } [a,b] \quad &\Rightarrow \quad f^{(n)} \text{ vanishes 1 time in } [a,b]. \end{aligned}$$

Problem 7.13 Let us consider the function $f(x) = x^{2/3}$ in the interval $[26,27]$. By the mean value theorem

$$27^{2/3} - 26^{2/3} = \frac{2}{3c^{1/3}}, \quad 26<c<27,$$

so

$$26^{2/3} = 9 - \frac{2}{3c^{1/3}}, \quad 26<c<27.$$

Approximating $c \approx 27$ we obtain

$$26^{2/3} \approx 9 - \frac{2}{9} = \frac{79}{9} = 8.777777\ldots$$

The exact value is $8.776382955\ldots$

Taking now $g(x) = \log x$ in $[1,3/2]$ we can write

$$\log(3/2) = \frac{1}{c}\left(\frac{3}{2} - 1\right) = \frac{1}{2c}, \quad 1 < c < \frac{3}{2}.$$

From this we conclude

$$\frac{1}{3} < \log(3/2) < \frac{1}{2} \quad \Leftrightarrow \quad 0.3333333\cdots < \log(3/2) < 0.5.$$

The exact vale is $0.405465108\ldots$

Problem 7.14

(i) We can obtain the limit

$$\ell = \lim_{x\to 0} \frac{e^x - \sin x - 1}{x^2}$$

by applying l'Hôpital's rule twice, as

$$\ell = \lim_{x\to 0} \frac{e^x + \sin x}{2} = \frac{1}{2}.$$

(ii) We can obtain the limit

$$\ell = \lim_{x\to 0} \frac{\log|\sin 7x|}{\log|x|}$$

by applying l'Hôpital's rule as

$$\ell = \lim_{x\to 0} \frac{7\cos 7x \sin x}{\sin 7x \cos x} = \lim_{x\to 0} \frac{7\cos 7x}{\cos x} \lim_{x\to 0} \frac{\sin x}{\sin 7x} = 7 \lim_{x\to 0} \frac{\sin x}{\sin 7x},$$

and then again,

$$\ell = 7 \lim_{x\to 0} \frac{\cos x}{7\cos 7x} = 1.$$

(iii) Writing the limit as

$$\ell = \lim_{x\to 1^+} \frac{\log(x-1)}{\frac{1}{\log x}}$$

it becomes a ∞/∞ indeterminacy, which we can sort out using l'Hôpital's rule. Thus,

$$\ell = \lim_{x\to 1^+} \frac{\frac{1}{x-1}}{-\frac{1}{x(\log x)^2}} = \lim_{x\to 1^+} \frac{-x(\log x)^2}{x-1} = -\lim_{x\to 1^+} \frac{(\log x)^2}{x-1}.$$

And we can solve this $0/0$ indeterminacy by applying l'Hôpital's rule once more to obtain

$$-\lim_{x\to 1^+} \frac{2\log x}{x} = 0.$$

Therefore $\ell = 0$.

(iv) This limit can be written as

$$\ell = \lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\log x/x} = \exp\left\{\lim_{x\to\infty} \frac{\log x}{x}\right\}.$$

This new limit can be obtain by applying l'Hôpital's rule as

$$\lim_{x\to\infty} \frac{1}{x} = 0,$$

therefore $\ell = 1$.

(v) The limit

$$\ell = \lim_{x\to 0} \frac{(1+x)^{1+x} - 1 - x - x^2}{x^3}$$

is a $0/0$ indeterminacy, which can be solved by applying l'Hôpital's rule three times. The denominator becomes then 6. As for the numerator, $(1-x-x^2)''' = 0$, so we have to take three derivatives of $g(x) = (1+x)^{1+x} = e^{(1+x)\log(1+x)}$. Thus,

$$\begin{aligned}
g'(x) &= g(x)\big[\log(1+x)+1\big],\\
g''(x) &= g(x)\big[\log(1+x)+1\big]^2 + \frac{g(x)}{1+x},\\
g'''(x) &= g(x)\big[\log(1+x)+1\big]^3 + 3g(x)\frac{\log(1+x)+1}{1+x} - \frac{g(x)}{(1+x)^2}.
\end{aligned}$$

Therefore

$$\ell = \frac{1}{6}\lim_{x\to 0}\left\{g(x)\big[\log(1+x)+1\big]^3 + 3g(x)\frac{\log(1+x)+1}{1+x} - \frac{g(x)}{(1+x)^2}\right\} = \frac{1}{2}.$$

(vi) We can change the variable x to $t = 1/x$. Then

$$\ell = \lim_{x\to\infty} x\left(\tan\frac{2}{x} - \tan\frac{1}{x}\right) = \lim_{t\to 0^+} \frac{\tan 2t - \tan t}{t}.$$

We can solve this $0/0$ indeterminacy by applying l'Hôpital's rule to obtain

$$\ell = \lim_{t\to 0^+}\left(\frac{2}{\cos^2 2t} - \frac{1}{\cos^2 t}\right) = 1.$$

Problem 7.15 First of all, $h(0) = 0$ because if $h(0) = c \neq 0$, then

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{h(x)}{x^2} = \pm\infty,$$

so f would not be continuous at $x = 0$.

Now, since the limit above is a $0/0$ indeterminacy we can try to apply l'Hôpital's rule and calculate

$$\lim_{x\to 0} \frac{h'(x)}{2x}.$$

As h is twice differentiable $h'(x) \to h'(0)$ as $x \to 0$. For the same reason as above $h'(0) = 0$, otherwise it would be $\pm\infty$ and

$$\lim_{x\to 0} \frac{h(x)}{x^2} = \pm\infty,$$

again in contradiction with the fact that f is continuous at $x = 0$.

Finally, once stablished that $h'(0) = 0$ we can rewrite

$$\lim_{x\to 0} \frac{h'(x)}{2x} = \frac{1}{2} \lim_{x\to 0} \frac{h'(x) - h'(0)}{x} = \frac{h''(0)}{2}.$$

This limit has to be 1 if f is to be continuous at $x = 0$, thus $h''(0) = 2$.

Problem 7.16

(i) We can change the variable x to $t = 1/x$ to tranform the limit

$$\ell = \lim_{x\to\infty} x \left[\left(1 + \frac{1}{x}\right)^x - e \right] = \lim_{t\to 0^+} \frac{(1+t)^{1/t} - e}{t}.$$

Since $(1+t)^{1/t} \to e$ as $t \to 0^+$ we face a $0/0$ indeterminacy. Let us apply l'Hôpital's rule and calculate

$$\ell = \lim_{t\to 0^+} (1+t)^{1/t} \left(\frac{1}{t(1+t)} - \frac{\log(1+t)}{t^2} \right) = e \lim_{t\to 0^+} \frac{t - (1+t)\log(1+t)}{t^2(1+t)}$$
$$= e \lim_{t\to 0^+} \frac{t - (1+t)\log(1+t)}{t^2},$$

another $0/0$ indeterminacy that can be solved by applying l'Hôpital's rule again twice. Doing it once we get

$$\ell = -\frac{e}{2} \lim_{t\to 0^+} \frac{\log(1+t)}{t},$$

and the second time we obtain

$$\ell = -\frac{e}{2} \lim_{t\to 0^+} \frac{1}{1+t} = -\frac{e}{2}.$$

(ii) Taking logarithms in the limit we can calculate it as

$$\log \ell = \lim_{x\to\infty} \left[x^2 \log\left(1 + \frac{1}{x}\right) - x \right].$$

Now we change the variable x to $t = 1/x$ and write

$$\log \ell = \lim_{t\to 0^+} \frac{\log(1+t) - t}{t^2},$$

a $0/0$ indeterminacy that can be solved by applying l'Hôpital's. Thus,

$$\log \ell = \lim_{t\to 0^+} \frac{\frac{1}{1+t}-1}{2t} = -\frac{1}{2}\lim_{t\to 0^+}\frac{t}{t(1+t)} = -\frac{1}{2}\lim_{t\to 0^+}\frac{1}{1+t} = -\frac{1}{2}.$$

Therefore $\ell = 1/\sqrt{e}$.

(iii) This is an indeterminacy 1^∞ which can be calculated as

$$\ell = \lim_{x\to\infty}\left(\frac{2^{1/x}+18^{1/x}}{2}\right)^x = e^c, \qquad c = \lim_{x\to\infty} x\left(\frac{2^{1/x}+18^{1/x}}{2}-1\right).$$

Now we change the variable x to $t = 1/x$ and write

$$c = \frac{1}{2}\lim_{t\to 0^+}\frac{2^t+18^t-2}{t} = \frac{1}{2}\left(\lim_{t\to 0^+}\frac{2^t-1}{t} + \lim_{t\to 0^+}\frac{18^t-1}{t}\right) = \frac{1}{2}\left(\frac{d}{dt}2^t\Big|_{t=0} + \frac{d}{dt}18^t\Big|_{t=0}\right)$$
$$= \frac{1}{2}(\log 2 + \log 18) = \log\sqrt{36} = \log 6.$$

Therefore $\ell = 6$.

(iv) This limit generalises the previous one. Again,

$$\ell = \lim_{x\to\infty}\left(\frac{1}{p}\sum_{k=1}^{p} a_k^{1/x}\right)^x = e^c, \qquad c = \lim_{x\to\infty} x\left(\frac{1}{p}\sum_{k=1}^{p} a_k^{1/x} - 1\right).$$

Now we change the variable x to $t = 1/x$ and write

$$c = \frac{1}{p}\lim_{t\to 0^+}\frac{\sum_k a_k^t - p}{t} = \frac{1}{p}\sum_{k=1}^{p}\lim_{t\to 0^+}\frac{a_k^t-1}{t} = \frac{1}{p}\sum_{k=1}^{p}\frac{d}{dt}a_k^t\Big|_{t=0} = \frac{1}{p}\sum_{k=1}^{p}\log a_k$$
$$= \log\left[(a_1a_2\cdots a_p)^{1/p}\right].$$

Therefore $\ell = (a_1a_2\cdots a_p)^{1/p}$.

Problem 7.17

(a) Suppose $f(0) = c \neq 0$. Then

$$\ell = \lim_{x\to 0}\frac{f(2x^3)}{5x^3} = \pm\infty,$$

in contradiction with the hypothesis. Thus $f(0) = 0$.

(b) Introduce the variable $t = 2x^3$. Then $t \to 0$ as $x \to 0$. Thus,

$$1 = \lim_{t\to 0}\frac{f(t)}{5t/2} = \frac{2}{5}\lim_{t\to 0}\frac{f(t)-f(0)}{t} = \frac{2}{5}f'(0),$$

hence $f'(0) = 5/2$.

(c) Applying l'Hôpital's rule, the limit

$$\ell = \lim_{x\to 0}\frac{(f\circ f)(2x)}{f^{-1}(3x)}$$

can be obtained through the derivatives of the functions at the numerator and denominator. But

$$\frac{d}{dx}(f\circ f)(2x) = \frac{d}{dx}f\big(f(2x)\big) = f'\big(f(2x)\big)f'(2x)2, \qquad \frac{d}{dx}f^{-1}(3x) = \frac{3}{f'\big(f^{-1}(3x)\big)},$$

and since $f(0)=0$ also $f^{-1}(0)=0$. Then

$$\begin{aligned}\ell &= \lim_{x\to 0}\frac{2f'(2x)f'\big(f(2x)\big)f'\big(f^{-1}(3x)\big)}{3} = \frac{2f'(0)f'\big(f(0)\big)f'\big(f^{-1}(0)\big)}{3} = \frac{2f'(0)^3}{3} \\ &= \frac{2}{3}\cdot\frac{5^3}{2^3} = \frac{125}{12}.\end{aligned}$$

Problem 7.18 Since $g(x)\to f^{-1}(1)=0$ as $x\to 0$, this limit is a $0/0$ indeterminacy. Thus we can apply l'Hôpital's rule to calculate it as

$$\ell = \lim_{x\to 0}\left(e^x+\cos x\, e^{-\sin x}\right) f'\left(f^{-1}(x+1)\right) = 2f'(0).$$

All that remains is to compute $f'(0)$. We can do that evaluating the equation defining $f(x)$ at $x=0$. This yields,

$$e^{-f(0)}f'(0)=2 \quad\Rightarrow\quad f'(0)=2e,$$

since $f(0)=1$. Thus $\ell=4e$.

Problem 7.19

(a) f is continuous in $\mathbb{R}$ because so are polynomials and the absolute value function. As for differentiability, we can express f in a piecewise description as

$$f(x)=\begin{cases}4x^3-x^4-1, & 0<x<4,\\ x^4-4x^3-1, & \text{otherwise,}\end{cases}$$

separating out the cases where $x^3(x-4)<0$ from those where $x^3(x-4)\geqslant 0$. Both pieces are differentiable (they are polynomials), so we must check the joints. Since

$$f'(x)=\begin{cases}12x^2-4x^3, & 0<x<4,\\ 4x^3-12x^2, & x<0 \text{ or } x>4,\end{cases}$$

we have $f'(0-)=f'(0^+)=0$, so f is differentiable at $x=0$, but $f'(4^-)=-64$, and $f'(4^+)=64$, so f is not differentiable at $x=4$.
Summarising, f is continuous in $\mathbb{R}$ and differentiable in $\mathbb{R}-\{4\}$.

(b) First of all we need to look where $f'(x)=0$. This means

$$4x^2(3-x)=0 \quad\Rightarrow\quad x=0,\ x=3.$$

If $x<0$ but close to $x=0$ then $f'(x)=4x^2(x-3)<0$; if $x>0$ but close to $x=0$ then $f'(x)=4x^2(3-x)>0$. Therefore f has a local minimum at $x=0$. On the other hand, if $x<3$ then $f'(x)=4x^2(3-x)>0$ and if $x>3$ then $f'(x)=4x^2(3-x)<0$, so f has a local maximum at $x=3$.
But this is not the whole story because f is not differentiable at $x=4$ —hence $x=4$ cannot be a solution to $f'(x)=0$. We need to check this point separately. Now, $f(4)=-1$, but for any $x\neq 4$ near $x=4$ we have $f(x)=|x^3(x-4)|-1>-1$, so $x=4$ is a local minimum.
Finally, -1 is the smallest value that $f(x)$ can take, and $f(0)=f(4)=-1$, so both, at $x=0$ and at $x=4$, function $f(x)$ reaches its absolute minimum. There is no absolute maximum though, because the function grows indefinitely as $x\to\pm\infty$.

(c) $f(0)=-1$ and $f(1)=2$, so Bolzano's theorem guarantees that there is at least one solution to $f(x)=0$ in $(0,1)$. On the other hand, in $(0,1)$ we have $f'(x)=4x^2(3-x)>0$ so the function is monotonically increasing. Therefore the solution is unique.

Problem 7.20

(a) The amount of material is proportional to the surface of the can, which is given by the formula $S = 2\pi r^2 + 2\pi rh$. But cans have all the same volume $V = \pi r^2 h$, so $h = V/\pi r^2$ and thefore

$$S = 2\pi\left(r^2 + \frac{V}{\pi r}\right).$$

Minimising the surface amounts to minimising the function

$$f(r) = r^2 + \frac{V}{\pi r}.$$

This is a differentiable function for all $r > 0$, so the minimum is reached at a solution of

$$f'(r) = 2r - \frac{V}{\pi r^2} = 0 \quad \Rightarrow \quad r^3 = \frac{V}{2\pi} \quad \Rightarrow \quad r = \left(\frac{V}{2\pi}\right)^{1/3}$$

and

$$h = \frac{V}{\pi r^2} = \left(\frac{4V}{\pi}\right)^{1/3}.$$

(b) Lead is proportional to the surface. If the side of the square base is a and the height h, then the surface will be $S = a^2 + 4ah$. The volume constraint, $32 = a^2 h$, implies $h = 32/a^2$, so

$$S = a^2 + \frac{128}{a} = f(a).$$

Now,

$$f'(a) = 2a - \frac{128}{a^2} \quad \Rightarrow \quad a^3 = 64 \quad \Rightarrow \quad a = 4, \quad h = 2.$$

(c) We can eliminate $y = 20 - x$, so the function to maximise is

$$f(x) = x^2(20-x)^3.$$

Now,

$$f'(x) = 2x(20-x)^3 - 3x^2(20-x)^2 = x(20-x)^2(40-2x-3x) = 5x(20-x)^2(8-x) = 0.$$

The two solutions $x = 0$, $x = 20$ clearly minimise the function. The maximum is then $x = 8$ and $y = 12$.

(d) If x is half the horizontal side of the rectangle, then

$$y = b\sqrt{1 - \frac{x^2}{a^2}}$$

is half the vertical side. Then the area of the rectangle is

$$A = 4xy = 4bx\sqrt{1 - \frac{x^2}{a^2}}.$$

Maximising this area is tantamount to maximising

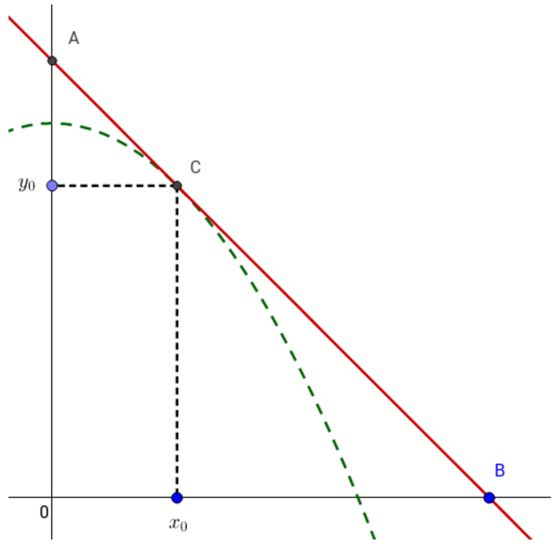
$$f(x) = \frac{A^2}{16b^2} = x^2 - \frac{x^4}{a^2},$$

which means solving the equation

$$f'(x) = 2x - \frac{4x^3}{a^2} = 2x\left(1 - \frac{2x^2}{a^2}\right) = 0.$$

One solution is $x = 0$ —which is obviously not the right one— and the other two solutions are $x = \pm a/\sqrt{2}$. Clearly the one that maximises the area has to be $x = a/\sqrt{2}$.

(e) The picture illustrates how to construct the described triangle:



We can select an arbitrary point on the parabola, $(x_0, 6-x_0^2)$. The slope of the tangent at that point will be $m=-2x_0$ (obtained differentiating $6-x^2$), so the equation of the tangent straight line will be

$$y = 6-x_0^2-2x_0(x-x_0) = 6+x_0^2-2x_0x.$$

Now, this straight line meets the Y axis at $A(0, 6+x_0^2)$, and the X axis at $B\big((6+x_0^2)/2x_0, 0\big)$, so the area of the triangle will be

$$A = \frac{(6+x_0^2)^2}{4x_0} = \frac{9}{x_0}+3x_0+\frac{x_0^3}{4} = f(x_0).$$

Minimising the area means solving

$$f'(x_0) = -\frac{9}{x_0^2}+3+\frac{3x_0^2}{4} = \frac{3(x_0^4+4x_0^2-12)}{4x_0^2} = \frac{3(x_0^2+6)(x_0^2-2)}{4x_0^2} = 0.$$

The only meaningful solution to this equation is $x_0=\sqrt{2}$.

(f) The area of the triangle at the base is $a^2\sqrt{3}/4$, and that of the lateral rectangles $3ah$, so the total cost will be

$$C = 0.20\times a^2\frac{\sqrt{3}}{4}+0.10\times 3ah = 0.10\times\sqrt{3}\left(\frac{a^2}{2}+\sqrt{3}ah\right).$$

Since $128 = ha^2\sqrt{3}/4$ we get $\sqrt{3}ah = 512/a$, so $C = 0.10\times\sqrt{3}f(a)$, where

$$f(a) = \frac{a^2}{2}+\frac{512}{a}.$$

The value of a minimising cost will be a solution of

$$f'(a) = a-\frac{512}{a^2} = 0 \quad\Rightarrow\quad a^3 = 512 \quad\Rightarrow\quad a = 8.$$

(g) For a given $0 \leqslant x \leqslant 2$ the corresponding y on the circunference is given by

$$y = \sqrt{1-(x-1)^2} = \sqrt{x(2-x)}.$$

Thus, the three points of the triangle are $A(0,0)$, $B\left(x, \sqrt{x(2-x)}\right)$, $C(x,0)$. The area of the triangle will then be $S = x\sqrt{x(2-x)}/2 = x^{3/2}(2-x)^{1/2}/2$. So maximising this area is tantamount to maximising

$$f(x) = 4S^2 = x^3(2-x) = 2x^3 - x^4.$$

The corresponding x will be a solution of

$$f'(x) = 6x^2 - 4x^3 = 2x^2(3-2x) = 0.$$

The only meaningful solution is $x = 3/2$.

(h) Triangle similarity implies

$$\frac{y_0+\beta}{x_0+\alpha} = \frac{\beta}{x_0} \quad \Rightarrow \quad x_0y_0 + \cancel{\beta x_0} = \cancel{\beta x_0} + \beta\alpha \quad \Rightarrow \quad \beta = \frac{x_0y_0}{\alpha}.$$

(i) The length of segment AB is

$$\ell = \sqrt{(x_0+\alpha)^2+(y_0+\beta)^2} = \sqrt{(x_0+\alpha)^2+\left(y_0+\frac{x_0y_0}{\alpha}\right)^2} = \sqrt{(x_0+\alpha)^2+\frac{y_0^2}{\alpha^2}(x_0+\alpha)^2}$$
$$= (x_0+\alpha)\sqrt{1+\frac{y_0^2}{\alpha^2}}.$$

So minimising ℓ is tantamount to minimising

$$f(\alpha) = \ell^2 = (x_0+\alpha)^2\left(1+\frac{y_0^2}{\alpha^2}\right).$$

Differentiating

$$f'(\alpha) = 2(x_0+\alpha)\left(1+\frac{y_0^2}{\alpha^2}\right) - 2(x_0+\alpha)^2\frac{y_0^2}{\alpha^3} = 2(x_0+\alpha)\left(1+\cancel{\frac{y_0^2}{\alpha^2}} - \frac{x_0y_0^2}{\alpha^3} - \cancel{\frac{y_0^2}{\alpha^2}}\right)$$
$$= 2(x_0+\alpha)\left(1-\frac{x_0y_0^2}{\alpha^3}\right) = 0.$$

This equation has the solution

$$\alpha = \left(x_0y_0^2\right)^{1/3}, \qquad \beta = \frac{x_0y_0}{\alpha} = \left(x_0^2y_0\right)^{1/3}.$$

(ii) The sum of segments OA and OB is

$$f(\alpha) = x_0+\alpha+y_0+\beta = x_0+y_0+\alpha+\frac{x_0y_0}{\alpha}.$$

Differentiating

$$f'(\alpha) = 1-\frac{x_0y_0}{\alpha^2} = 0 \quad \Rightarrow \quad \alpha = (x_0y_0)^{1/2}, \quad \beta = \frac{x_0y_0}{\alpha} = (x_0y_0)^{1/2}.$$

(iii) The area of the triangle is

$$A = \frac{1}{2}(x_0+\alpha)(y_0+\beta) = \frac{1}{2}(x_0+\alpha)\left(y_0+\frac{x_0y_0}{\alpha}\right) = \frac{y_0}{2}\frac{(x_0+\alpha)^2}{\alpha} = \frac{y_0}{2}\left(\frac{x_0^2}{\alpha}+2x_0+\alpha\right).$$

Minimising the area implies minimising

$$f(\alpha) = \frac{2A}{y_0} = \frac{x_0^2}{\alpha}+2x_0+\alpha.$$

Differentiating

$$f'(\alpha) = -\frac{x_0^2}{\alpha^2}+1 = 0 \quad \Rightarrow \quad \alpha = x_0, \quad \beta = \frac{x_0y_0}{\alpha} = y_0.$$

Problem 7.21

(a) For $a = 1$ the inequality becomes a trivial equality. For $a > 1$ take the function

$$f(x) = (1+x)^a - 1 - ax.$$

Differentiating,

$$f'(x) = a(1+x)^{a-1} - a = 0 \quad \Rightarrow \quad (1+x)^{a-1} = 1 \quad \Rightarrow \quad x = 0,$$

so $x = 0$ is a local extremum. From the second derivative,

$$f''(x) = a(a-1)(1+x)^{a-2} \quad \Rightarrow \quad f''(0) = a(a-1) > 0$$

we conclude that $x = 0$ is a minimum —the absolute minimum if $x > -1$—, therefore $f(x) \geqslant f(0) = 0$ for every $x > -1$. This means

$$(1+x)^a \geqslant 1+ax.$$

(b) Take the function

$$f(x) = e^x - 1 - x.$$

Differentiating,

$$f'(x) = e^x - 1 = 0 \quad \Rightarrow \quad x = 0,$$

so $x = 0$ is a local extremum. From the second derivative,

$$f''(x) = e^x \quad \Rightarrow \quad f''(0) = 1 > 0,$$

we conclude that $x = 0$ is a minimum —which is absolute in this case because there is no other one in $\mathbb{R}$. Therefore $f(x) \geqslant f(0) = 0$ for every $x \in \mathbb{R}$, i.e.,

$$e^x \geqslant 1+x.$$

(c) Take the function

$$f(x) = \log(1+x) - \frac{x}{1+x}.$$

Differentiating,

$$f'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{x}{(1+x)^2} = 0 \quad \Rightarrow \quad x = 0,$$

so $x=0$ is a local extremum. From the second derivative,

$$f''(x) = -\frac{1}{(1+x)^2} + \frac{2}{(1+x)^3} = \frac{1-x}{(1+x)^3} \quad \Rightarrow \quad f''(0) = 1 > 0,$$

we conclude that $x=0$ is a minimum —which is absolute in this case because there is no other one when $x > -1$. Therefore $f(x) \geqslant f(0) = 0$ for every $x > -1$. This proves the first inequality. As for the second, take

$$g(x) = x - \log(1+x)$$

and differentiate:

$$g'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} = 0 \quad \Rightarrow \quad x = 0,$$

so $x=0$ is a local extremum. From the second derivative,

$$f''(x) = \frac{1}{(1+x)^2} \quad \Rightarrow \quad f''(0) = 1 > 0,$$

we conclude that $x=0$ is a minimum —again absolute—, so $f(x) \geqslant f(0) = 0$ for every $x > -1$. This proves the second inequality.

Problem 7.22

(a) Take the function

$$f(x) = \frac{\log x}{x}.$$

Differentiating,

$$f'(x) = \frac{1-\log x}{x^2} = 0 \quad \Rightarrow \quad x = e.$$

From the second derivative,

$$f''(x) = \frac{2\log x - 3}{x^3} \quad \Rightarrow \quad f''(e) = -\frac{1}{e^3} < 0,$$

so $x=e$ is the absolute maximum for $x>0$. Thus $f(x) < f(e)$ for all $x>0$, $x \neq e$, which means

$$\frac{\log x}{x} < \frac{1}{e}.$$

(b) Multiplying the inequality by ex it becomes $e\log x < x$, and taking exponentials

$$x^e < e^x.$$

Problem 7.23

(i) The polynomial $f(x) = x^7 + 4x - 3 \sim x^7$ as $x \to \pm\infty$, so $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$. Thus $f(x) = 0$ at at least one point. What we need to know is to figure out how many times $f(x)$ bends up and down and from that determining the number of times it crosses the X axis. Now,

$$f'(x) = 7x^6 + 4 > 0$$

for all $x \in \mathbb{R}$, therefore $f(x)$ increases monotonically. The conclusion is that there is only *one* solution.

(ii) Similarly to the previous exercise, $f(x) = x^5 - 5x + 6 \sim x^5$ as $x \to \pm\infty$, so $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$. Thus $f(x) = 0$ at at least one point. Now,

$$f'(x) = 5x^4 - 5 = 0 \quad \Rightarrow \quad x = \pm 1,$$

and from the second derivative

$$f''(x) = 20x^3 \quad \Rightarrow \quad f''(1) = 20 > 0, \quad f''(-1) = -20 < 0,$$

so we conclude that $x = -1$ is a local minimum and $x = 1$ a local maximum. But $f(1) = 2 > 0$ and $f(-1) = 10 > 0$, so the local minimum is above the X axis. In conclusion, there is only *one* solution.

(iii) $f(x) = x^4 - 4x^3 - 1 \sim x^4$ as $x \to \pm\infty$, so $f(x) \to \infty$ when $x \to \pm\infty$. It is not guaranteed that there is even a single solution. From the derivative,

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3) = 0$$

we conclude that $x = 0$ and $x = 3$ may be extrema. $f'(x) < 0$ around $x = 0$ (at both sides), so it is an inflection point. However, close to $x = 3$ we have $f'(x) < 0$ for $x < 3$ and $f'(x) > 0$ for $x > 3$, so at $x = 3$ the polynomial reaches its absolute minimum $f(3) = -28$. Since this value is below the X axis, $f(x)$ has to cross it twice. Therefore there are *two* solutions to the equation.

(iv) The function $f(x) = 2x - 1 - \sin x \sim 2x$ as $x \to \pm\infty$, so $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$. Thus $f(x) = 0$ at at least one point. Now,

$$f'(x) = 2 - \cos x > 0 \quad \text{for all } x \in \mathbb{R},$$

so $f(x)$ monotonically increases. Therefore there is only *one* solution.

(v) Let us first rewite the equation. Taking logarithms the equation becomes

$$f(x) = x \log x - \log 2 = 0.$$

$f(1) = -\log 2 < 0$ and $f(x) \to \infty$ as $x \to \infty$, so $f(x)$ vanishes at one point at least. Now,

$$f'(x) = \log x + 1,$$

which is $f'(x) < 0$ for $x < 1/e$ and $f'(x) > 0$ for $x > 1/e$. In other words, $f'(x) > 0$ in the interval $[1, \infty)$, so $f(x)$ monotonically increases in that interval. Therefore there is only *one* solution.

(vi) Writing the equation

$$f(x) = x^2 + \log x = 0$$

we have $f(1) = 1 > 0$, and $f(x) \sim x^2$ as $x \to \pm\infty$, so $f(x) \to \infty$ as $x \to \pm\infty$. There is no guarantee that the equation has even a single solution in that interval. From the derivative,

$$f'(x) = 2x + \frac{1}{x} = \frac{2x^2 + 1}{x}$$

we conclude that $f'(x) > 0$ in $(1, \infty)$, so $f(x)$ increases monotonically. Therefore the equation has *no* solution in that interval.