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Calculus I

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Unit 8. Taylor Expansions

Exercises

Problems

Problem 8.1 Write the Taylor polynomial $P_{5,0}(x)$ for these functions:

(i) $e^x \sin x$;
(ii) $e^{-x^2} \cos 2x$;
(iii) $\sin x \cos 2x$;
(iv) $e^x \log(1-x)$;
(v) $\sin^2 x$;
(vi) $\dfrac{1}{1-x^3}$.

Problem 8.2 Write the polynomial $x^4 - 5x^3 + x^2 - 3x + 4$ in powers of $x - 4$.

Problem 8.3 Write the Taylor polynomial $P_{n,a}(x)$ for these functions around the specified a:

(i) $f(x) = 1/x$ around $a = -1$;
(ii) $f(x) = xe^{-2x}$ around $a = 0$;
(iii) $f(x) = (1+e^x)^2$ around $a = 0$;
(iv) $f(x) = \sin x$ around $a = \pi$.

Problem 8.4 Consider the function

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0 & x = 0. \end{cases}$$

(i) Prove by induction that $f^{(n)}(x) = Q_n(1/x)e^{-1/x^2}$ for $x \neq 0$, where $Q_n(t)$ is some polynomial.
(ii) Prove by induction that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.
(iii) Write the Taylor polynomial $P_{n,0}(x)$ of $f(x)$. What can you conclude from that?

Problem 8.5 Prove that

(i) $\sin x = o(x^\alpha)$ $(x \to 0)$ for all $\alpha < 1$;
(ii) $\log(1+x^2) = o(x)$ $(x \to 0)$;
(iii) $\log x = o(x)$ $(x \to \infty)$;
(iv) $\tan x - \sin x = o(x^2)$ $(x \to 0)$.

Problem 8.6 Calculate the following limits using Taylor's theorem:

(i) $\displaystyle\lim_{x\to 0} \frac{e^x - \sin x - 1}{x^2}$;
(ii) $\displaystyle\lim_{x\to 0} \frac{\sin x - x + x^3/6}{x^5}$;
(iii) $\displaystyle\lim_{x\to 0} \frac{\cos x - \sqrt{1-x}}{\sin x}$;
(iv) $\displaystyle\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$;
(v) $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x(1-\cos 3x)}$;
(vi) $\displaystyle\lim_{x\to 0} \frac{\cos x + e^x - x - 2}{x^3}$;
(vii) $\displaystyle\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$;
(viii) $\displaystyle\lim_{x\to 0} \frac{1}{x}\left(\frac{1}{x} - \cot x\right)$;
(ix) $\displaystyle\lim_{x\to \infty} x^{3/2}\left(\sqrt{x+1} + \sqrt{x-1} - 2\sqrt{x}\right)$;
(x) $\displaystyle\lim_{x\to \infty} \left[x - x^2 \log\left(1 + \frac{1}{x}\right)\right]$.

Problem 8.7 If $f(x) = -\dfrac{x}{2} - \dfrac{x^2}{4} + o(x^2)$ $(x \to 0)$, calculate

$$\lim_{x\to 0} \frac{\log[1+f(x)] + x/2}{x^2}.$$

Problem 8.8 Prove that the function

$$f(x) = \begin{cases} \dfrac{1}{x} - \dfrac{1}{e^x - 1}, & \text{if } x \neq 0, \\ \dfrac{1}{2}, & \text{if } x = 0, \end{cases}$$

is differentiable at $x = 0$ by calculating $f'(0)$ from the definition.

Problem 8.9 Determine the first nonzero order in the Taylor expansion of the following functions:

(i) $f(x) = \tan(\sin x) - \sin(\tan x)$;

(ii) $f(x) = \dfrac{1}{R^2} - \dfrac{1}{(R+x)^2}$;

(iii) $f(x) = \sqrt[3]{\dfrac{1+x}{1-x}} - \sqrt[3]{\dfrac{1-x}{1+x}}$.

Problem 8.10 Consider the function

$$f(x) = \frac{1-\cos x}{1+\cos x}.$$

This function is even and $f(0) = 0$, so its Taylor expansion up to 7th order will be

$$f(x) = Ax^2 + Bx^4 + Cx^6 + o(x^7), \quad (x \to 0).$$

Then

$$1 - \cos x = \left[Ax^2 + Bx^4 + Cx^6 + o(x^7)\right](1 + \cos x).$$

Using the Taylor expansion of $\cos x$ up to 7th order find the coefficients A, B, and C from this equation.

Problem 8.11 Find coefficients a and b so that

(i) $x - (a + b\cos x)\sin x = o(x^4)$ $(x \to 0)$;

(ii) $\cot x - \dfrac{1+ax^2}{x+bx^3} = o(x^4)$ $(x \to 0)$.

Problem 8.12 Find constants $a, b, c, d \in \mathbb{R}$ such that

$$e^x = \frac{1+ax+bx^2}{1+cx+dx^2} + o(x^4) \quad (x \to 0).$$

Problem 8.13 Given that $\sqrt{1+x} = 1 + \frac{x}{2} + o(x)$ $(x \to 0)$, prove:

(i) $\lim\limits_{n\to\infty} \sin\left(\pi\sqrt{1+n^2}\right) = 0$;

(ii) $\sum\limits_{n=0}^{\infty} \sin^2\left(\pi\sqrt{1+n^2}\right) < \infty$.

Problem 8.14 Calculate the Taylor polynomial $P_{4,0}(x)$ for $f(x) = 1 + x^3 \sin x$. Given the result, does f have a local maximum, minimum or inflection point at $x = 0$?

Problem 8.15 Use a Taylor polynomial of the specified degree to provide an approximation to these numbers, and give an upper bound for the error incurred:

(i) $\dfrac{1}{\sqrt{1.1}}$, degree 3; (ii) $\sqrt[3]{28}$, degree 2.

Problem 8.16 Given the function $f(x) = \cos x + e^x$,

(i) find its Taylor polynomial $P_{3,0}(x)$;

(ii) estimate an upper bound for the error incurred if $-1/4 \leqslant x \leqslant 1/4$.

Problem 8.17 What is the smallest degree Taylor polynomial necessary to approximate the function $f(x) = e^x$ in $[-1, 1]$ with at least three exact decimal places?

Problem 8.18 Determine the convergence radius of the following power series, and specify the interval where they converge absolutely:

(i) $\sum_{n=1}^{\infty} \frac{x^n}{2^n n^2}$;

(ii) $\sum_{n=1}^{\infty} \frac{n!x^n}{n^n}$;

(iii) $\sum_{n=1}^{\infty} \frac{x^n}{n10^{n-1}}$;

(iv) $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$;

(v) $\sum_{n=0}^{\infty} (3-2x)^n$;

(vi) $\sum_{n=1}^{\infty} \frac{(x-2)^n}{\sqrt{2n}}$.

Problem 8.19 Expand the function $f(x) = \frac{1}{(1-x)^k}$ for $k = 1$, 2, and 3.

Problem 8.20 Consider the power series

$$\frac{1}{x^2+x+1} = \sum_{n=0}^{\infty} a_n x^n.$$

What are the values of the coefficients a_{300}, a_{301}, and a_{302}?
HINT: Recall that $1-x^3 = (1-x)(x^2+x+1)$.

Problem 8.21 Calculate the derivatives $f^{(100)}(0)$ and $f^{(231)}(0)$ of the function $f(x) = \log\left(1+x^2\right)$.

Problem 8.22 Determine the convergence radius of the following power series, and calculate their sums:

(i) $\sum_{n=1}^{\infty} \frac{x^n}{n}$;

(ii) $\sum_{n=0}^{\infty} (n+1)2^{-n}x^n$.

Problem 8.23 Expand in power series the following functions, specifying the domain of validity of those expansions:

(i) $f(x) = \sin^2 x$;

(ii) $f(x) = \log\sqrt{\frac{1+x}{1-x}}$;

(iii) $f(x) = \frac{x}{a+bx}$;

(iv) $f(x) = \frac{1}{2-x^2}$;

(v) $f(x) = \frac{1+x-(1-x)e^{2x}}{e^x}$.

Problem 8.24 Sum the following series:

(i) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$;

(ii) $\sum_{n=1}^{\infty} \frac{n}{2^n}$;

(iii) $\sum_{n=1}^{\infty} \frac{1}{n2^n}$;

(iv) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

Problem 8.25 Given the function $f(x) = \sum_{n=1}^{\infty} \frac{n^x}{n!}$, compute the values $f(0)$, $f(1)$, and $f(2)$.

Problem 8.26 Find a function $f(x)$ that can be expaded in power series of x and such that it satisfies the equation $f'(x) = f(x) + x$ with the condition $f(0) = 2$.

Problem 8.27 Prove that if f and g are twice differentiable, convex functions, and f is increasing, then $h = f \circ g$ is convex.

Problem 8.28 Discuss the convexity of the following functions:

(i) $f(x) = (x-2)x^{2/3}$;

(ii) $f(x) = |x|e^{|x|}$;

(iii) $f(x) = \log(x^2-6x+8)$.

Problem 8.29

(i) Sketch the graph of the function $f(x) = x + \log|x^2-1|$.

(ii) Based on the previous graph, plot function $g(x) = |x| + \log|x^2-1|$ and $h(x) = \left|x + \log|x^2-1|\right|$.

Problem 8.30 Sketch a plot of the following functions:

(i) $f(x) = e^x \sin x$;
(ii) $f(x) = \sqrt{x^2-1} - 1$;
(iii) $f(x) = xe^{1/x}$;
(iv) $f(x) = x^2 e^x$;
(v) $f(x) = (x-2)x^{2/3}$;
(vi) $f(x) = (x^2-1)\log\left(\dfrac{1+x}{1-x}\right)$;
(vii) $f(x) = \dfrac{x}{\log x}$;
(viii) $f(x) = \dfrac{x^2-1}{x^2+1}$;
(ix) $f(x) = \dfrac{e^{1/x}}{1-x}$;
(x) $f(x) = \log[(x-1)(x-2)]$;
(xi) $f(x) = \dfrac{e^x}{x(x-1)}$;
(xii) $f(x) = 2\sin x + \cos 2x$;
(xiii) $f(x) = \dfrac{x-2}{\sqrt{4x^2+1}}$;
(xiv) $f(x) = \sqrt{|x-4|}$;
(xv) $f(x) = \dfrac{1}{1+e^x}$;
(xvi) $f(x) = \dfrac{e^{2x}}{e^x-1}$;
(xvii) $f(x) = e^{-x}\sin x$;
(xviii) $f(x) = x^2 \sin\dfrac{1}{x}$.

Problem 8.31 Draw the graph of the following functions:

(i) $f(x) = \min\{\log|x^3-3|, \log|x+3|\}$;
(ii) $f(x) = \dfrac{1}{|x|-1} - \dfrac{1}{|x-1|}$;
(iii) $f(x) = \dfrac{1}{1+|x|} - \dfrac{1}{1+|x-a|}$, $(a>0)$;
(iv) $f(x) = x\sqrt{x^2-1}$;
(v) $f(x) = \arctan\log|x^2-1|$;
(vi) $f(x) = 2\arctan x + \arcsin\left(\dfrac{2x}{1+x^2}\right)$.

Problem 8.32 Plot the function

$$f(x) = \begin{cases} \dfrac{e^{1/x}}{1+x}, & x \neq 0, \\ 0 & x = 0, \end{cases}$$

and discuss how many real solutions has the equation $\dfrac{e^{1/x}}{1+x} = x^3$.

Problem 8.33 Given the function $f(x) = \dfrac{1+x}{3+x^2}$ plot the functions $g(x) = \sup_{y>x} f(y)$ and $h(x) = \inf_{y>x} f(y)$.

Problem 8.34 Determine the equations of the tangents to $f(x) = \log(1+x^2)$ at its inflection points and plot them along with the graph of $f(x)$.