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Calculus I

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Unit 9. Primitives

Exercises

Problems

Problem 9.1 Obtain the following immediate (or nearly so) primitives:

(i) $\displaystyle\int \frac{dx}{\cos^2 x}$; (ii) $\displaystyle\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx$; (iii) $\displaystyle\int \frac{x}{(x^2+1)^{5/2}}\,dx$; (iv) $\displaystyle\int \frac{1+\sin x}{1+\cos x}\,dx$; (v) $\displaystyle\int \frac{dx}{1-\sin x}\,dx$; (vi) $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\,dx$; (vii) $\displaystyle\int \frac{1+\sqrt{1-\sqrt{x}}}{\sqrt{x}}\,dx$; (viii) $\displaystyle\int \frac{\cos^3 x}{\sin^4 x}\,dx$; (ix) $\displaystyle\int x^3\sqrt{1-x^2}\,dx$.

HINTS: (iv) multiply and divide by $1-\cos x$ and expand; (v) idem with $1+\sin x$; (vii) alternatively $t=\sqrt{1-\sqrt{x}}$; (viii) $\cos^3 x = (1-\sin^2 x)\cos x$ and expand; (ix) write $x^3 = x(x^2-1)+x$ and expand.

Problem 9.2 Obtain the primitives of the following rational functions:

(i) $\displaystyle\int \frac{x^2}{(x-1)^3}\,dx$; (ii) $\displaystyle\int \frac{dx}{(x-1)^2(x^2+x+1)}$; (iii) $\displaystyle\int \frac{2x^2+3}{x^2(x-1)}\,dx$; (iv) $\displaystyle\int \frac{2}{x^2-2x+2}\,dx$; (v) $\displaystyle\int \frac{4x^4-x^3-46x^2-20x+153}{x^3-2x^2-9x+18}\,dx$; (vi) $\displaystyle\int \frac{x^5-2x^3}{x^4-2x^2+1}\,dx$.

HINTS: (ii) $x^2+x+1=(x+1/2)^2+3/4$; (v) $x^3-2x^2-9x+18=(x-2)(x-3)(x+3)$; (vi) $x^4-2x^2+1=(x-1)^2(x+1)^2$.

Problem 9.3 Obtain the following primitives doing an appropriate change of variable:

(i) $\displaystyle\int x^2\sqrt{x-1}\,dx$; (ii) $\displaystyle\int x^2\sin\sqrt{x^3}\,dx$; (iii) $\displaystyle\int \cos(\log x)\,dx$; (iv) $\displaystyle\int \sin(\log x)\,dx$; (v) $\displaystyle\int \cos^2(\log x)\,dx$; (vi) $\displaystyle\int \frac{\sqrt{x}+1}{x+3}\,dx$; (vii) $\displaystyle\int \frac{(x+1)^3}{\sqrt{1-(x+1)^2}}\,dx$; (viii) $\displaystyle\int \frac{x^3}{(1+x^2)^3}\,dx$; (ix) $\displaystyle\int \frac{dx}{(2+x)\sqrt{1+x}}$; (x) $\displaystyle\int \frac{dx}{1+\sqrt[3]{1-x}}$; (xi) $\displaystyle\int \frac{e^{4x}}{e^{2x}+2e^x+2}\,dx$; (xii) $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}\,dx$; (xiii) $\displaystyle\int \sqrt{e^x-1}\,dx$; (xiv) $\displaystyle\int \frac{\sin^2 x\cos^5 x}{\tan^3 x}\,dx$; (xv) $\displaystyle\int \frac{dx}{3+\sqrt{2x+5}}$; (xvi) $\displaystyle\int \sqrt{\frac{x-1}{x+1}}\,dx$; (xvii) $\displaystyle\int \sqrt{\sqrt{x}+1}\,dx$; (xviii) $\displaystyle\int \frac{\sqrt{x+2}}{1+\sqrt{x+2}}\,dx$; (xix) $\displaystyle\int \sqrt{2+e^x}\,dx$; (xx) $\displaystyle\int \frac{\sin x+3\cos x}{\sin x+2\cos x}\,dx$; (xxi) $\displaystyle\int \frac{\sin x+3\cos x}{\sin x\cos x+2\sin x}\,dx$; (xxii) $\displaystyle\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x}}\,dx$; (xxiii) $\displaystyle\int \frac{dx}{(x+1)\sqrt[3]{x+2}}$; (xxiv) $\displaystyle\int \frac{dx}{e^x-4e^{-x}}\,dx$.

HINTS: (i) $t=\sqrt{x-1}$ (or int. by parts twice); (ii) $t^2=x^3$; (iii)–(v) $t=\log x$; (vi) $t=\sqrt{x}$; (vii) $t=\sqrt{1-(x+1)^2}$; (viii) $t=1+x^2$; (ix) $t^2=1+x$; (x) $t^3=1-x$; (xi) $t=e^x$; (xii) $t^2=e^{2x}-1$; (xiii) $t^2=e^x-1$; (xiv) $t=\cos x$; (xv) $t=3+\sqrt{2x+5}$; (xvi) $t=\sqrt{(x-1)/(x+1)}$; (xvii) $t=\sqrt{\sqrt{x}+1}$; (xviii) $t=\sqrt{x+2}$; (xix) $t=\sqrt{2+e^x}$; (xx) $t=\tan x$; (xxi) $t=\tan(x/2)$; (xxii) $t=\sqrt{1+x^{1/3}}$; (xxiii) $t^3=x+2$; (xxiv) $t=e^x$.

Problem 9.4 Obtain the following primitives with the help of some trigonometric identity:

(i) $\int \sin^2 x\,dx$; (ii) $\int \cos^2 x\,dx$; (iii) $\int \sin^4 x\,dx$; (iv) $\int \cos^4 x\,dx$; (v) $\int \cos^6 x\,dx$;

(vi) $\int \sin^2 x \cos^2 x\,dx$; (vii) $\int \tan^2 x\,dx$; (viii) $\int \tan^4 x\,dx$; (ix) $\int \frac{dx}{\cos^4 x}$; (x) $\int \sin^5 x\,dx$;

(xi) $\int \cos^3 x \sin^2 x\,dx$; (xii) $\int \sec^6 x\,dx$; (xiii) $\int \sin^3 x \cos^2 x\,dx$; (xiv) $\int \tan^3 x\,dx$; (xv) $\int \tan^3 x \sec^4 x\,dx$.

HINTS: Identities to use: $2\cos^2 x = 1 + \cos 2x$; $2\sin^2 x = 1 - \cos 2x$; $\cos^2 x + \sin^2 x = 1$; $\sec^2 x = 1 + \tan^2 x$.

Problem 9.5 Integrate by parts to obtain the following primitives:

(i) $\int x\tan^2(2x)\,dx$; (ii) $\int e^x \sin \pi x\,dx$; (iii) $\int e^x \cos 2x\,dx$; (iv) $\int \sec^3 x\,dx$;

(v) $\int \tan^2(3x)\sec^3(3x)\,dx$; (vi) $\int e^{\sin x}\cos^3 x\,dx$; (vii) $\int x^2 \log x\,dx$; (viii) $\int x^m \log x\,dx$;

(ix) $\int (\log x)^3\,dx$; (x) $\int x(\log x)^2\,dx$; (xi) $\int \frac{x\log x}{(1+x^2)^2}\,dx$; (xii) $\int \arctan \sqrt[3]{x}\,dx$.

Problem 9.6 Obtain the following primitives by performing a trigonometric substitution:

(i) $\int \frac{x^2+1}{\sqrt{x^2-1}}\,dx$; (ii) $\int \frac{x^2}{(x^2+1)^{5/2}}\,dx$;

(iii) $\int \frac{x^2}{(1-x^2)^{3/2}}\,dx$; (iv) $\int \frac{dx}{x^2\sqrt{1-x^2}}$;

(v) $\int \frac{dx}{x^2\sqrt{9-x^2}}$.

Problem 9.7 Find recurrence formulas for the following integrals:

(i) $I_m = \int \sin^m x\,dx \longrightarrow I_m = -\frac{1}{m}\sin^{m-1} x \cos x + \frac{m-1}{m} I_{m-2}$;

(ii) $I_m = \int (\log x)^m\,dx \longrightarrow I_m = x(\log x)^m - mI_{m-1}$;

(iii) $I_m = \int x^m e^{-x}\,dx \longrightarrow I_m = -x^m e^{-x} + mI_{m-1}$;

(iv) $I_m = \int \tan^m x\,dx \longrightarrow I_m = \frac{1}{m-1}\tan^{m-1} x - I_{m-2}$;

(v) $I_m = \int \sec^m x\,dx \longrightarrow I_m = \frac{1}{m-1}\tan x \sec^{m-2} x + \frac{m-2}{m-1} I_{m-2}$;

(vi) $I_m = \int x^m e^{x^2}\,dx \longrightarrow I_m = \frac{1}{2}x^{m-1}e^{x^2} - \frac{m-1}{2} I_{m-2}$;

(vii) $I_{m,n} = \int \sin^m x \cos^n x\,dx \longrightarrow I_{m,n} = -\frac{1}{m+n}\sin^{m-1} x \cos^{n+1} x + \frac{m-1}{m+n} I_{m-2,n}$.

Problem 9.8 Without calculating the integral, prove that

$$\int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}\,dx = Ax + B\log|c\cos x + d\sin x| + \text{const.}$$

by determining the constants A and B as functions of a, b, c, and d.