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Calculus I

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Unit 8. Taylor Expansions

Solutions

D.8 Taylor Expansions

Problem 8.1

(i) There are two ways to solve these exercises. The first one amounts to applying Taylor's formula for $P_{n,a}(x)$. For the case of $f(x) = e^x \sin x$ we have

$$\begin{aligned}
f(x) &= e^x \sin x, & f(0) &= 0,\\
f'(x) &= e^x(\sin x + \cos x), & f'(0) &= 1,\\
f''(x) &= 2e^x \cos x, & f''(0) &= 2,\\
f'''(x) &= 2e^x(\cos x - \sin x), & f'''(0) &= 2,\\
f^{(4)}(x) &= -4e^x \sin x, & f^{(4)}(0) &= 0,\\
f^{(5)}(x) &= -4e^x(\sin x + \cos x), & f^{(5)}(0) &= -4,
\end{aligned}$$

thus

$$P_{5,0}(x) = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30}.$$

The alternative way —the one we will follow here— amounts to relying upon known Taylor expansions and operate with them. For instance in this case we know that when $x \to 0$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + o(x^5), \qquad \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5),$$

therefore, multiplying the two expressions —and collecting any power higher than x^5 as $o(x^5)$— we obtain

$$\begin{aligned}
e^x \sin x &= \left[1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + o(x^5)\right]\left[x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)\right]\\
&= \left[x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)\right] + \left[x^2 - \cancel{\frac{x^4}{6}} + o(x^5)\right] + \left[\frac{x^3}{2} - \frac{x^5}{12} + o(x^5)\right]\\
&\quad + \left[\cancel{\frac{x^4}{6}} + o(x^5)\right] + \left[\frac{x^5}{24} + o(x^5)\right]\\
&= x + x^2 + \left(\frac{1}{2} - \frac{1}{6}\right)x^3 + \left(\frac{1}{120} + \frac{1}{24} - \frac{1}{12}\right)x^5 + o(x^5)\\
&= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + o(x^5),
\end{aligned}$$

and we get to the same result.

(ii) Now

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} + o(x^5), \qquad \cos 2x = 1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{24} + o(x^5) = 1 - 2x^2 + \frac{2}{3}x^4 + o(x^5),$$

so multiplying and collecting equal powers,

$$\begin{aligned}
e^{-x^2}\cos 2x &= \left[1 - x^2 + \frac{x^4}{2} + o(x^5)\right]\left[1 - 2x^2 + \frac{2}{3}x^4 + o(x^5)\right]\\
&= 1 - (1+2)x^2 + \left(\frac{1}{2} + 2 + \frac{2}{3}\right)x^4 + o(x^5)\\
&= 1 - 3x^2 + \frac{19}{6}x^4 + o(x^5).
\end{aligned}$$

Thus

$$P_{5,0}(x) = 1 - 3x^2 + \frac{19}{6}x^4.$$

(iii) Using the trigonometric identity

$$\sin\theta\cos\phi = \frac{1}{2}\left[\sin(\theta+\phi) + \sin(\theta-\phi)\right]$$

we can write

$$\sin x \cos 2x = \frac{1}{2}\left(\sin 3x - \sin x\right).$$

Now, since for $z \to 0$

$$\sin z = z - \frac{z^3}{6} + \frac{z^5}{120} + o(z^5),$$

then

$$\sin x \cos 2x = \frac{1}{2}\left(3x - \frac{9}{2}x^3 + \frac{81}{40}x^5 - x + \frac{x^3}{6} - \frac{x^5}{120}\right) + o(x^5)$$

(iv) In this case

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + o(x^5), \qquad \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} + o(x^5),$$

so

$$\begin{aligned}
e^x\log(1-x) &= -x\left[1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + o(x^4)\right]\left[1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + o(x^4)\right] \\
&= -x\left[1 + \left(1 + \frac{1}{2}\right)x + \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{3}\right)x^2 + \left(\frac{1}{6} + \frac{1}{4} + \frac{1}{3} + \frac{1}{4}\right)x^3\right. \\
&\quad \left. + \left(\frac{1}{24} + \frac{1}{12} + \frac{1}{6} + \frac{1}{4} + \frac{1}{5}\right)x^4 + o(x^4)\right] \\
&= -x - \frac{3}{2}x^2 - \frac{4}{3}x^3 - x^4 - \frac{89}{120}x^5 + o(x^5).
\end{aligned}$$

Therefore

$$P_{5,0}(x) = -x - \frac{3}{2}x^2 - \frac{4}{3}x^3 - x^4 - \frac{89}{120}x^5.$$

(v) Since $\sin^2 x = (1-\cos 2x)/2$,

$$\sin^2 x = \frac{1}{2}\left[\cancel{1} - \cancel{1} + \frac{(2x)^2}{2} - \frac{(2x)^4}{24} + o(x^5)\right] = x^2 - \frac{x^4}{3} + o(x^5),$$

hence

$$P_{5,0}(x) = x^2 - \frac{x^4}{3}.$$

(vi) We know that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots,$$

therefore

$$\frac{1}{1-x^3} = 1 + x^3 + o(x^5),$$

which implies $P_{5,0}(x) = 1 + x^3$.

Problem 8.2 The Taylor polynomial $P_{4,4}(x)$ of $P(x) = x^4 - 5x^3 + x^2 - 3x + 4$ is obtained through

$$\begin{aligned}
&P(x) = x^4 - 5x^3 + x^2 - 3x + 4, && P(4) = -56,\\
&P'(x) = 4x^3 - 15x^2 + 2x - 3, && P'(4) = 21,\\
&P''(x) = 12x^2 - 30x + 2, && P''(4) = 74,\\
&P'''(x) = 24x - 30, && P'''(4) = 66,\\
&P^{(4)}(x) = 24, && P^{(4)}(4) = 24.
\end{aligned}$$

Hence

$$P(x) = -56 + 21(x-4) + 37(x-4)^2 + 11(x-4)^3 + (x-4)^4.$$

Problem 8.3

(i) The polynomial must be expressed in powers of $t = x+1$, so if we write

$$\frac{1}{x} = \frac{1}{t-1} = -\frac{1}{1-t} = -1 - t - t^2 - \cdots - t^n + \cdots$$

we immediately obtain $P_{n,-1}(x) = -1 - (x+1) - (x+1)^2 - \cdots - (x+1)^n$.

(ii) Since

$$\begin{aligned}
e^{-2x} &= 1 + (-2x) + \frac{(-2x)^2}{2} + \cdots + \frac{(-2x)^{n-1}}{(n-1)!} + o(x^{n-1})\\
&= 1 - 2x + 2x^2 + \cdots + (-1)^{n-1}\frac{2^{n-1}}{(n-1)!}x^{n-1} + o(x^{n-1})
\end{aligned}$$

then

$$xe^{-2x} = x - 2x^2 + 2x^3 + \cdots + (-1)^{n-1}\frac{2^{n-1}}{(n-1)!}x^n + o(x^n).$$

Thus

$$P_{n,0}(x) = x - 2x^2 + 2x^3 + \cdots + (-1)^{n-1}\frac{2^{n-1}}{(n-1)!}x^n.$$

(iii) We can expand $(1+e^x)^2 = 1 + 2e^x + e^{2x}$, so

$$\begin{aligned}
(1+e^x)^2 =& 1 + 2\left[1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + o(x^n)\right] + \left[1 + 2x + \frac{(2x)^2}{2} + \cdots + \frac{(2x)^n}{n!} + o(x^n)\right]\\
=& 4 + 4x + 3x^2 + \cdots + \frac{2+2^n}{n!}x^n + o(x^n),
\end{aligned}$$

from which

$$P_{n,0}(x) = 4 + 4x + 3x^2 + \cdots + \frac{2+2^n}{n!}x^n.$$

(iv) We must express the polynomial in powers of $t = x - \pi$, therefore $\sin x = \sin(\pi + t) = -\sin t$, and

$$\sin x = -t + \frac{t^3}{6} - \frac{t^5}{120} + \cdots + (-1)^n \frac{t^{2n-1}}{(2n-1)!} + o(t^{2n-1}).$$

Thus

$$P_{2n,\pi}(x) = P_{2n-1,\pi}(x) = -(x-\pi) + \frac{(x-\pi)^3}{6} - \frac{(x-\pi)^5}{120} + \cdots + (-1)^n \frac{(x-\pi)^{2n-1}}{(2n-1)!}.$$

Problem 8.4

(i) For $x \neq 0$,

$$f'(x) = \frac{2}{x^3} e^{-1/x^2}.$$

so $Q_1(t) = 2t^3$. Suppose now that $f^{(n)}(x) = Q_n(1/x)e^{-1/x^2}$ for $x \neq 0$ and some $n \in \mathbb{N}$. Differentiating once more,

$$f^{(n+1)} = \left[-\frac{1}{x^2} Q_n'(1/x) + \frac{2}{x^3} Q_n(1/x) \right] e^{-1/x^2} = Q_{n+1}(1/x)e^{-1/x^2},$$

where $Q_{n+1}(t) = -t^2 Q_n'(t) + 2t^3 Q_n(t)$ is a polynomial if so is $Q_n(t)$. This proves the result for all $n \in \mathbb{N}$.

(ii) First of all,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{1}{x} e^{-1/x^2} = \lim_{t\to\pm\infty} te^{-t^2} = 0.$$

Suppose now that $f^{(n)}(0) = 0$ for some $n \in \mathbb{N}$. From (i),

$$f^{(n+1)}(0) = \lim_{x\to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x} = \lim_{x\to 0} \frac{1}{x} Q_n(1/x)e^{-1/x^2} = \lim_{t\to\pm\infty} tQ_n(t)e^{-t^2} = 0.$$

This proves that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

(iii) Since we have proven that $f^{(n)}(0) = 0$ for every $n \in \mathbb{N}$, this means that $P_{n,0}(x) = 0$ for every $n \in \mathbb{N}$. In other words, the best polynomial approximation to $f(x)$ at $x = 0$ is just 0. The conclusion we get from this fact is that the reminder of this function must be $R_{n,0}(x) = f(x)$ for any $n \in \mathbb{N}$.
This is one example of a function that does not have a Taylor series which converges to it.

Problem 8.5

(i) Since $\sin x \sim x$ when $x \to 0$,

$$\lim_{x\to 0} \frac{\sin x}{x^\alpha} = \lim_{x\to 0} \frac{x}{x^\alpha} = \lim_{x\to 0} x^{1-\alpha} = 0$$

because $1 - \alpha > 0$.

(ii) Since $\log(1 + x^2) \sim x^2$ when $x \to 0$,

$$\lim_{x\to 0} \frac{\log(1+x^2)}{x} = \lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0.$$

(iii) We need to calculate the limit

$$\lim_{x\to\infty} \frac{\log x}{x}.$$

Since this is an indeterminacy $\frac{\infty}{\infty}$ we can apply l'Hôpital and calculate instead

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

(iv) Since

$$\lim_{x\to 0} \frac{\tan x - \sin x}{x^2}$$

is a $\frac{0}{0}$ indeterminacy we can apply l'Hôpital and calculate instead

$$\lim_{x\to 0} \frac{1+\tan^2 x - \cos x}{2x}.$$

And we apply l'Hôpital again because this is still a $\frac{0}{0}$ indeterminacy:

$$\lim_{x\to 0} \frac{2\tan x(1+\tan^2 x)+\sin x}{2} = 0.$$

Problem 8.6

(i) When $x \to 0$ e have

$$e^x = 1+x+\frac{x^2}{2}+o(x^2), \qquad \sin x = x+o(x^2),$$

thus

$$\lim_{x\to 0} \frac{e^x - \sin x - 1}{x^2} = \lim_{x\to 0} \frac{1+x+\frac{x^2}{2}+o(x^2)-x-1}{x^2} = \lim_{x\to 0} \frac{\frac{x^2}{2}+o(x^2)}{x^2}$$
$$= \lim_{x\to 0} \left(\frac{1}{2}+o(1)\right) = \frac{1}{2}.$$

(ii) When $x \to 0$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5),$$

so

$$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5} = \lim_{x\to 0} \frac{x-\frac{x^3}{6}+\frac{x^5}{120}+o(x^5)-x+\frac{x^3}{6}}{x^5} = \lim_{x\to 0} \frac{\frac{x^5}{120}+o(x^5)}{x^5}$$
$$= \lim_{x\to 0} \left(\frac{1}{120}+o(1)\right) = \frac{1}{120}.$$

(iii) When $x \to 0$ the denominator $\sin x = x + o(x)$. On the other hand, $\cos x = 1 + o(x)$ and $\sqrt{1-x} = 1 - \frac{x}{2} + o(x)$, so

$$\lim_{x\to 0} \frac{\cos x - \sqrt{1-x}}{\sin x} = \lim_{x\to 0} \frac{1+o(x)-1+\frac{x}{2}}{x+o(x)} = \lim_{x\to 0} \frac{\frac{x}{2}+o(x)}{x+o(x)} = \lim_{x\to 0} \frac{\frac{1}{2}+o(1)}{1+o(1)} = \frac{1}{2}.$$

(iv) When $x \to 0$

$$\tan x = x + \frac{x^3}{3} + o(x^3), \qquad \sin x = x - \frac{x^3}{6} + o(x^3),$$

so

$$\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x\to 0} \frac{x + \frac{x^3}{3} + o(x^3) - x + \frac{x^3}{6}}{x^3} = \lim_{x\to 0} \frac{\frac{x^3}{2} + o(x^3)}{x^3} = \lim_{x\to 0} \left(\frac{1}{2} + o(1)\right) = \frac{1}{2}.$$

(v) When $x \to 0$

$$\cos 3x = 1 - \frac{9}{2}x^2 + o(x^2), \qquad \sin x = x - \frac{x^3}{6} + o(x^3),$$

so

$$\lim_{x\to 0} \frac{x - \sin x}{x(1-\cos 3x)} = \lim_{x\to 0} \frac{\frac{x^3}{6} + o(x^3)}{x\left(\frac{9}{2}x^2 + o(x^2)\right)} = \lim_{x\to 0} \frac{\frac{1}{6} + o(1)}{\frac{9}{2} + o(1)} = \frac{1}{27}.$$

(vi) When $x \to 0$

$$\cos x = 1 - \frac{x^2}{2} + o(x^3), \qquad e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3),$$

so

$$\lim_{x\to 0} \frac{\cos x + e^x - x - 2}{x^3} = \lim_{x\to 0} \frac{1 - \frac{x^2}{2} + o(x^3) + 1 + x + \frac{x^2}{2} + \frac{x^3}{6} - x - 2}{x^3} = \lim_{x\to 0} \frac{\frac{x^3}{6} + o(x^3)}{x^3}$$
$$= \lim_{x\to 0} \left(\frac{1}{6} + o(1)\right) = \frac{1}{6}.$$

(vii) First of all

$$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0} \frac{\sin x - x}{x \sin x}.$$

Now, $\sin x = x + o(x^2)$ when $x \to 0$, so

$$\lim_{x\to 0} \frac{\sin x - x}{x \sin x} = \lim_{x\to 0} \frac{o(x^2)}{x^2 + o(x^3)} = \lim_{x\to 0} \frac{o(1)}{1 + o(x)} = \lim_{x\to 0} \frac{o(1)}{1 + o(1)} = 0.$$

(Remember that $o(x^n)$ can be replaced by $o(x^m)$ when $x \to 0$ if $n > m$.)

(viii) To begin with, since $\cot x = \cos x / \sin x$,

$$\lim_{x\to 0} \frac{1}{x} \left(\frac{1}{x} - \cot x\right) = \lim_{x\to 0} \frac{\sin x - x\cos x}{x^2 \sin x}.$$

Now,

$$\sin x = x - \frac{x^3}{6} + o(x^3) = x + o(x), \qquad x\cos x = x\left[1 - \frac{x^2}{2} + o(x^2)\right] = x - \frac{x^3}{2} + o(x^3),$$

hence

$$\lim_{x\to 0} \frac{\sin x - x\cos x}{x^2 \sin x} = \lim_{x\to 0} \frac{x - \frac{x^3}{6} + o(x^3) - x + \frac{x^3}{2}}{x^3 + o(x^3)} = \lim_{x\to 0} \frac{\frac{x^3}{3} + o(x^3)}{x^3 + o(x^3)} = \lim_{x\to 0} \frac{\frac{1}{3} + o(1)}{1 + o(1)} = \frac{1}{3}.$$

(ix) We cannot apply Taylor when $x \to \infty$, but we can rewrite the limit by changing the variable to $t = 1/x$. Then

$$\ell = \lim_{x\to\infty} x^{3/2}\left(\sqrt{x+1}+\sqrt{x-1}-2\sqrt{x}\right) = \lim_{t\to 0^+} \frac{\sqrt{\frac{1}{t}+1}+\sqrt{\frac{1}{t}-1}-\frac{2}{\sqrt{t}}}{t^{3/2}},$$

and multiplying numerator and denominator by $\sqrt{t}$,

$$\ell = \lim_{t\to 0^+} \frac{\sqrt{1+t}+\sqrt{1-t}-2}{t^2}.$$

Now we know that when $t \to 0$

$$(1+t)^\alpha = 1+\alpha t+\frac{\alpha(\alpha-1)}{2}t^2+o(t^2),$$

so setting $\alpha = 1/2$,

$$\sqrt{1+t} = 1+\frac{t}{2}-\frac{t^2}{8}+o(t^2), \qquad \sqrt{1-t} = 1-\frac{t}{2}-\frac{t^2}{8}+o(t^2).$$

Thus,

$$\ell = \lim_{t\to 0^+} \frac{1+\frac{t}{2}-\frac{t^2}{8}+o(t^2)+1-\frac{t}{2}-\frac{t^2}{8}-2}{t^2} = \lim_{t\to 0^+} \frac{-\frac{t^2}{4}+o(t^2)}{t^2} = \lim_{t\to 0^+}\left(-\frac{1}{4}+o(1)\right) = -\frac{1}{4}.$$

(x) Changing from x to $t = 1/x$,

$$\ell = \lim_{x\to\infty}\left[x-x^2\log\left(1+\frac{1}{x}\right)\right] = \lim_{t\to 0^+}\left[\frac{1}{t}-\frac{\log(1+t)}{t^2}\right] = \lim_{t\to 0^+}\frac{t-\log(1+t)}{t^2}.$$

If we now write

$$\log(1+t) = t-\frac{t^2}{2}+o(t^2) \quad (t\to 0),$$

then

$$\ell = \lim_{t\to 0^+}\frac{t-t+\frac{t^2}{2}+o(t^2)}{t^2} = \lim_{t\to 0^+}\frac{\frac{t^2}{2}+o(t^2)}{t^2} = \lim_{t\to 0^+}\left(\frac{1}{2}+o(1)\right) = \frac{1}{2}.$$

Problem 8.7 To begin with, when $y \to 0$

$$\log(1+y) = y-\frac{y^2}{2}+o(y^2).$$

In our case

$$y = f(x) = -\frac{x}{2}-\frac{x^2}{4}+o(x^2),$$

which clearly goes to 0 when $x \to 0$. Then

$$y^2 = f(x)^2 = \left[-\frac{x}{2}-\frac{x^2}{4}+o(x^2)\right]\left[-\frac{x}{2}-\frac{x^2}{4}+o(x^2)\right] = \frac{x^2}{4}+o(x^2),$$

and $o(y^2) = o(x^2)$ because $y = -\frac{x}{2} + o(x)$. Therefore

$$\log[1+f(x)] = \left(-\frac{x}{2} - \frac{x^2}{4} + o(x^2)\right) - \frac{1}{2}\left(\frac{x^2}{4} + o(x^2)\right) + o(x^2) = -\frac{x}{2} - \frac{3}{8}x^2 + o(x^2).$$

If we now substitute

$$\lim_{x\to 0} \frac{\log[1+f(x)] + \frac{x}{2}}{x^2} = \lim_{x\to 0} \frac{-\frac{x}{2} - \frac{3}{8}x^2 + o(x^2) + \frac{x}{2}}{x^2} = \lim_{x\to 0} \frac{-\frac{3}{8}x^2 + o(x^2)}{x^2}$$
$$= \lim_{x\to 0}\left(-\frac{3}{8} + o(1)\right) = -\frac{3}{8}.$$

Problem 8.8 From the definition,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{\frac{1}{x} - \frac{1}{e^x - 1} - \frac{1}{2}}{x} = \lim_{x\to 0} \frac{(2-x)(e^x-1) - 2x}{2x^2(e^x - 1)}.$$

Now, $e^x - 1 = x + o(x)$ when $x \to 0$. This means that the denominator is $2x^2(e^x - 1) = 2x^3 + o(x^3)$ when $x \to 0$, and so we need to expand the numerator up to order $o(x^3)$. We need more terms of the exponential:

$$e^x - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3).$$

Substituting in the numerator we have

$$(2-x)(e^x - 1) - 2x = 2x + x^2 + \frac{x^3}{3} + o(x^3) - x^2 - \frac{x^3}{2} + o(x^3) - 2x = -\frac{x^3}{6} + o(x^3).$$

Then

$$f'(0) = \lim_{x\to 0} \frac{-\frac{x^3}{6} + o(x^3)}{2x^3 + o(x^3)} = \lim_{x\to 0} \frac{-\frac{1}{6} + o(1)}{2 + o(1)} = -\frac{1}{12}.$$

Problem 8.9

(i) The difficulty of this problem is that we don't know beforehand to which order we need to do the Taylor expansions of the functions involved in order to get the first nonzero term. It turns out that the first order is the seventh. Thus we need the expansions for $x \to 0$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + o(x^7), \qquad \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + o(x^7).$$

Since $\sin x$ is the argument of $\tan(\sin x)$ we will need to calculate the expansions of $\sin^3 x$, $\sin^5 x$, and $\sin^7 x$. So,

$$\begin{aligned}
\sin^2 x &= \sin x \cdot \sin x = x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 + o(x^7), \\
\sin^3 x &= \sin^2 x \cdot \sin x = x^3 - \frac{x^5}{2} + \frac{13}{120}x^7 + o(x^7), \\
\sin^5 x &= \sin^2 x \cdot \sin^3 x = x^5 - \frac{5}{6}x^7 + o(x^7), \\
\sin^7 x &= \sin^2 x \cdot \sin^5 x = x^7 + o(x^7).
\end{aligned}$$

Besides $o(\sin^7 x) = o(x^7)$. Likewise, $\tan x$ is the argument of $\sin(\tan x)$, therefore

$$\begin{aligned}
\tan^2 x &= \tan x \cdot \tan x = x^2 + \frac{2}{3}x^4 + \frac{17}{45}x^6 + o(x^7),\\
\tan^3 x &= \tan^2 x \cdot \tan x = x^3 + x^5 + \frac{11}{15}x^7 + o(x^7),\\
\tan^5 x &= \tan^2 x \cdot \tan^3 x = x^5 + \frac{5}{3}x^7 + o(x^7),\\
\tan^7 x &= \tan^2 x \cdot \tan^5 x = x^7 + o(x^7),
\end{aligned}$$

and $o(\tan^7 x) = o(x^7)$.
Then,

$$\begin{aligned}
\tan(\sin x) =& \sin x + \frac{1}{3}\sin^3 x + \frac{2}{15}\sin^5 x + \frac{17}{315}\sin^7 x + o(x^7),\\
=& \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + o(x^7)\right) + \frac{1}{3}\left(x^3 - \frac{x^5}{2} + \frac{13}{120}x^7 + o(x^7)\right)\\
&+ \frac{2}{15}\left(x^5 - \frac{5}{6}x^7 + o(x^7)\right) + \frac{17}{315}\left(x^7 + o(x^7)\right) + o(x^7)\\
=& x + \frac{x^3}{6} - \frac{x^5}{40} - \frac{107}{5040}x^7 + o(x^7).
\end{aligned}$$

Similarly

$$\begin{aligned}
\sin(\tan x) =& \tan x - \frac{1}{6}\tan^3 x + \frac{1}{120}\tan^5 x - \frac{1}{5040}\tan^7 x + o(x^7),\\
=& \left(x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + o(x^7)\right) - \frac{1}{6}\left(x^3 + x^5 + \frac{11}{15}x^7 + o(x^7)\right)\\
&+ \frac{1}{120}\left(x^5 + \frac{5}{3}x^7 + o(x^7)\right) - \frac{1}{5040}\left(x^7 + o(x^7)\right) + o(x^7)\\
=& x + \frac{x^3}{6} - \frac{x^5}{40} - \frac{55}{1008}x^7 + o(x^7).
\end{aligned}$$

Accordingly, substracting these two expansions,

$$f(x) = \tan(\sin x) - \sin(\tan x) = \frac{x^7}{30} + o(x^7) \quad (x \to 0).$$

(ii) We can write

$$f(x) = \frac{1}{R^2}\left[1 - (1+z)^{-2}\right], \quad z = \frac{x}{R},$$

and use the expansion $(1+z)^{-2} = 1 - 2z + o(z)$. Then

$$(1+z)^{-2} = 1 - 2\frac{x}{R} + o(x),$$

and

$$f(x) = \frac{1}{R^2}\left[1 - 1 + 2\frac{x}{R} + o(x)\right] = 2\frac{x}{R^3} + o(x).$$

(iii) We can rewrite the function as

$$f(x) = (1+x)^{1/3}(1-x)^{-1/3} - (1-x)^{1/3}(1+x)^{-1/3}$$

and then use, when $x \to 0$,

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + o(x^2).$$

For $\alpha = 1/3$ this leads to

$$(1+x)^{1/3} = 1 + \frac{x}{3} - \frac{x^2}{9} + o(x^2), \qquad (1-x)^{1/3} = 1 - \frac{x}{3} - \frac{x^2}{9} + o(x^2),$$

and for $\alpha = -1/3$

$$(1+x)^{-1/3} = 1 - \frac{x}{3} + \frac{2}{9}x^2 + o(x^2), \qquad (1-x)^{-1/3} = 1 + \frac{x}{3} + \frac{2}{9}x^2 + o(x^2).$$

Multiplying,

$$\begin{aligned}
(1+x)^{1/3}(1-x)^{-1/3} &= \left[1 + \frac{x}{3} - \frac{x^2}{9} + o(x^2)\right]\left[1 + \frac{x}{3} + \frac{2}{9}x^2 + o(x^2)\right] \\
&= 1 + \frac{2}{3}x + \frac{2}{9}x^2 + o(x^2), \\
(1-x)^{1/3}(1+x)^{-1/3} &= \left[1 - \frac{x}{3} - \frac{x^2}{9} + o(x^2)\right]\left[1 - \frac{x}{3} + \frac{2}{9}x^2 + o(x^2)\right] \\
&= 1 - \frac{2}{3}x + \frac{2}{9}x^2 + o(x^2),
\end{aligned}$$

and substracting,

$$f(x) = \frac{4}{3}x + o(x^2).$$

Problem 8.10 First of all, as $x \to 0$,

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + o(x^7),$$

so

$$\begin{aligned}
\frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} + o(x^7) &= \left[Ax^2 + Bx^4 + Cx^6 + o(x^7)\right]\left[2 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + o(x^7)\right] \\
&= 2Ax^2 + \left(2B - \frac{A}{2}\right)x^4 + \left(2C - \frac{B}{2} + \frac{A}{24}\right)x^6 + o(x^7).
\end{aligned}$$

The two expansions coincide if

$$\left.\begin{aligned}
2A &= \frac{1}{2}, \\
2B - \frac{A}{2} &= -\frac{1}{24}, \\
2C - \frac{B}{2} + \frac{A}{24} &= \frac{1}{720},
\end{aligned}\right\} \quad \Rightarrow \quad A = \frac{1}{4}, \quad B = \frac{1}{24}, \quad C = \frac{17}{2880}.$$

Problem 8.11

(i) Let $f(x) = x - (a + b\cos x)\sin x = x - a\sin x - b\sin x\cos x = x - a\sin x - (b/2)\sin 2x$. Up to fourth order,

$$\sin x = x - \frac{x^3}{6} + o(x^4), \qquad \sin 2x = 2x - \frac{4}{3}x^3 + o(x^4),$$

therefore

$$f(x) = x - ax + \frac{a}{6}x^3 - bx + \frac{2b}{3}x^3 + o(x^4) = (1 - a - b)x + \frac{a + 4b}{6}x^3 + o(x^4).$$

This function will be $f(x) = o(x^4)$ if and only if

$$\begin{cases} a + b = 1, \\ a + 4b = 0, \end{cases} \qquad \Rightarrow \qquad a = \frac{4}{3}, \quad b = -\frac{1}{3}.$$

(ii) Both $\cot x$ and the rational function diverge when $x \to 0$, so we can multiply by x and write the equivalent equation

$$f(x) = x\cot x - \frac{1 + ax^2}{1 + bx^2} = o(x^5) \quad (x \to 0),$$

where neither of the two functions involved is singular at $x = 0$. Take first the rational function. This is a product of two functions, namely

$$\frac{1 + ax^2}{1 + bx^2} = (1 + ax^2)(1 + bx^2)^{-1}.$$

But $(1 + z)^{-1} = 1 - z + z^2 - z^3 + o(z^3)$, $(z \to 0)$, therefore

$$(1 + bx^2)^{-1} = 1 - bx^2 + b^2x^4 - b^3x^6 + o(x^6) = 1 - bx^2 + b^2x^4 + o(x^5)$$

(we don't need to keep powers higher than x^5 in the expansion). Substituting

$$\begin{aligned} \frac{1 + ax^2}{1 + bx^2} &= \left[1 - bx^2 + b^2x^4 + o(x^5)\right] + \left[ax^2 - abx^4 + o(x^5)\right] \\ &= 1 + (a - b)x^2 - b(a - b)x^4 + o(x^5). \end{aligned}$$

As for $x\cot x$, we can also write it as the product of two functions

$$x\cot x = x\frac{\cos x}{\sin x} = \cos x \left(\frac{\sin x}{x}\right)^{-1}.$$

Now,

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5), \qquad \frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} + o(x^5) = 1 - y,$$

where $y = \frac{x^2}{6} - \frac{x^4}{120} + o(x^5)$. Then

$$\begin{aligned} \left(\frac{\sin x}{x}\right)^{-1} &= 1 + y + y^2 + y^3 + o(y^3) = 1 + \left[\frac{x^2}{6} - \frac{x^4}{120} + o(x^5)\right] + \left[\frac{x^4}{36} + o(x^5)\right] + o(x^5) \\ &= 1 + \frac{x^2}{6} + \frac{7}{360}x^4 + o(x^5) \end{aligned}$$

(where we have taken into account that $y^3+o(y^3)\propto x^6+o(x^6)=o(x^5)$). Accordingly,

$$x\cot x=\left[1-\frac{x^2}{2}+\frac{x^4}{24}+o(x^5)\right]\left[1+\frac{x^2}{6}+\frac{7}{360}x^4+o(x^5)\right]=1-\frac{x^2}{3}-\frac{x^4}{45}+o(x^5)$$

and

$$\begin{aligned} f(x)&=\left[1-\frac{x^2}{3}-\frac{x^4}{45}+o(x^5)\right]-\left[1+(a-b)x^2-b(a-b)x^4+o(x^5)\right]\\ &=\left(b-a-\frac{1}{3}\right)x^2+\left(b(a-b)-\frac{1}{45}\right)x^4+o(x^5). \end{aligned}$$

We will have $f(x)=o(x^5)$ if, and only if,

$$\begin{cases} b-a=\frac{1}{3}, \\ b(a-b)=\frac{1}{45}, \end{cases} \quad\Rightarrow\quad a=-\frac{2}{5},\quad b=-\frac{1}{15}.$$

The interest of this exercise is to show that when $x\to 0$,

$$\cot x=\frac{1-\frac{2}{5}x^2}{x-\frac{1}{15}x^3}+o(x^4),$$

which provides a reasonable approximation of $\cot x$ as a rational function near $x=0$.

Problem 8.12 We can transform the expression into

$$e^x(1+cx+dx^2)=1+ax+bx^2+o(x^4),\quad (x\to 0),$$

because $o(x^4)(1+cx+dx^2)=o(x^4)+o(x^5)+o(x^6)=o(x^4)$. Using the expasion for the exponential

$$e^x=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+o(x^4),\quad (x\to 0),$$

and multiplying we get

$$e^x(1+cx+dx^2)=1+(1+c)x+\left(\frac{1}{2}+c+d\right)x^2+\left(\frac{1}{6}+\frac{c}{2}+d\right)x^3+\left(\frac{1}{24}+\frac{c}{6}+\frac{d}{2}\right)x^4+o(x^4).$$

Comparing the two sides of the first equation we obtain

$$1+c=a, \qquad \frac{1}{6}+\frac{c}{2}+d=0,$$
$$\frac{1}{2}+c+d=b, \qquad \frac{1}{24}+\frac{c}{6}+\frac{d}{2}=0.$$

The two equations on the right can be rewritten as

$$\begin{cases} c+2d=-\frac{1}{3}, \\ c+3d=-\frac{1}{4}, \end{cases} \quad\Rightarrow\quad d=\frac{1}{12},\quad c=-\frac{1}{2},$$

and from these values we get $a=\frac{1}{2}$, $b=\frac{1}{12}$ from the equations on the left. Hence

$$e^x=\frac{1+\frac{x}{2}+\frac{x^2}{12}}{1-\frac{x}{2}+\frac{x^2}{12}}+o(x^4),\quad (x\to 0).$$

Problem 8.13

(i) First of all we can write

$$\sqrt{1+n^2} = n\sqrt{1+\frac{1}{n^2}} = n\left[1+\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)\right] = n+\frac{1}{2n}+o\left(\frac{1}{n}\right) \quad (n\to\infty).$$

Let us denote $\varepsilon_n = \frac{\pi}{2n}+o\left(\frac{1}{n}\right)$. Then

$$\sin\pi\sqrt{1+n^2} = \sin(\pi n+\varepsilon_n) = (-1)^n\sin\varepsilon_n = (-1)^n\left[\varepsilon_n+o(\varepsilon_n)\right].$$

But $o(\varepsilon_n) = o\left(\frac{1}{n}\right)$, so

$$\sin\pi\sqrt{1+n^2} = (-1)^n\frac{\pi}{2n}+o\left(\frac{1}{n}\right) = (-1)^n\frac{\pi}{2n}[1+o(1)] \quad (n\to\infty).$$

Thus

$$\lim_{n\to\infty}\sin\pi\sqrt{1+n^2} = 0.$$

(ii) From the previous result

$$\sin^2\pi\sqrt{1+n^2} = \frac{\pi^2}{4n^2}[1+o(1)] \quad (n\to\infty);$$

in other words, $\sin^2\pi\sqrt{1+n^2}\sim\frac{\pi^2}{4n^2}$ when $n\to\infty$. The convergence of the series follows from the fact that $\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty$.

Problem 8.14 Since $\sin x = x+o(x)$, then $f(x) = 1+x^4+o(x^4)$, when $x\to 0$. Thus $P_{4,0}(x) = 1+x^4$. Accordingly f has a local minimum at $x=0$.

Problem 8.15

(i) Let us consider the function

$$f(x) = \frac{1}{\sqrt{1+x}}.$$

The value we want to obtain is $f(0.1)$. The Taylor expansion for this function near $a=0$ follows from

$$\begin{aligned}
f(x) &= (1+x)^{-1/2}, & f(0) &= 1,\\
f'(x) &= -\frac{1}{2}(1+x)^{-3/2}, & f'(0) &= -\frac{1}{2},\\
f''(x) &= \frac{3}{4}(1+x)^{-5/2}, & f''(0) &= \frac{3}{4},\\
f'''(x) &= -\frac{15}{8}(1+x)^{-7/2}, & f'''(0) &= -\frac{15}{8},\\
f^{(4)}(x) &= \frac{105}{16}(1+x)^{-9/2}, & &
\end{aligned}$$

which implies

$$P_{3,0}(x) = 1-\frac{x}{2}+\frac{3}{8}x^2-\frac{5}{16}x^3, \qquad R_{3,0}(x) = \frac{35}{128}\left(\frac{1}{\sqrt{1+\theta x}}\right)^9 x^4, \quad 0<\theta<1.$$

Now $P_{3,0}(0.1) = 0.9534375$ and since $\sqrt{1+\theta x} > 1$ for every $x > 0$,

$$|R_{3,0}(x)| < \frac{35}{128}x^4 \quad \Rightarrow \quad |R_{3,0}(0.1)| < 2.7 \times 10^{-5}.$$

Hence $1/\sqrt{1.1} = 0.9534(3)$ —where the figure in brackets may be affected by the error. (The exact value is $1/\sqrt{1.1} = 0.953462589\ldots$)

(ii) Consider the function $f(x) = \sqrt[3]{27+x} =$. Then $\sqrt[3]{28} = f(1)$. To ontain the second degree Taylor expansion around $a = 0$ we calculate

$$\begin{aligned}
f(x) &= (27+x)^{1/3}, & f(0) &= 3, \\
f'(x) &= \frac{1}{3}(27+x)^{-2/3}, & f'(0) &= \frac{1}{27}, \\
f''(x) &= -\frac{2}{9}(27+x)^{-5/3}, & f''(0) &= -\frac{2}{2187}, \\
f'''(x) &= \frac{10}{27}(27+x)^{-8/3}, & &
\end{aligned}$$

from which

$$P_{2,0}(x) = 3 + \frac{x}{27} - \frac{x^2}{2187}, \qquad R_{2,0}(x) = \frac{5}{81}\frac{x^3}{\left(\sqrt[3]{27+\theta x}\right)^8}, \quad 0 < \theta < 1.$$

Now $P_{2,0}(1) = 3.03657979$ and since $\sqrt[3]{27+\theta x} > \sqrt[3]{27} = 3$ for every $x > 0$,

$$|R_{2,0}(x)| < \frac{5x^3}{531441} \qquad \Rightarrow \qquad |R_{2,0}(1)| < \frac{5}{531441} = 0.9408 \times 10^{-5}.$$

Hence $\sqrt[3]{28} = 3.0365(8)$. (As a matter of fact $\sqrt[3]{28} = 3.036588972\ldots$)

Problem 8.16

(i) Since for $x \to 0$

$$\cos x = 1 - \frac{x^2}{2} + o(x^3), \qquad e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3),$$

then

$$P_{3,0}(x) = 2 + x + \frac{x^3}{6}.$$

(ii) First of all $(\cos x)^{(4)} = \cos x$ and $(e^x)^{(4)} = e^x$, so $f^{(4)}(x) = f(x)$. Therefore

$$R_{3,0}(x) = \frac{\cos\theta x + e^{\theta x}}{24}x^4, \quad 0 < \theta < 1.$$

Now $|\cos\theta x| \leqslant 1$ and $e^{\theta x} \leqslant \max\{e^x, 1\}$. Thus for $-1/4 \leqslant x \leqslant 1/4$

$$|R_{3,0}(x)| < \frac{1+e^{1/4}}{24}\left(\frac{1}{4}\right)^4 = 3.72 \times 10^{-4}.$$

Problem 8.17 The reminder of the Taylor expansion of $f(x) = e^x$ around $a = 0$ is

$$R_{n,0}(x) = \frac{e^{\theta x}}{(n+1)!}x^{n+1}, \quad 0 < \theta < 1,$$

so an upper bound for $-1 \leqslant x \leqslant 1$ will be

$$|R_{n,0}(x)| < \frac{e}{(n+1)!}.$$

If we want to have three exact decimal places the error should be smaller than 10^{-3}, so we must look for the smallest n for which $(n+1)! > 10^3$. Since $6! = 720$ and $7! = 5040$ then $n = 6$.

Problem 8.18

(i)

$$\frac{1}{\rho} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{2^n n^2}} = \frac{1}{2} \qquad \Rightarrow \qquad \rho = 2.$$

For $x = 2$

$$\sum_{n=1}^{\infty} \frac{2^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

so the interval of absolute convergence is $[-2,2]$.

(ii)

$$\rho = \lim_{n\to\infty} \frac{n!}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)!} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

For $x = e$

$$\frac{n!e^n}{n^n} \sim \sqrt{2\pi n}$$

so the series does not converge absolutely at $x = \pm e$. Therefore the inveral of absolute convergence is $(-e,e)$.

(iii)

$$\frac{1}{\rho} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n10^{n-1}}} = \frac{1}{10} \qquad \Rightarrow \qquad \rho = 10.$$

For $x = 10$

$$\sum_{n=1}^{\infty} \frac{10^n}{n10^{n-1}} = 10, \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

so the interval of absolute convergence is $(-10,10)$.

(iv)

$$\rho = \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} = 1.$$

For $x = 1$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty,$$

so the interval of absolute convergence is $(-1,1)$.

(v) We can rewrite the series as

$$\sum_{n=0}^{\infty}(-2)^n\left(x-\frac{3}{2}\right)^n,$$

so the radius of convergence is

$$\frac{1}{\rho}=\lim_{n\to\infty}\sqrt[n]{2^n}=2 \qquad \Rightarrow \qquad \rho=\frac{1}{2}.$$

When $x-3/2=\pm 1/2$

$$\sum_{n=0}^{\infty}2^n\left|x-\frac{3}{2}\right|^n=\sum_{n=0}^{\infty}2^n\frac{1}{2^n}=\sum_{n=0}^{\infty}1=\infty.$$

Therefore the interval of absolute convergence is given by $|x-3/2|<1/2$, i.e., $(1,2)$.

(vi)

$$\rho=\lim_{n\to\infty}\frac{\sqrt{2(n+1)}}{\sqrt{2n}}=1,$$

and for $|x-2|=1$

$$\sum_{n=1}^{\infty}\frac{|x-2|^n}{\sqrt{2n}}=\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n}}=\infty,$$

so the interval of absolute convergence is given by $|x-2|<1$, i.e., $(1,3)$.

Problem 8.19 We can rewrite the function as $f(x)=(1-x)^{-k}$, which matches the function

$$(1+t)^{\alpha}=\sum_{n=0}^{\infty}\binom{\alpha}{n}t^n, \qquad \binom{\alpha}{n}=\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!},$$

for $t=-x$ and $\alpha=-k$. Thus

$$\frac{1}{(1-x)^k}=\sum_{n=0}^{\infty}\binom{-k}{n}(-1)^n x^n.$$

Now

$$\begin{aligned}(-1)^n\binom{-k}{n}&=(-1)^n\frac{(-k)(-k-1)\cdots(-k-n+1)}{n!}=(-1)^n(-1)^n\frac{k(k+1)\cdots(k+n-1)}{n!}\\&=\frac{(k+n-1)!}{n!(k-1)!}=\binom{k+n-1}{k-1}.\end{aligned}$$

Therefore

$$\frac{1}{(1-x)^k}=\sum_{n=0}^{\infty}\binom{n+k-1}{k-1}x^n.$$

For $k=1$ the coefficients are $\binom{n}{0}=1$, so

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n,$$

as expected.

For $k=2$ the coefficients are $\binom{n+1}{1}=n+1$, so

$$\frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}(n+1)x^n.$$

Finally, for $k=3$ the coefficients are $\binom{n+2}{2}=(n+2)(n+1)/2$, so

$$\frac{1}{(1-x)^3}=\frac{1}{2}\sum_{n=0}^{\infty}(n+2)(n+1)x^n.$$

An alternative way of getting the same result goes as follows. We know the result for $k=1$ because it is the geometric series,

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n.$$

Now differentiating this equation we obtain

$$\frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}(n+1)x^n,$$

which is the result for $k=2$. And differentiating again,

$$\frac{2}{(1-x)^3}=\sum_{n=0}^{\infty}(n+2)(n+1)x^n,$$

which, divided by 2, yields the result for $k=3$.

Problem 8.20 Using the hint we write

$$\frac{1}{x^2+x+1}=\frac{1-x}{1-x^3}=\frac{1}{1-x^3}-\frac{x}{1-x^3}.$$

Now,

$$\frac{1}{1-x^3}=\sum_{n=0}^{\infty}\left(x^3\right)^n=\sum_{n=0}^{\infty}x^{3n},\qquad \frac{x}{1-x^3}=\sum_{n=0}^{\infty}x^{3n+1},$$

therefore

$$\frac{1}{x^2+x+1}=\sum_{n=0}^{\infty}x^{3n}-\sum_{n=0}^{\infty}x^{3n+1}.$$

In other words, the coefficients a_n are such that $a_{3n}=1$, $a_{3n+1}=-1$, and $a_{3n+2}=0$. Accordingly $a_{300}=1$, $a_{301}=-1$, $a_{302}=0$.

Problem 8.21 For a function $f(x)$ that can be expanded as a Taylor series

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n.$$

Therefore the coefficient of x^n in this series is $f^{(n)}(0)/n!$. Since

$$\log(1+u)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}u^n$$

then

$$\log(1+x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^{2n}.$$

Therefore $f^{(231)}(0) = 0$ because the coefficients of the odd terms are all zero, and the coefficient of x^{100} is

$$\frac{(-1)^{51}}{50} = -\frac{1}{50} \qquad \Rightarrow \qquad \frac{f^{(100)}(0)}{100!} = -\frac{1}{50} \qquad \Rightarrow \qquad f^{(100)}(0) = -\frac{100!}{50}.$$

Problem 8.22

(i) Let us denote $f(x)$ the sum of the series. Then

$$f'(x) = \sum_{n=1} x^{n-1} = \sum_{n=0} x^n = \frac{1}{1-x}, \qquad |x| < 1.$$

Therefore $f(x) = -\log(1-x) + c$. To determine the constant we just calculate $f(0)$. From the series $f(0) = 0$, and from the latter expression $f(0) = c$. Therfore $c = 0$ and

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x), \qquad |x| < 1.$$

(ii) First of all we can write the series as

$$\sum_{n=0}^{\infty} (n+1)\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} (n+1)t^n,$$

where $t = x/2$. Now

$$\sum_{n=0}^{\infty} (n+1)t^n = \sum_{n=1}^{\infty} nt^{n-1} = \sum_{n=1}^{\infty} (t^n)' \underset{(*)}{=} \left(\sum_{n=0}^{\infty} t^n\right)' = \left(\frac{1}{1-t}\right)' = \frac{1}{(1-t)^2}$$

(in (*) we have added the term $n = 0$ to the sum because it is a constant, and the derivative of a constant is zero). Therefore

$$\sum_{n=0}^{\infty} (n+1)2^{-n}x^n = \frac{1}{(1-x/2)^2} = \frac{4}{(2-x)^2}, \qquad |x| < 2.$$

(The radius of convergence of the geometric series is 1, i.e., converges for $|t| < 1$; since $t = x/2$, our series —which is the derivative of the geometric— converges for $|x| < 2$.)

Problem 8.23

(i) To begin with

$$f(x) = \sin^2 x = \frac{1}{2}(1 - \cos 2x),$$

and since

$$\cos t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!}, \qquad t \in \mathbb{R},$$

substituting we obtain

$$f(x) = \frac{1}{2}\left[1 - \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!}\right] = \frac{1}{2}\left[-\sum_{n=1}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!}\right] = \frac{1}{2}\sum_{n=1}^{\infty} (-1)^{n+1} 2^{2n} \frac{x^{2n}}{(2n)!}$$
$$= \sum_{n=1}^{\infty} (-1)^{n+1} 2^{2n-1} \frac{x^{2n}}{(2n)!}, \qquad x \in \mathbb{R}.$$

(ii) We can rewrite

$$f(x) = \log\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\log(1+x) - \frac{1}{2}\log(1-x)$$

and use

$$\log(1+t) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{t^n}{n}, \quad |t| < 1,$$

to obtain

$$f(x) = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{x^n}{n} = \sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}+1}{2}\right]\frac{x^n}{n}, \quad |x| < 1.$$

But

$$\frac{(-1)^{n+1}+1}{2} = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

Therefore

$$f(x) = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}, \quad |x| < 1.$$

(iii) We can rewrite

$$f(x) = \frac{x}{a}\cdot\frac{1}{1+bx/a}.$$

Now since

$$\frac{1}{1-t} = \sum_{n=0}^{\infty}t^n, \quad |t| < 1,$$

then

$$f(x) = \frac{x}{a}\sum_{n=0}^{\infty}(-1)^n\frac{b^n}{a^n}x^n = \sum_{n=0}^{\infty}(-1)^n\frac{b^n}{a^{n+1}}x^{n+1} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{b^{n-1}}{a^n}x^n, \quad |x| < \left|\frac{a}{b}\right|.$$

(iv) We can express

$$f(x) = \frac{1}{2}\frac{1}{1-x^2/2} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x^2}{2}\right)^n = \sum_{n=0}^{\infty}\frac{x^{2n}}{2^{n+1}},$$

and the converge requires $x^2/2 < 1$, i.e., $|x| < \sqrt{2}$.

(v) We can rewrite

$$f(x) = (1+x)e^{-x} - (1-x)e^x.$$

At this point we can already expand the exponential, so that

$$\begin{aligned}(1-x)e^x = (1-x)\sum_{n=0}^{\infty}\frac{x^n}{n!} &= \sum_{n=0}^{\infty}\frac{x^n}{n!} - \sum_{n=0}^{\infty}\frac{x^{n+1}}{n!} = \sum_{n=0}^{\infty}\frac{x^n}{n!} - \sum_{n=1}^{\infty}\frac{x^n}{(n-1)!} \\ &= 1 + \sum_{n=1}^{\infty}\left(\frac{1}{n!} - \frac{1}{(n-1)!}\right)x^n = 1 + \sum_{n=1}^{\infty}\left(\frac{1}{n!} - \frac{n}{n!}\right)x^n = 1 + \sum_{n=1}^{\infty}\frac{1-n}{n!}x^n.\end{aligned}$$

The expansion of $(1+x)e^{-x}$ is the same one but changing x by $-x$, i.e.,

$$(1+x)e^{-x} = 1+\sum_{n=1}^{\infty}\frac{1-n}{n!}(-x)^n = 1+\sum_{n=1}^{\infty}(-1)^n\frac{1-n}{n!}x^n.$$

Substracting both expansions

$$\begin{aligned} f(x) = (1+x)e^{-x}-(1-x)e^{x} &= \sum_{n=1}^{\infty}(-1)^n\frac{1-n}{n!}x^n - \sum_{n=1}^{\infty}\frac{1-n}{n!}x^n \\ &= \sum_{n=1}^{\infty}\frac{n-1}{n!}x^n - \sum_{n=1}^{\infty}(-1)^n\frac{n-1}{n!}x^n = \sum_{n=1}^{\infty}[1-(-1)^n]\frac{n-1}{n!}x^n. \end{aligned}$$

But $1-(-1)^n = 2$ when n is odd and $=0$ when n is even, therefore

$$f(x) = \sum_{k=0}^{\infty}\frac{4k}{(2k+1)!}x^{2k+1}.$$

An alternative derivation arises from realising that

$$f(x) = e^{-x}+xe^{-x}-e^{x}+xe^{x} = 2x\cosh x - 2\sinh x,$$

for then

$$\begin{aligned} f(x) = 2x\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} - 2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} &= 2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n)!} - 2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} \\ &= 2\sum_{n=0}^{\infty}\left(\frac{1}{(2n)!}-\frac{1}{(2n+1)!}\right)x^{2n+1} = 2\sum_{n=0}^{\infty}\left(\frac{2n+1}{(2n+1)!}-\frac{1}{(2n+1)!}\right)x^{2n+1} \\ &= 2\sum_{n=0}^{\infty}\frac{2n}{(2n+1)!}x^{2n+1} = \sum_{n=0}^{\infty}\frac{4n}{(2n+1)!}x^{2n+1}. \end{aligned}$$

Problem 8.24

(i)

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!} = \sum_{n=0}^{\infty}\frac{(-1/2)^n}{n!} = \sum_{n=0}^{\infty}\frac{x^n}{n!}\bigg|_{x=-1/2} = e^{-1/2}.$$

(ii)

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{n}{2^n} = \sum_{n=1}^{\infty}n\left(\frac{1}{2}\right)^n = \frac{1}{2}\sum_{n=1}^{\infty}n\left(\frac{1}{2}\right)^{n-1} &= \frac{1}{2}\sum_{n=1}^{\infty}nx^{n-1}\bigg|_{x=1/2} = \frac{1}{2}\left(\sum_{n=0}^{\infty}x^n\right)'\bigg|_{x=1/2} \\ &= \frac{1}{2}\left(\frac{1}{1-x}\right)'\bigg|_{x=1/2} = \frac{1}{2(1-x)^2}\bigg|_{x=1/2} = \frac{1}{2(1-1/2)^2} = 2. \end{aligned}$$

(iii)

$$\sum_{n=1}^{\infty}\frac{1}{n2^n} = \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty}\frac{x^n}{n}\bigg|_{x=1/2} = -\log(1-x)\Big|_{x=1/2} = -\log(1/2) = \log 2.$$

(iv)

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}\bigg|_{x=1} = \arctan 1 = \frac{\pi}{4}.$$

Problem 8.25 For $x=0$

$$f(0)=\sum_{n=1}^{\infty}\frac{1}{n!}=e-1.$$

For $x=1$

$$f(1)=\sum_{n=1}^{\infty}\frac{n}{n!}=\sum_{n=1}^{\infty}\frac{1}{(n-1)!}=\sum_{n=0}^{\infty}\frac{1}{n!}=e.$$

For $x=2$

$$f(2)=\sum_{n=1}^{\infty}\frac{n^2}{n!}=\sum_{n=1}^{\infty}\frac{n}{(n-1)!}=\sum_{n=0}^{\infty}\frac{n+1}{n!}=\sum_{n=1}^{\infty}\frac{n}{n!}+\sum_{n=0}^{\infty}\frac{1}{n!}=f(1)+e=2e.$$

Problem 8.26 Since $f(0)=2$ the series must be

$$f(x)=2+\sum_{n=1}^{\infty}a_nx^n.$$

Now,

$$f'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n,$$

so $f'(x)=f(x)+x$ implies

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n=x+2+\sum_{n=1}^{\infty}a_nx^n,$$

or equivalently

$$a_1+2a_2x+\sum_{n=2}^{\infty}(n+1)a_{n+1}x^n=2+(1+a_1)x+\sum_{n=2}^{\infty}a_nx^n.$$

From this equality we get $a_1=2$, $a_2=(1+a_1)/2=3/2$ and for $n>1$

$$a_{n+1}=\frac{a_n}{n+1}.$$

The iteration yields

$$a_n=\frac{1}{n}a_{n-1}=\frac{1}{n(n-1)}a_{n-2}=\frac{1}{n(n-1)(n-2)}a_{n-3}=\cdots=\frac{1}{n(n-1)(n-2)\cdots4\cdot3}a_2.$$

The denominator is $n!/2$, so

$$a_n=\frac{2a_2}{n!}=\frac{3}{n!},\quad n>1.$$

Therefore

$$f(x)=2+2x+3\sum_{n=2}^{\infty}\frac{x^n}{n!}=2+2x+3(e^x-1-x)=3e^x-1-x.$$

It is straightforward to check that this function satisfies both $f(0)=2$ and $f'(x)=f(x)+x$.

Problem 8.27 Let us compute two derivatives of h:

$$h' = (f' \circ g)g', \qquad h'' = (f' \circ g)'g' + (f' \circ g)g'' = (f'' \circ g)(g')^2 + (f' \circ g)g''.$$

Since f is convex $f'' \circ g > 0$; since f is increasing $f' \circ g > 0$; since g is convex $g'' > 0$; and of course $(g')^2 \geqslant 0$. Therefore $h'' > 0$, hence h is convex.

Problem 8.28

(i) $f(x) = x^{5/3} - 2x^{2/3}$, so

$$f'(x) = \frac{5}{3}x^{2/3} - \frac{4}{3}x^{-1/3}, \qquad f''(x) = \frac{10}{9}x^{-1/3} + \frac{4}{9}x^{-4/3} = \frac{10}{9}x^{-4/3}\left(x + \frac{2}{5}\right).$$

Since $x^{-4/3} > 0$ for all $x \neq 0$, then $f(x)$ is concave for $x < -2/5$ and convex in $-2/5 < x < 0$ and $x > 0$. At $x = -2/5$ it has an inflection point, and at $x = 0$ the function has a nondifferentiable cusp.

(ii) $f(x)$ is not differentiable at $x = 0$. Now, for $x > 0$

$$f(x) = xe^x, \qquad f'(x) = (x+1)e^x, \qquad f''(x) = (x+2)e^x,$$

so the funtion is always convex. On the other hand, the function is even (because $f(-x) = f(x)$), so it is convex also for $x < 0$.

(iii) $x^2 - 6x + 8 = (x-2)(x-4)$, so the domain of this function is $(-\infty, 2) \cup (4, \infty)$. On the other hand, in its domain

$$f(x) = \log(x^2 - 6x + 8) = \log|x^2 - 6x + 8| = \log|x-2| + \log|x-4|,$$

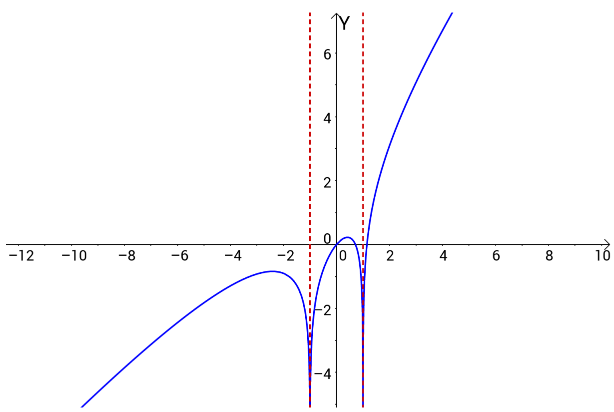
so

$$f'(x) = \frac{1}{x-2} + \frac{1}{x-4}, \qquad f''(x) = -\frac{1}{(x-2)^2} - \frac{1}{(x-4)^2},$$

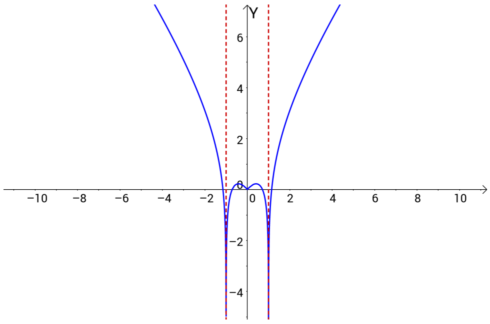
and then we have $f''(x) < 0$ in the whole domain of the function. Thus $f(x)$ is concave.

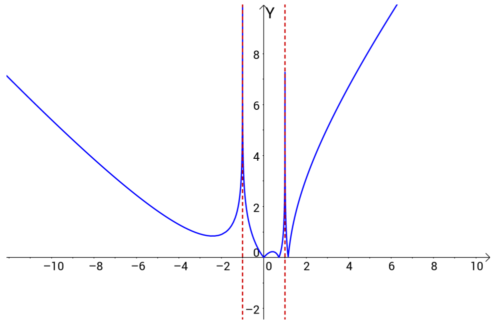
Problem 8.29

(i) $f(x) = x + \log|x^2 - 1|$:



(ii) $g(x) = f(|x|)$ $h(x) = |f(x)|$:

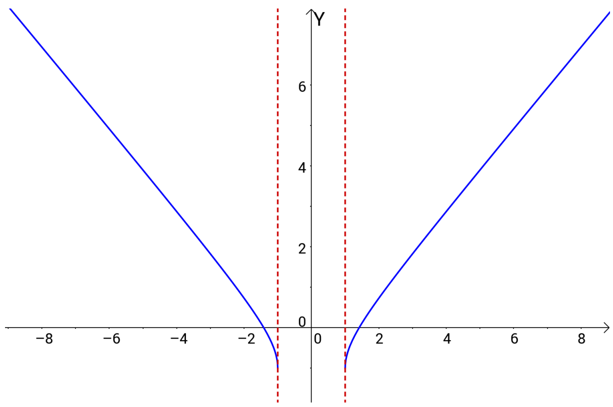




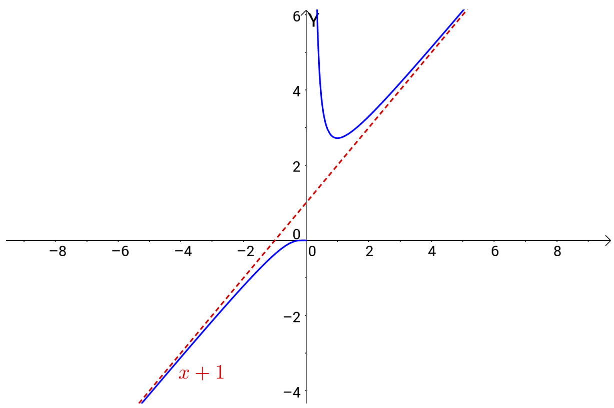
Problem 8.30

(i) $f(x) = e^x \sin x$: this function oscillates between $y = e^x$ and $y = -e^x$, crossing the X axis at $x = n\pi$, where $n \in \mathbb{Z}$.

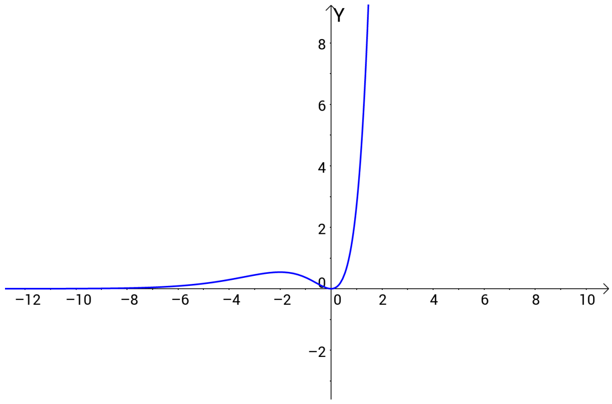
(ii) $f(x) = \sqrt{x^2 - 1} - 1$:



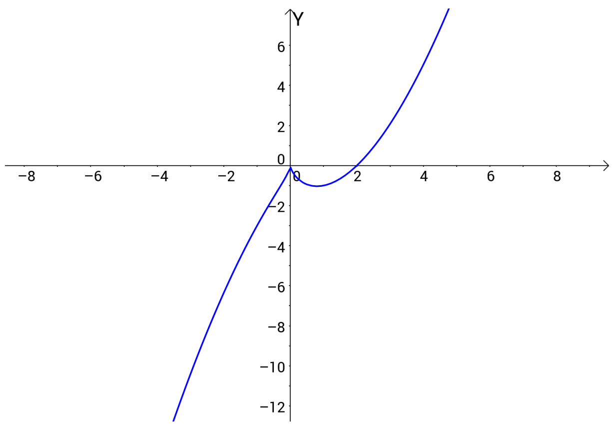
(iii) $f(x) = xe^{1/x}$:



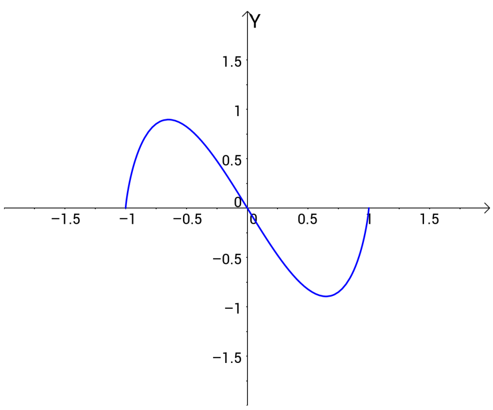
(iv) $f(x) = x^2 e^x$:



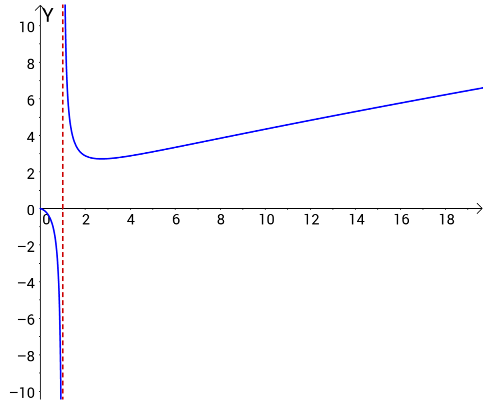
(v) $f(x) = (x-2)x^{2/3}$:



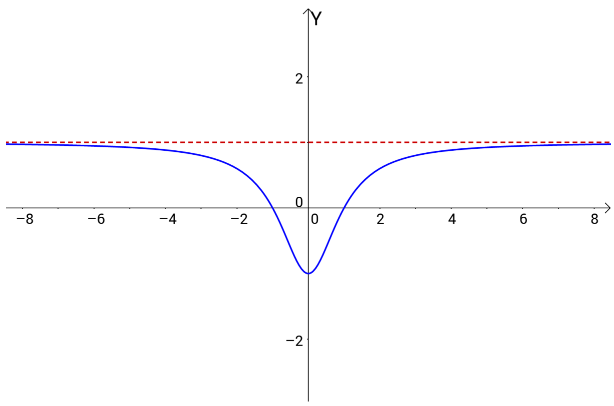
(vi) $f(x) = (x^2-1)\log\left(\dfrac{1+x}{1-x}\right)$:



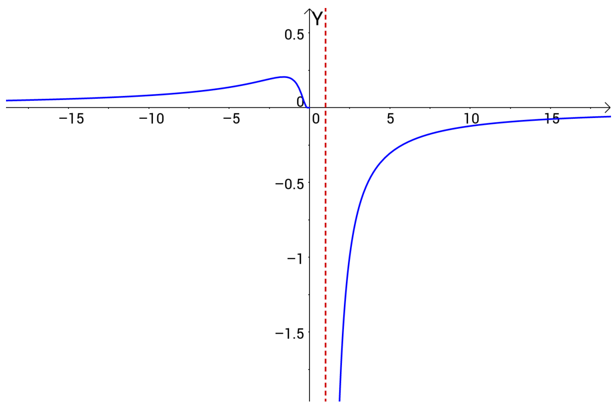
(vii) $f(x) = \dfrac{x}{\log x}$:



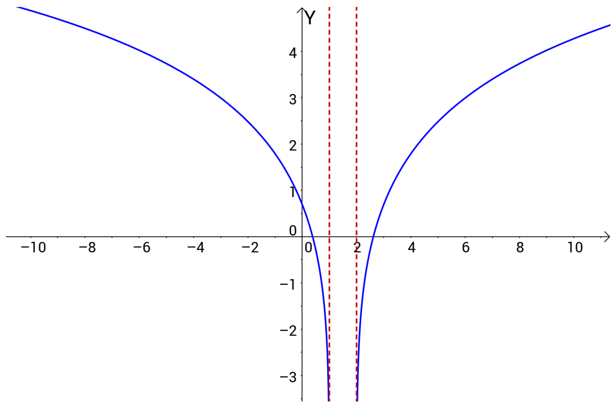
(viii) $f(x) = \dfrac{x^2-1}{x^2+1}$:



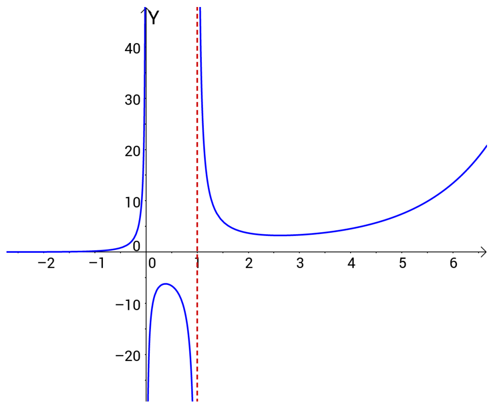
(ix) $f(x) = \dfrac{e^{1/x}}{1-x}$:



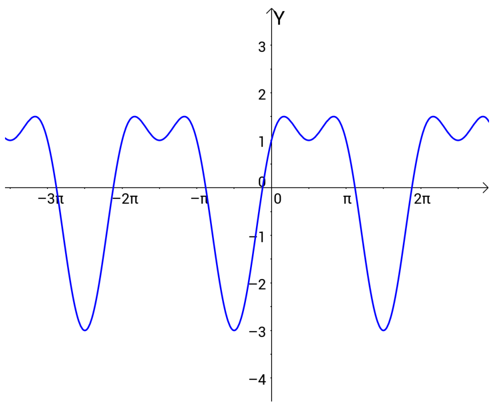
(x) $f(x) = \log\left[(x-1)(x-2)\right]$:



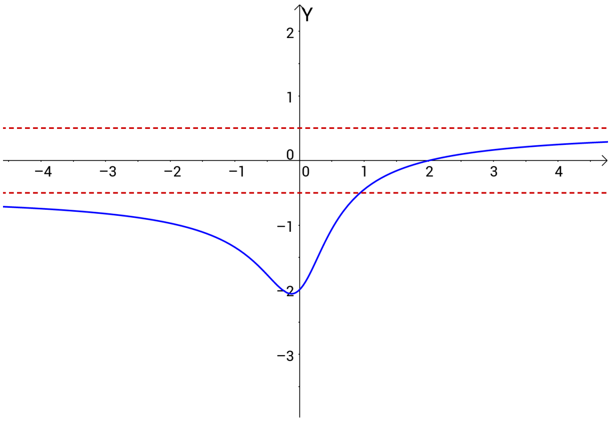
(xi) $f(x) = \dfrac{e^x}{x(x-1)}$:



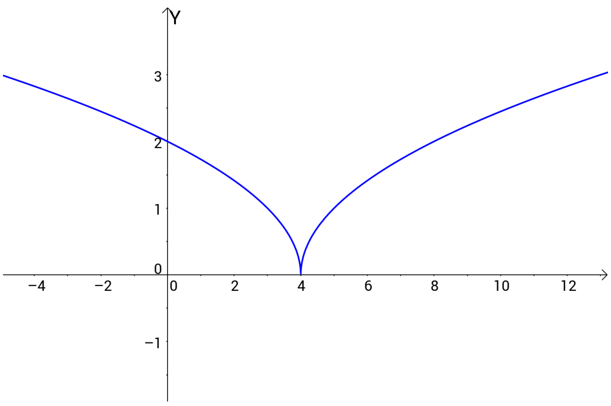
(xii) $f(x) = 2\sin x + \cos 2x$:



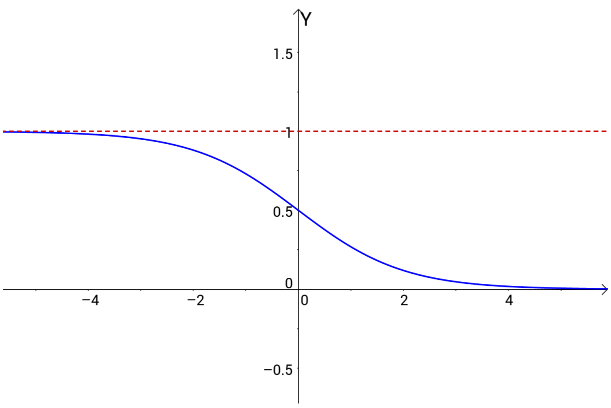
(xiii) $f(x) = \dfrac{x-2}{\sqrt{4x^2+1}}$:



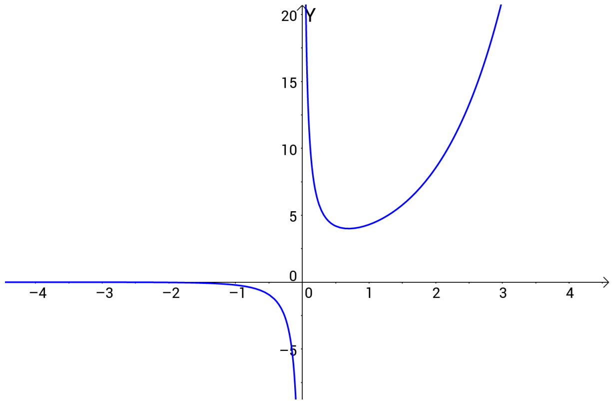
(xiv) $f(x) = \sqrt{|x-4|}$:



(xv) $f(x) = \dfrac{1}{1+e^x}$:



(xvi) $f(x) = \dfrac{e^{2x}}{e^x - 1}$:



(xvii) $f(x) = e^{-x}\sin x$: this function oscillates between $y = e^{-x}$ and $y = -e^{-x}$, crossing the X axis at $x = n\pi$, where $n \in \mathbb{Z}$.

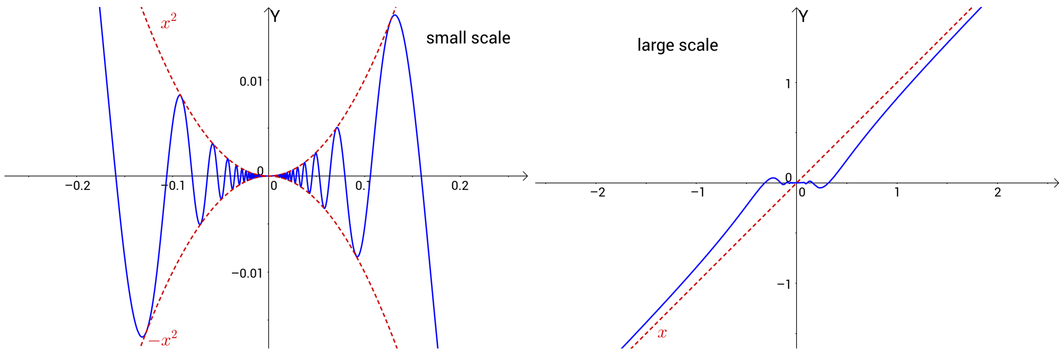
(xviii) $f(x) = x^2 \sin\frac{1}{x}$: this function has an oblique asymptote because

$$\sin\frac{1}{x} = \frac{1}{x} + o\left(\frac{1}{x^2}\right) \qquad (x \to \pm\infty)$$

(given that $\sin t = t + o(t^2)$ $(t \to 0)$); hence

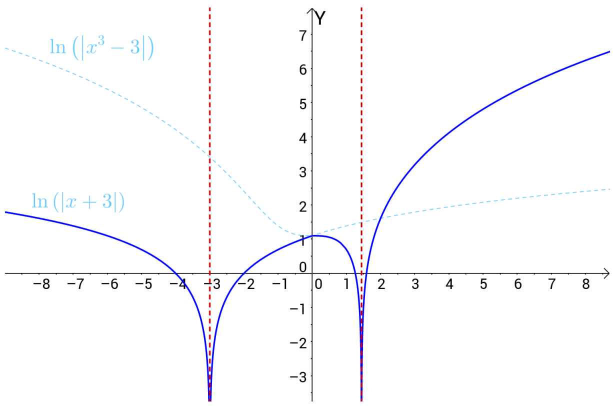
$$f(x) = x^2\left[\frac{1}{x} + o\left(\frac{1}{x^2}\right)\right] = x + o(1) \qquad (x \to \pm\infty).$$

Therefore the function looks different on a small scale and on a large scale. On a small scale it is an oscillatory function between $-x^2$ and x^2 that crosses the X axis at $x = \pm\frac{1}{n\pi}$, for all $n \in \mathbb{Z} - \{0\}$; on a large scale it asymptotes to $y = x$:



Problem 8.31

(i) $f(x) = \min\{\log|x^3 - 3|, \log|x + 3|\}$:



(ii) $f(x) = \dfrac{1}{|x|-1} - \dfrac{1}{|x-1|}$: this function has a different form for $x > 1$, for $0 < x < 1$ and for $x < 0$.
For $x > 1$

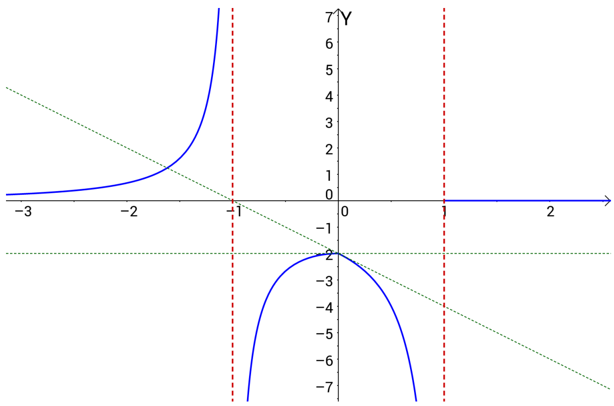
$$f(x) = \frac{1}{x-1} - \frac{1}{x-1} = 0.$$

For $0 < x < 1$ we have $|x-1| = -(x-1)$ so

$$f(x) = \frac{1}{x-1} + \frac{1}{x-1} = \frac{2}{x-1}.$$

For $x < 0$ we have $|x| - 1 = -(x+1)$ and $|x-1| = -(x-1)$, so

$$f(x) = -\frac{1}{x+1} + \frac{1}{x-1} = \frac{2}{x^2-1}.$$



(iii) $f(x) = \dfrac{1}{1+|x|} - \dfrac{1}{1+|x-a|}$ $(a > 0)$: this function also has different definitions depending on whether $x > a$, $0 < x < a$, or $x < 0$.
For $x > a$

$$f(x) = \frac{1}{1+x} - \frac{1}{1+x-a} = \frac{-a}{(x+1)(x-a+1)},$$

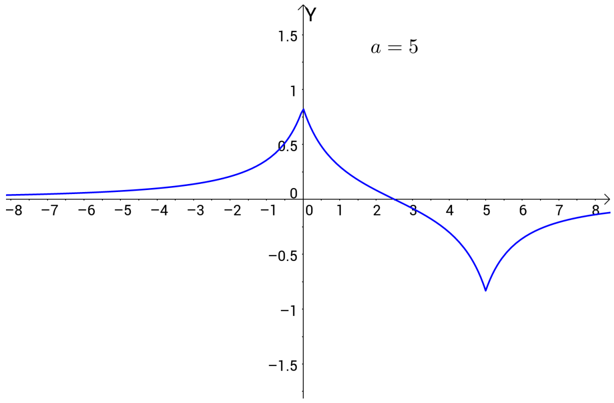
which has two vertical asymptotes, $x = -1$ and $x = a-1$, both out of the region $x > a$.
For $0 < x < a$

$$f(x) = \frac{1}{1+x} - \frac{1}{1+a-x} = \frac{2x-a}{(x+1)(x-a-1)},$$

which again has two asymptotes, $x = -1$ and $x = a+1$, both out of the region $0 < x < a$.
For $x < 0$

$$f(x) = \frac{1}{1-x} - \frac{1}{1+a-x} = \frac{a}{(x-1)(x-a-1)},$$

which also has two asymptotes, $x = 1$ and $x = a+1$, both out of the region $x < 0$.
Here is a plot for $a = 5$ (which is generic):

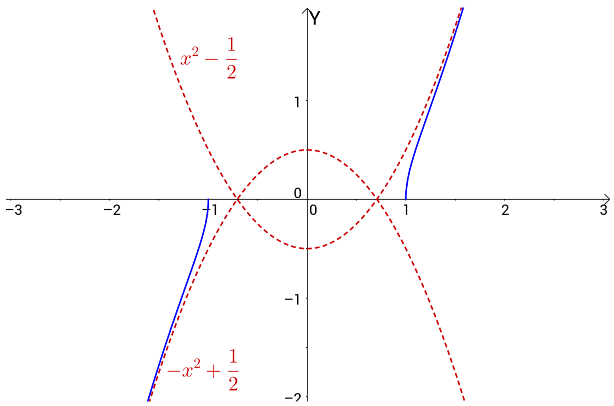


(iv) $f(x) = x\sqrt{x^2-1}$: notice that

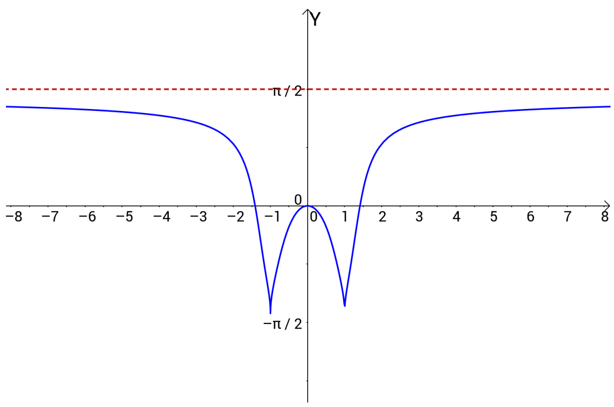
$$f(x) = x|x|\sqrt{1-\frac{1}{x^2}},$$

and since $\sqrt{1-t} = 1 - t/2 + o(t)$ $(t \to 0)$, when $x \to \pm\infty$,

$$f(x) = x|x|\left[1 - \frac{1}{2x^2} + o\left(\frac{1}{x^2}\right)\right] = x|x| - \frac{|x|}{2x} + o(1) = \begin{cases} x^2 - \frac{1}{2} + o(1) & (x \to \infty), \\ -x^2 + \frac{1}{2} + o(1) & (x \to -\infty). \end{cases}$$



(v) $f(x) = \arctan \log |x^2 - 1|$: when $x \to \pm 1$ the logarithm diverges to $-\infty$, so $f(x) \to -\pi/2$. In other words, even though the function is not well defined in $x = \pm 1$, at these two points it has an *avoidable* discontinuity which can be remedied by setting $f(\pm 1) = -\pi/2$. On the other hand, as $x \to \pm\infty$ the logarithm diverges to ∞ and therefore $f(x) \to \pi/2$.



(vi) $f(x) = 2\arctan x + \arcsin\left(\dfrac{2x}{1+x^2}\right)$: the domain of this function is $\mathbb{R}$ because so is the domain of $\arctan x$ and the argument of the arcsin is within $[-1,1]$. To see this

$$(x-1)^2 \geqslant 0 \quad\Leftrightarrow\quad x^2-2x+1 \geqslant 0 \quad\Leftrightarrow\quad x^2+1 \geqslant 2x \quad\Leftrightarrow\quad \frac{2x}{x^2+1} \leqslant 1,$$

$$(x+1)^2 \geqslant 0 \quad\Leftrightarrow\quad x^2+2x+1 \geqslant 0 \quad\Leftrightarrow\quad x^2+1 \geqslant -2x \quad\Leftrightarrow\quad -\frac{2x}{x^2+1} \leqslant 1$$

$$\Leftrightarrow\quad \frac{2x}{x^2+1} \geqslant -1.$$

if we calculate $f'(x)$, using the fact that

$$\left(\frac{2x}{1+x^2}\right)' = \frac{2(1+x^2)-(2x)^2}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2},$$

we obtain

$$f'(x) = \frac{2}{1+x^2} + \frac{1}{\sqrt{1-\frac{4x^2}{(1+x^2)^2}}}\frac{2(1-x^2)}{(1+x^2)^2}.$$

But

$$1-\frac{4x^2}{(1+x^2)^2} = \frac{1+2x^2+x^4-4x^2}{(1+x^2)^2} = \frac{1-2x^2+x^4}{(1+x^2)^2} = \frac{(1-x^2)^2}{(1+x^2)^2},$$

so

$$f'(x) = \frac{2}{1+x^2} + \frac{(1+x^2)}{|1-x^2|}\cdot\frac{2(1-x^2)}{(1+x^2)^2} = \frac{2}{1+x^2}\left[1+\frac{1-x^2}{|1-x^2|}\right].$$

Now

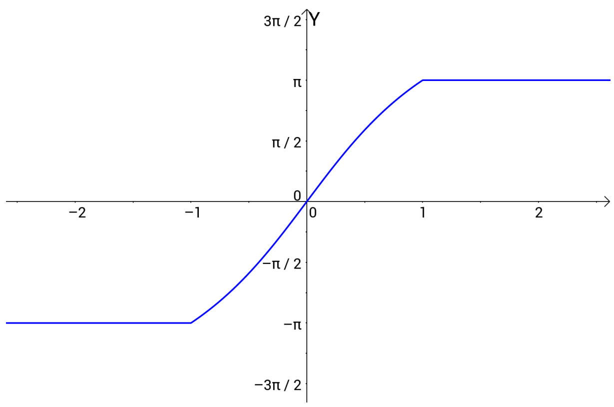
$$\frac{1-x^2}{|1-x^2|} = \begin{cases} 1, & |x|<1, \\ -1, & |x|>1, \end{cases}$$

therefore

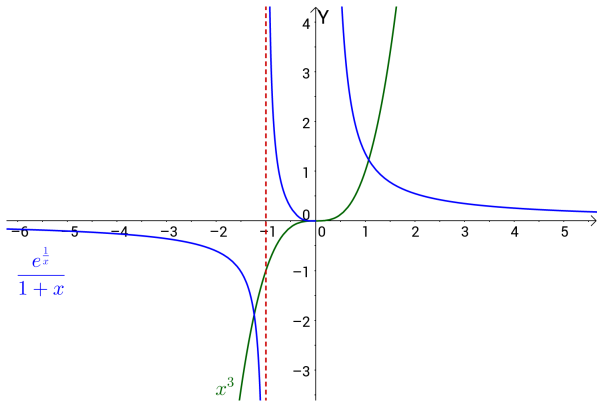
$$f'(x) = \begin{cases} \dfrac{4}{1+x^2}, & |x|<1, \\ 0, & |x|>1. \end{cases}$$

Function $f(x)$ is thus constant if $|x|>1$ and strictly increasing if $|x|<1$. Besides, $f(x)$ is obviously continuous because so are all functions involved, so the constant values it takes for $x \geqslant 1$ and $x \leqslant -1$ can be found as

$$f(\pm 1) = \pm 2\arctan 1 + \pm\arcsin 1 = \pm\left(2\cdot\frac{\pi}{4}+\frac{\pi}{2}\right) = \pm\pi.$$



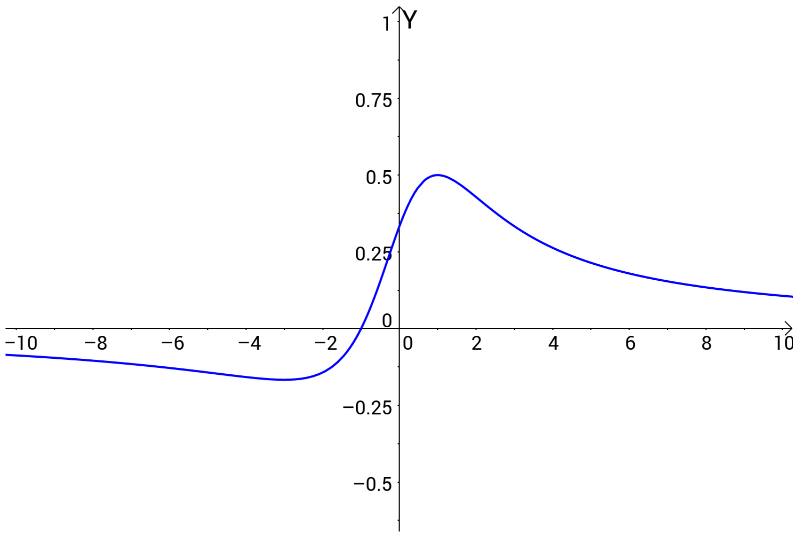
Problem 8.32



Problem 8.33 Let us calculate the derivative of $f(x)$:

$$f'(x) = \frac{3+x^2-(1+x)2x}{(3+x^2)^2} = \frac{3-2x-x^2}{(3+x^2)^2} = \frac{(3+x)(1-x)}{(3+x^2)^2},$$

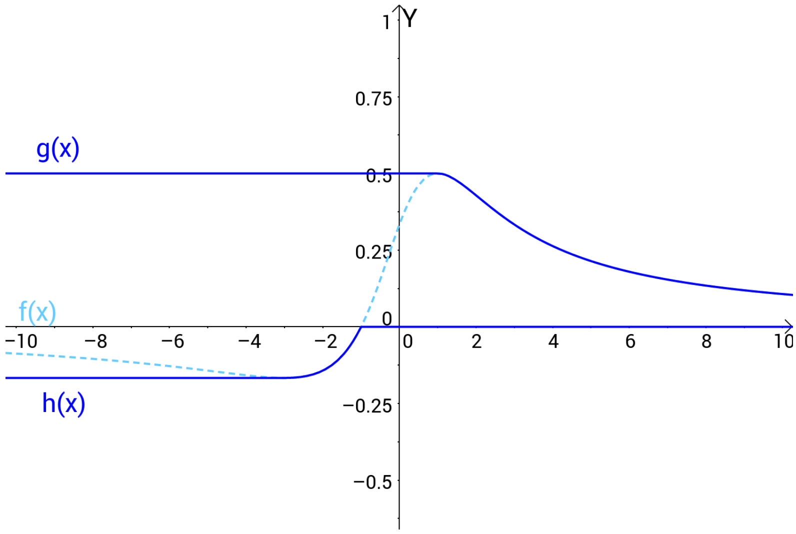
so the function increases for $-3 < x < 1$ and decreases for $x < -3$ and $x > 1$, hence it has a local maximum at $x = 1$ and a local minimum at $x = -3$. The X axis is a horizontal asymptote, and $f(1) = 1/2$, $f(-3) = -1/6$. Besides $f(x) = 0$ for $x = -1$ only. Here is a plot of the function:



Therefore

$$g(x) = \sup_{y>x} f(y) = \begin{cases} \frac{1}{2}, & x < 1, \\ f(x), & x \geqslant 1, \end{cases} \qquad h(x) = \inf_{y>x} f(y) = \begin{cases} -\frac{1}{6}, & x < -3, \\ f(x), & -3 \geqslant x \leqslant -1, \\ 0, & -1 < x. \end{cases}$$

Here is a plot of these functions:



Problem 8.34 We first need to calculate two derivatives of $f(x)$:

$$f'(x) = \frac{2x}{1+x^2}, \qquad f''(x) = \frac{2(1-x^2)}{(1+x^2)^2},$$

Thus $x = \pm 1$ are inflection points of $f(x)$ (because $f''(x) < 0$ on one side of them and $f''(x) > 0$ on the other side). The slopes at these two points are $f'(1) = 1$, $f'(-1) = -1$, and the coordinates of those points are $\big(1, f(1)\big) = (1, \log 2)$, $\big(-1, f(-1)\big) = (-1, \log 2)$. Therefore the two straight tangents are

$$y = \log 2 + (x-1), \qquad y = \log 2 - (x+1).$$

The plot of $f(x)$ along with these two tangents goes as follows: