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Calculus I

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Unit 9. Primitives

Solutions

D.9 Primitives

Problem 9.1

(i) $\cos^{-2} x = \sec^2 x = 1 + \tan^2 x = (\tan x)'$; hence

$$\int \frac{dx}{\cos^2 x} = \tan x + c.$$

(ii) $\sin x - \cos x = -(\cos x + \sin x)'$; hence

$$\int \frac{\sin x - \cos x}{\sin x + \cos x}\, dx = -\log|\sin x + \cos x| + c.$$

(iii) $2x = (x^2+1)'$; hence

$$\int \frac{x}{(x^2+1)^{5/2}}\, dx = -\frac{1}{3}\left(x^2+1\right)^{-3/2} + c.$$

(iv) $\dfrac{1+\sin x}{1+\cos x} = \dfrac{(1+\sin x)(1-\cos x)}{1-\cos^2 x} = \dfrac{1+\sin x - \cos x - \sin x \cos x}{\sin^2 x}$, thus

$$\begin{aligned}\int \frac{1+\sin x}{1+\cos x}\, dx &= \int \csc^2 x\, dx + \int \csc x\, dx - \int \frac{\cos x}{\sin^2 x}\, dx - \int \cot x\, dx \\ &= -\cot x - \log|\csc x + \cot x| + \csc x - \log|\sin x| \\ &= \frac{1-\cos x}{\sin x} - \log(1+\cos x) + c.\end{aligned}$$

(v) $\dfrac{1}{1-\sin x} = \dfrac{1+\sin x}{1-\sin^2 x} = \dfrac{1+\sin x}{\cos^2 x}$, thus

$$\int \frac{dx}{1-\sin x} = \int \sec^2 x\, dx + \int \frac{\sin x}{\cos^2 x}\, dx = \tan x + \sec x + c = \frac{\sin x + 1}{\cos x} + c.$$

(vi) $2x = (1+x^2)'$; hence

$$\int \frac{x}{\sqrt{1+x^2}}\, dx = \sqrt{1+x^2} + c.$$

(vii) As $\left(1-\sqrt{x}\right)' = -\dfrac{1}{2\sqrt{x}}$,

$$\begin{aligned}\int \frac{1+\sqrt{1-\sqrt{x}}}{\sqrt{x}}\, dx &= -2\int \left(1+\sqrt{1-\sqrt{x}}\right)\left(1-\sqrt{x}\right)'\, dx \\ &= -2\left(1-\sqrt{x}\right) - \frac{4}{3}\left(1-\sqrt{x}\right)^{3/2} + c = 2\sqrt{x} - \frac{4}{3}\left(1-\sqrt{x}\right)^{3/2} + c'\end{aligned}$$

(where $c' = c - 2$).

(viii) $\dfrac{\cos^3 x}{\sin^4 x} = \dfrac{\cos^2 x}{\sin^4 x}(\sin x)' = \dfrac{1-\sin^2 x}{\sin^4 x}(\sin x)'$, hence

$$\int \frac{\cos^3 x}{\sin^4 x}\, dx = \int \left(\frac{1}{\sin^4 x} - \frac{1}{\sin^2 x}\right)(\sin x)'\, dx = -\frac{1}{3}\csc^3 x + \csc x + c.$$

(ix) $x^3\sqrt{1-x^2} = x(x^2-1)\sqrt{1-x^2} + x\sqrt{1-x^2} = -x(1-x^2)^{3/2} + x(1-x^2)^{1/2}$. Since $(1-x^2)' = -2x$, then

$$\int x^3\sqrt{1-x^2}\, dx = \frac{1}{5}\left(1-x^2\right)^{5/2} - \frac{1}{3}\left(1-x^2\right)^{3/2} + c.$$

Problem 9.2

(i) $-\dfrac{1}{2(x-1)^2}-\dfrac{2}{x-1}+\log|x-1|+c;$

(ii) $-\dfrac{1}{3(x-1)}-\dfrac{1}{3}\log|x-1|+\dfrac{1}{6}\log(x^2+x+1)+\dfrac{1}{3\sqrt{3}}\arctan\left(\dfrac{2x+1}{\sqrt{3}}\right)+c;$

(iii) $5\log|x-1|-3\log|x|+\dfrac{3}{x}+c;$

(iv) $2\arctan(x-1)+c;$

(v) $2x^2+7x+3\log|x-2|-4\log|x-3|+5\log|x+3|+c;$

(vi) $\dfrac{x^2}{2}+\dfrac{1}{4(x-1)}-\dfrac{1}{4(x+1)}+c.$

Problem 9.3

(i) $\dfrac{2}{7}(x-1)^{7/2}+\dfrac{4}{5}(x-1)^{5/2}+\dfrac{2}{3}(x-1)^{3/2}+c;$

(ii) $\dfrac{2}{3}\left(-x^{3/2}\cos x^{3/2}+\sin x^{3/2}\right)+c;$

(iii) $\dfrac{x}{2}\cos(\log x)+\dfrac{x}{2}\sin(\log x)+c;$

(iv) $-\dfrac{x}{2}\cos(\log x)+\dfrac{x}{2}\sin(\log x)+c;$

(v) $\dfrac{x}{2}+\dfrac{x}{10}\cos(2\log x)+\dfrac{x}{5}\sin(2\log x)+c;$

(vi) $2\sqrt{x}+\log|x+3|-2\sqrt{3}\arctan\sqrt{\dfrac{x}{3}}+c;$

(vii) $\dfrac{1}{3}\left[1-(x+1)^2\right]^{3/2}-\left[1-(x+1)^2\right]^{1/2}+c;$

(viii) $-\dfrac{1}{2(1+x^2)}+\dfrac{1}{4(1+x^2)^2}+c;$

(ix) $2\arctan\sqrt{1+x}+c;$

(x) $-\dfrac{3}{2}(1-x)^{2/3}+3(1-x)^{1/3}-3\log\left|1+(1-x)^{1/3}\right|+c;$

(xi) $\dfrac{e^{2x}}{2}-2e^x+\log\left(e^{2x}+2e^x+2\right)+2\arctan(e^x+1)+c;$

(xii) $\arctan\sqrt{e^{2x}-1}+c;$

(xiii) $2\sqrt{e^x-1}-2\arctan\sqrt{e^x-1}+c;$

(xiv) $\dfrac{1}{7}\cos^7 x+\dfrac{1}{5}\cos^5 x+\dfrac{1}{3}\cos^3 x+\cos x+\dfrac{1}{2}\log(1-\cos x)-\dfrac{1}{2}\log(1+\cos x)+c;$

(xv) $\sqrt{2x+5}-3\log\left(3+\sqrt{2x+5}\right)+c;$

(xvi) $-\dfrac{1}{t-1}+\log|t-1|-\dfrac{1}{t+1}-\log|t+1|+c$, with $t=\sqrt{(x-1)/(x+1)}$;

(xvii) $\dfrac{4}{5}\left(\sqrt{x}+1\right)^{5/2}-\dfrac{4}{3}\left(\sqrt{x}+1\right)^{3/2}+c;$

(xviii) $x-2\sqrt{x+2}+2\log\left(\sqrt{x+2}+1\right)+c;$

(xix) $2\sqrt{2+e^x}+\sqrt{2}\log\left(\sqrt{2+e^x}-\sqrt{2}\right)-\sqrt{2}\log\left(\sqrt{2+e^x}+\sqrt{2}\right)+c;$

(xx) $\frac{1}{5}\log|\tan x+2|-\frac{1}{10}\log\left(\tan^2 x+1\right)+\frac{7}{5}x+c;$

(xxi) $\log\left|\tan\frac{x}{2}\right|-2\log\left(\tan^2\frac{x}{2}+3\right)+\frac{2}{\sqrt{3}}\arctan\left(\frac{\tan(x/2)}{\sqrt{3}}\right)+c;$

(xxii) $\frac{6}{5}\left(1+x^{1/3}\right)^{5/2}-2\left(1+x^{1/3}\right)^{3/2}+c;$

(xxiii) $\log\left|(x+2)^{1/3}-1\right|-\frac{1}{2}\log\left[(x+2)^{2/3}+(x+2)^{1/3}+1\right]+\sqrt{3}\arctan\left(\frac{2(x+2)^{1/3}+1}{\sqrt{3}}\right)+c;$

(xxiv) $\frac{1}{4}\log|e^x-2|-\frac{1}{4}\log(e^x+2)+c.$

Problem 9.4

(i) $\sin^2 x=\frac{1}{2}-\frac{\cos 2x}{2}$, hence

$$\int\sin^2 x\,dx=\frac{x}{2}-\frac{1}{4}\sin 2x+c.$$

(ii) $\cos^2 x=\frac{1}{2}+\frac{\cos 2x}{2}$, hence

$$\int\cos^2 x\,dx=\frac{x}{2}+\frac{1}{4}\sin 2x+c.$$

(iii) $\sin^4 x=\frac{1}{4}(1-\cos 2x)^2=\frac{1}{4}-\frac{1}{2}\cos 2x+\frac{1}{4}\cos^2 2x=\frac{1}{4}-\frac{1}{2}\cos 2x+\frac{1}{8}+\frac{1}{8}\cos 4x=\frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$, hence

$$\int\sin^4 x\,dx=\frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+c.$$

(iv) $\cos^4 x=\frac{1}{4}(1+\cos 2x)^2=\frac{1}{4}+\frac{1}{2}\cos 2x+\frac{1}{4}\cos^2 2x=\frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$, hence

$$\int\cos^4 x\,dx=\frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+c.$$

(v) $\cos^6 x=\frac{1}{8}(1+\cos 2x)^3=\frac{1}{8}+\frac{3}{8}\cos 2x+\frac{3}{8}\cos^2 2x+\frac{1}{8}\cos^3 2x=\frac{5}{16}+\frac{3}{8}\cos 2x+\frac{3}{16}\cos 4x+\frac{1}{16}(1-\sin^2 2x)(\sin 2x)'$, hence

$$\begin{aligned}\int\cos^6 x\,dx&=\frac{5}{16}x+\frac{3}{16}\sin 2x+\frac{3}{64}\sin 4x+\frac{1}{16}\sin 2x-\frac{1}{48}\sin^3 2x+c\\&=\frac{5}{16}x+\frac{1}{4}\sin 2x+\frac{3}{64}\sin 4x-\frac{1}{48}\sin^3 2x+c.\end{aligned}$$

(vi) $\sin^2 x\cos^2 x=\frac{1}{4}-\frac{1}{4}\cos^2 2x=\frac{1}{8}-\frac{1}{8}\cos 4x$, hence

$$\int\sin^2 x\cos^2 x\,dx=\frac{x}{8}-\frac{1}{32}\sin 4x+c.$$

(vii) $\tan^2 x=(\tan x)'-1$, hence

$$\int\tan^2 x\,dx=\tan x-x+c.$$

(viii) $\tan^4 x = \tan^2 x(\tan x)' - \tan^2 x = \tan^2 x(\tan x)' - (\tan x)' + 1$, hence

$$\int \tan^4 x\,dx = \frac{1}{3}\tan^3 x - \tan x + x + c.$$

(ix) $\dfrac{1}{\cos^4 x} = (1+\tan^2 x)(\tan x)'$, hence

$$\int \frac{dx}{\cos^4 x}\,dx = \tan x + \frac{1}{3}\tan^3 x + c.$$

(x) $\sin^5 x = -\sin^4 x(\cos x)' = -(1-\cos^2 x)^2(\cos x)' = (-1+2\cos^2 x - \cos^4 x)(\cos x)'$, thus

$$\int \sin^5 x\,dx = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c.$$

(xi) $\cos^3\sin^2 x = (1-\sin^2 x)\sin^2 x(\sin x)' = (\sin^2 x - \sin^4 x)(\sin x)'$, so

$$\int \cos^3\sin^2 x\,dx = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c.$$

(xii) $\sec^6 x = (1+\tan^2 x)^2(\tan x)' = (1+2\tan^2 x + \tan^4 x)(\tan x)'$, hence

$$\int \sec^6 x\,dx = \tan x + \frac{2}{3}\tan^3 x + \frac{1}{5}\tan^5 x + c.$$

(xiii) $\sin^3 x\cos^2 x = -(1-\cos^2 x)\cos^2 x(\cos x)' = (\cos^4 x - \cos^2 x)(\cos x)'$, therefore

$$\int \sin^3 x\cos^2 x\,dx = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + c.$$

(xiv) $\tan^3 x = \tan x[(\tan x)' - 1] = \tan x(\tan x)' - \tan x$, thus

$$\int \tan^3 x\,dx = \frac{1}{2}\tan^2 x + \log|\cos x| + c.$$

(xv) $\tan^3 x\sec^4 x = \tan^3 x(1+\tan^2 x)(\tan x)' = (\tan^3 x + \tan^5 x)(\tan x)'$, thus

$$\int \tan^3 x\sec^4 x\,dx = \frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x + c.$$

Problem 9.5

(i) $\tan^2(2x) = \dfrac{1}{2}(\tan 2x)' - 1$, so

$$\int x\tan^2(2x)\,dx = \frac{1}{2}\int x(\tan 2x)'\,dx - \int x\,dx = -\frac{x^2}{2} + \frac{x}{2}\tan 2x - \frac{1}{2}\int \tan 2x\,dx$$
$$= -\frac{x^2}{2} + \frac{x}{2}\tan 2x + \frac{1}{4}\log|\cos 2x| + c.$$

(ii) Since $(e^x)' = e^x$,

$$\int e^x\sin\pi x\,dx = e^x\sin\pi x - \pi\int e^x\cos\pi x\,dx = e^x\sin\pi x - \pi e^x\cos\pi x - \pi^2\int e^x\sin\pi x\,dx.$$

Therefore

$$(1+\pi^2)\int e^x\sin\pi x\,dx = e^x(\sin\pi x - \pi\cos\pi x)$$

and finally

$$\int e^x\sin\pi x\,dx = \frac{e^x}{1+\pi^2}(\sin\pi x - \pi\cos\pi x) + c.$$

(iii) Since $(e^x)' = e^x$,

$$\int e^x \cos 2x\,dx = e^x \cos 2x + 2\int e^x \sin 2x\,dx = e^x \cos 2x + 2e^x \sin 2x - 4\int e^x \cos 2x\,dx.$$

Then

$$5\int e^x \cos 2x\,dx = e^x(\cos 2x + 2\sin 2x)$$

and

$$\int e^x \cos 2x\,dx = \frac{e^x}{5}(\cos 2x + 2\sin 2x) + c.$$

(iv) Since $\sec^2 x = 1 + \tan^2 x = (\tan x)'$ and $(\sec x)' = \sec x \tan x$,

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x \tan x - \int \tan^2 x \sec x\,dx = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx\\ &= \sec x \tan x - \int \sec^3 x\,dx + \log|\sec x + \tan x|,\end{aligned}$$

therefore

$$2\int \sec^3 x\,dx = \sec x \tan x + \log|\sec x + \tan x|$$

and finally

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\log|\sec x + \tan x| + c.$$

(v) First of all $\tan^2(3x)\sec^3(3x) = \frac{1}{3}\tan^2(3x)(\tan 3x)'\sec 3x = \frac{1}{9}\left[\tan^3(3x)\right]'\sec 3x$. Thus,

$$\begin{aligned}I(x) = \int \tan^2(3x)\sec^3(3x)\,dx &= \frac{1}{9}\tan^3(3x)\sec 3x - \frac{1}{9}\int \tan^3(3x)(\sec 3x)'\,dx\\ &= \frac{1}{9}\tan^3(3x)\sec 3x - \frac{1}{3}\int \tan^4(3x)\sec 3x\,dx.\end{aligned}$$

But $\tan^2(3x) = \sec^2(3x) - 1$, so

$$\begin{aligned}\int \tan^4(3x)\sec 3x\,dx &= \int \tan^2(3x)\sec^3(3x)\,dx - \int \tan^2(3x)\sec 3x\,dx\\ &= I(x) - \int \sec^3(3x)\,dx + \int \sec 3x\,dx.\end{aligned}$$

And from the previous exercise,

$$\int \sec^3(3x)\,dx = \frac{1}{6}\sec 3x \tan 3x + \frac{1}{6}\log|\sec 3x + \tan 3x|,$$
$$\int \sec 3x\,dx = \frac{1}{3}\log|\sec 3x + \tan 3x|.$$

Therefore,

$$\int \tan^4(3x)\sec 3x\,dx = I(x) - \frac{1}{6}\sec 3x \tan 3x + \frac{1}{6}\log|\sec 3x + \tan 3x|.$$

This yields

$$I(x) = \frac{1}{9}\tan^3(3x)\sec 3x - \frac{1}{3}I(x) + \frac{1}{18}\sec 3x \tan 3x - \frac{1}{18}\log|\sec 3x + \tan 3x|,$$

from which

$$\frac{4}{3}I(x) = \frac{1}{9}\tan^3(3x)\sec 3x + \frac{1}{18}\sec 3x \tan 3x - \frac{1}{18}\log|\sec 3x + \tan 3x|,$$

which leads to

$$\begin{aligned} I(x) &= \frac{1}{12}\tan^3(3x)\sec 3x + \frac{1}{24}\sec 3x \tan 3x - \frac{1}{24}\log|\sec 3x + \tan 3x| + c \\ &= \frac{1}{12}\tan 3x \sec^3(3x) - \frac{1}{24}\sec 3x \tan 3x - \frac{1}{24}\log|\sec 3x + \tan 3x| + c. \end{aligned}$$

(The last line is obtain by replacing $\tan^2(3x) = \sec^2(3x) - 1$.)

(vi) Since $e^{\sin x}\cos x = \left(e^{\sin x}\right)'$,

$$\begin{aligned} \int e^{\sin x}\cos^3 x\,dx &= e^{\sin x}\cos^2 x + 2\int e^{\sin x}\cos x \sin x\,dx \\ &= e^{\sin x}\cos^2 x + 2e^{\sin x}\sin x - 2\int e^{\sin x}\cos x\,dx \\ &= e^{\sin x}\cos^2 x + 2e^{\sin x}\sin x - 2e^{\sin x} + c = e^{\sin x}\left(\cos^2 x + 2\sin x - 2\right) + c \\ &= e^{\sin x}\left(-\sin^2 x + 2\sin x - 1\right) + c = -e^{\sin x}(1-\sin x)^2 + c. \end{aligned}$$

(vii) $\displaystyle\int x^2 \log x\,dx = \frac{x^3}{3}\log x - \frac{1}{3}\int x^3 \frac{1}{x}\,dx = \frac{x^3}{3}\log x - \frac{x^3}{9} + c.$

(viii) If $m \neq -1$,

$$\int x^m \log x\,dx = \frac{x^{m+1}}{m+1}\log x - \frac{1}{m+1}\int x^{m+1}\frac{1}{x}\,dx = \frac{x^{m+1}}{m+1}\log x - \frac{x^{m+1}}{(m+1)^2} + c.$$

If $m = -1$,

$$\int \frac{1}{x}\log x\,dx = \int \log x(\log x)'\,dx = \frac{1}{2}(\log x)^2 + c.$$

(ix) Taking $1 = (x)'$,

$$\int (\log x)^3\,dx = x(\log x)^3 - 3\int \cancel{x}(\log x)^2 \frac{1}{\cancel{x}}\,dx.$$

Similarly,

$$\int (\log x)^2\,dx = x(\log x)^2 - 2\int \cancel{x}\log x \frac{1}{\cancel{x}}\,dx = x(\log x)^2 - 2x\log x + 2x.$$

Thus,

$$\int (\log x)^3\,dx = x(\log x)^3 - 3x(\log x)^2 + 6x\log x - 6x + c.$$

(x) $\displaystyle\int x(\log x)^2\,dx = \frac{x^2}{2}(\log x)^2 - \int x^2 \log x \frac{1}{x}\,dx.$ Also,

$$\int x\log x\,dx = \frac{x^2}{2}\log x - \frac{1}{2}\int x^2 \frac{1}{x}\,dx = \frac{x^2}{2}\log x - \frac{x^2}{4}.$$

Therefore

$$\int x(\log x)^2\,dx = \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{x^2}{4} + c.$$

(xi) On the one hand, $\frac{x}{(1+x^2)^2} = -\frac{1}{2}\left(\frac{1}{1+x^2}\right)'$, so

$$\int \frac{x\log x}{(1+x^2)^2}\,dx = -\frac{1}{2}\frac{\log x}{1+x^2} + \frac{1}{2}\int \frac{dx}{x(1+x^2)}\,dx.$$

On the other hand,

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2} \quad \Rightarrow \quad 1 = A(1+x^2) + Bx^2 + Cx.$$

Setting $x = 0$ we obtain $A = 1$. Thus,

$$1 = 1 + x^2 + Bx^2 + Cx \quad \Rightarrow \quad -x^2 = Bx^2 + CX \quad \Rightarrow \quad B = -1, \quad C = 0.$$

Hence,

$$\int \frac{dx}{x(1+x^2)}\,dx = \int \frac{dx}{x} - \int \frac{x}{1+x^2}\,dx = \log x - \frac{1}{2}\log(1+x^2).$$

(Notice: we do not need to write $\log|x|$ because the $\log x$ of the integrand forces $x > 0$.) Therefore,

$$\int \frac{x\log x}{(1+x^2)^2}\,dx = -\frac{\log x}{2(1+x^2)} + \frac{1}{2}\log x - \frac{1}{4}\log(1+x^2) + c.$$

(xii) Using $1 = (x)'$ we get

$$\int \arctan\sqrt[3]{x}\,dx = x\arctan\sqrt[3]{x} - \frac{1}{3}\int x\frac{x^{-2/3}}{1+x^{2/3}}\,dx = x\arctan\sqrt[3]{x} - \frac{1}{3}\int \frac{x^{1/3}}{1+x^{2/3}}\,dx.$$

Now,

$$\frac{x^{1/3}}{1+x^{2/3}} = x^{-1/3}\frac{x^{2/3}}{1+x^{2/3}} = x^{-1/3}\left(1 - \frac{1}{1+x^{2/3}}\right) = \frac{3}{2}\left(1 - \frac{1}{1+x^{2/3}}\right)\left(x^{2/3}\right)',$$

thus

$$\frac{1}{3}\int \frac{x^{1/3}}{1+x^{2/3}}\,dx = \frac{x^{2/3}}{2} - \frac{1}{2}\log\left(1+x^{2/3}\right).$$

Accordingly,

$$\int \arctan\sqrt[3]{x}\,dx = x\arctan\sqrt[3]{x} - \frac{\sqrt[3]{x^2}}{2} + \frac{1}{2}\log\left(1+\sqrt[3]{x^2}\right) + c.$$

Problem 9.6

(i) Using the change of variable $x = \sec t$, with $dx = \sec t\tan t\,dt$,

$$\int \frac{x^2+1}{\sqrt{x^2-1}}\,dx = \int \frac{\sec^2 t + 1}{\cancel{\tan t}}\sec t\cancel{\tan t}\,dt = \int \sec^3 t\,dt + \int \sec t\,dt.$$

Using

$$\int \sec^3 t\,dt = \frac{1}{2}\sec t\tan t + \frac{1}{2}\log|\sec t + \tan t|,$$

$$\int \sec t\,dt = \log|\sec t + \tan t|,$$

$$\int \frac{x^2+1}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\sec t\tan t + \frac{3}{2}\log|\sec t + \tan t| + c = \frac{1}{2}x\sqrt{x^2-1} + \frac{3}{2}\log\left|x + \sqrt{x^2-1}\right| + c.$$

(ii) Using the change of variable $x = \tan t$, with $dx = \sec^2 t\,dt$, and the identity $\sec^2 t = 1 + \tan^2 t$,

$$\int \frac{x^2}{(x^2+1)^{5/2}}\,dx = \int \frac{\tan^2 t}{\sec^5 t}\sec^2 t\,dt = \int \tan^2 t\cos^3 t\,dt = \int \sin^2 t\cos t\,dt = \frac{1}{3}\sin^3 t.$$

Since $\sin t = \dfrac{\tan t}{\sqrt{1+\tan^2 t}}$,

$$\int \frac{x^2}{(x^2+1)^{5/2}}\,dx = \frac{x^3}{3(x^2+1)^{3/2}} + c.$$

(iii) Using the change of variable $x = \sin t$, with $dx = \cos t\,dt$,

$$\int \frac{x^2}{(1-x^2)^{3/2}}\,dx = \int \frac{\sin^2 t}{\cos^3 t}\cos t\,dt = \int \tan^2 t\,dt = \tan t - t + c = \frac{x}{\sqrt{1-x^2}} - \arcsin x + c.$$

(iv) Using the change of variable $x = \sin t$, with $dx = \cos t\,dt$,

$$\int \frac{dx}{x^2\sqrt{1-x^2}} = \int \frac{\cancel{\cos t}}{\sin^2 t\,\cancel{\cos t}}\,dt = \int \csc^2 t\,dt = -\cot t + c = -\frac{\sqrt{1-x^2}}{x} + c.$$

(v) Using the change of variable $x = 3y$, with $dx = 3\,dy$,

$$\int \frac{dx}{x^2\sqrt{9-x^2}} = \frac{3}{27}\int \frac{dy}{y^2\sqrt{1-y^2}} = -\frac{\sqrt{1-y^2}}{9y} + c = -\frac{\sqrt{9-x^2}}{9x} + c.$$

Problem 9.7

(i) First of all we write

$$I_m = \int \sin^m x\,dx = \int \sin^2 x\sin^{m-2} x\,dx = \int (1-\cos^2 x)\sin^{m-2} x\,dx$$
$$= I_{m-2} - \int \cos^2 x\sin^{m-2} x\,dx.$$

Now, since

$$\frac{d}{dx}\sin^{m-1} x = (m-1)\sin^{m-2} x\cos x,$$

we can integrate by parts

$$\int \cos^2 x\sin^{m-2} x\,dx = \frac{1}{m-1}\sin^{m-1} x\cos x - \frac{1}{m-1}\int \sin^{m-1} x(-\sin x)\,dx$$
$$= \frac{1}{m-1}\sin^{m-1} x\cos x + \frac{1}{m-1}I_m.$$

Therefore

$$I_m = I_{m-2} - \frac{1}{m-1}I_m - \frac{1}{m-1}\sin^{m-1} x\cos x,$$

which can be rewritten as

$$\left(1 + \frac{1}{m-1}\right)I_m = I_{m-2} - \frac{1}{m-1}\sin^{m-1} x\cos x \quad\Rightarrow\quad I_m = \frac{m-1}{m}I_{m-2} - \frac{1}{m}\sin^{m-1} x\cos x.$$

(ii) Integrating by parts,

$$I_m = \int (\log x)^m \, dx = x(\log x)^m - \int \not{x} m(\log x)^{m-1} \frac{1}{\not{x}} \, dx = x(\log x)^m - mI_{m-1}.$$

(iii) Integrating by parts,

$$I_m = \int x^m e^{-x} \, dx = -x^m e^{-x} + \int m x^{m-1} e^{-x} \, dx = -x^m e^{-x} + mI_{m-1}.$$

(iv) First of all,

$$\tan^m x = \tan^{m-2} x \tan^2 x = \tan^{m-2} x(\tan^2 x + 1 - 1) = \tan^{m-2} x(\tan x)' - \tan^{m-2} x.$$

Thus,

$$I_m = \int \tan^m x \, dx = -I_{m-2} + \int \tan^{m-2} x(\tan x)' \, dx.$$

Now, integrating by parts,

$$\begin{aligned} \int \tan^{m-2} x(\tan x)' \, dx &= \tan^{m-1} x - \int (m-2) \tan^{m-3} x(1 + \tan^2 x) \tan x \, dx \\ &= \tan^{m-1} x - (m-2)(I_{m-2} + I_m). \end{aligned}$$

Therefore,

$$I_m = -I_{m-2} + \tan^{m-1} x - (m-2)(I_{m-2} + I_m) \quad \Rightarrow \quad (m-1)I_m = -(m-1)I_{m-2} + \tan^{m-1} x,$$

so

$$I_m = -I_{m-2} + \frac{1}{m-1} \tan^{m-1} x.$$

(v) First of all

$$\sec^m x = \sec^{m-2} x \sec^2 x = \sec^{m-2} x(\tan x)',$$

so integrating by parts,

$$I_m = \int \sec^m x \, dx = \tan x \sec^{m-2} x - \int (m-2) \sec^{m-3} x(\sec x \tan x) \tan x \, dx.$$

But

$$\sec^{m-3} x(\sec x \tan x) \tan x = \sec^{m-2} x \tan^2 x = \sec^{m-2} x(\sec^2 x - 1),$$

therefore

$$I_m = \tan x \sec^{m-2} x - (m-2)(I_m - I_{m-2}) \quad \Rightarrow \quad (m-1)I_m = \tan x \sec^{m-2} x + (m-2)I_{m-2},$$

and finally

$$I_m = \frac{1}{m-1} \tan x \sec^{m-2} x + \frac{m-2}{m-1} I_{m-2}.$$

(vi) First of all

$$\sin^m x\cos^n x = \sin^{m-1} x(-\cos x)'\cos^n x, = -\sin^{m-1} x\frac{1}{n+1}\left(\cos^{n+1} x\right)',$$

so integrating by parts,

$$\begin{aligned} I_{m,n} &= \int \sin^m x\cos^n x\,dx \\ &= -\frac{1}{n+1}\sin^{m-1} x\cos^{n+1} x + \frac{1}{n+1}\int (m-1)\sin^{m-2} x\cos x\cos^{n+1} x\,dx \\ &= -\frac{1}{n+1}\sin^{m-1} x\cos^{n+1} x + \frac{m-1}{n+1}\int \sin^{m-2} x\cos^{n+2} x\,dx. \end{aligned}$$

But

$$\begin{aligned} \sin^{m-2} x\cos^{n+2} x &= \sin^{m-2} x\cos^n x\cos^2 x = \sin^{m-2} x\cos^n x(1-\sin^2 x) \\ &= \sin^{m-2} x\cos^n x - \sin^m x\cos^n x, \end{aligned}$$

therefore

$$I_{m,n} = -\frac{1}{n+1}\sin^{m-1} x\cos^{n+1} x + \frac{m-1}{n+1}(I_{m-2,n} - I_{m,n}),$$

from which

$$\frac{m+n}{n+1} I_{m,n} = -\frac{1}{n+1}\sin^{m-1} x\cos^{n+1} x + \frac{m-1}{n+1} I_{m-2,n},$$

and finally

$$I_{m,n} = -\frac{1}{m+n}\sin^{m-1} x\cos^{n+1} x + \frac{m-1}{m+n} I_{m-2,n}.$$

Problem 9.8 There is no need to calculate the integral. If the right-hand side is the primitive of the integrand then

$$\left(Ax + B\log|c\cos x + d\sin x|\right)' = \frac{a\cos x + b\sin x}{c\cos x + d\sin x},$$

in other words,

$$A + B\frac{-c\sin x + d\cos x}{c\cos x + d\cos x} = \frac{a\cos x + b\sin x}{c\cos x + d\sin x},$$

which when reduced to a single fraction becomes

$$\frac{(Ac+Bd)\cos x + (Ad-Bc)\sin x}{c\cos x + d\cos x} = \frac{a\cos x + b\sin x}{c\cos x + d\sin x}.$$

The two functions are the same if, and only if,

$$\left.\begin{aligned} Ac + Bd = a, \\ Ad - Bc = b, \end{aligned}\right\} \quad \Rightarrow \quad A = \frac{ac+bd}{c^2+d^2}, \quad B = \frac{ad-bc}{c^2+d^2}.$$