Calculus I

Differential and Integral Calculus of a Single Variable

José A. Cuesta

UNPUBLISHED

WWW.UC3M.ES

First edition, 2015
Second edition, 2017
Third edition, 2018
Fourth edition, 2019
Fifth edition, 2020
Sixth edition, 2021
Seventh edition, 2022

Contents

Real Numbers and Functions

1. The Real Line

In a loose sense, Calculus can be defined as the "algebra of real numbers". Real numbers are therefore its most basic ingredient. At present, we are all familiar with real numbers, but constructing them was a long process that took us more than two-thousand years. For the most intuitive numbers, those that are a human universal and that we use to count, are what mathematicians call *natural numbers*. Any other set of numbers is a construction deliberately introduced to solve a problem.

What we require of a number set so that it can be algebraically manipulated is that it satisfies a set of properties that we globally refer to as an *ordered field*. So let us start by setting up those properties and exploring what they allow us to do.

1.1 Ordered Fields

An ordered field is a set of elements —that we call *numbers*— with two binary operations: *addition* (denoted with "+") and *multiplication* (denoted with "·"), which satisfy two set of properties: *field axioms* and *order axioms*.

Definition 1.1.1 — Field. For all x, y, and z the following properties hold:

I. Addition axioms:

1. $x+y=y+x$ ***commutativity***
2. $x+(y+z)=(x+y)+z$ ***associativity***
3. There is a number 0 such that $x+0=x$ ***zero***
4. For each x there is a number, denoted $-x$, such that $x+(-x)=0$ ***inverse***

II. Multiplication axioms:

5. $x\cdot y=y\cdot x$ ***commutativity***
6. $x\cdot(y\cdot z)=(x\cdot y)\cdot z$ ***associativity***
7. There is a number 1 such that $x\cdot 1=x$ ***unity***
8. For each $x\neq 0$ there is a number, denoted x^{-1}, such that $x\cdot x^{-1}=1$ ***reciprocal***
9. $x\cdot(y+z)=x\cdot y+x\cdot z$ ***distributive law***
10. $1\neq 0$ ***nontriviality***

R Another way of representing the multiplicative reciprocal is using a division bar: $xy^{-1} = x/y = \frac{x}{y}$.

Any set with two binary opeartions satisfying these axioms is called a field. But to be an ordered field there must be also a relation, denoted "$\leqslant$" (and read "smaller than or equal to"), with the following properties:

Definition 1.1.2 For all x, y, and z the following properties hold:

III. Order axioms:

11. $x \leqslant x$ ***reflexivity***

12. If $x \leqslant y$ and $y \leqslant x$ then $x = y$ ***antisymmetry***

13. If $x \leqslant y$ and $y \leqslant z$ then $x \leqslant z$ ***transitivity***

14. Either $x \leqslant y$ or $y \leqslant x$ ***linear ordering***

15. If $x \leqslant y$ then $x + z \leqslant y + z$ ***compatibility with addition***

16. If $0 \leqslant x$ and $0 \leqslant y$ then $0 \leqslant x \cdot y$ ***compatibility with multiplication***

Properties 11–13 define an *ordering* (sometimes call *partial ordering*). Property 14 defines a *linear* or *total ordering,* meaning that any two elements can be compared and decided which one is on the left and which one on the right (alternatively, it means that we can place the elements on a straight line in such a way that $x < y$ means that x is 'to the left of' y). An ordered field is a field satisfying all properties 11–16. The last two properties are important in order to manipulate inequalities.

We will also introduce some other order symbols: "$\geqslant$", read "greater than or equal to", so that $x \geqslant y$ is equivalent to $y \leqslant x$; "$<$", read "smaller than", so that $x < y$ is equivalent to $x \leqslant y$ but $x \neq y$; "$>$", read "greater than", so that $x > y$ is equivalent to $x \geqslant y$ but $x \neq y$.

We can use properties 11, 12, and 14 to proof the following:

Proposition 1.1.1 — Law of trichotomy. For every pair of elements x and y of an ordered field, one and only one of the following relations hold: $x < y$, $x = y$, or $x > y$.

The next proposition list a set of properties that can be proven to follow from the axioms of an ordered field. You will recognise them as the standard algebraic operations we perform when manipulating equations. It is important to make clear that they strongly depend on the axioms above, hence the importance to have sets of numbers that are ordered fields in order to do algebra.

R Notice that in general we will omit the symbol "$\cdot$" in multiplications, unless it is necessary to avoid ambiguitiy.

Proposition 1.1.2 The following properties hold in any ordered field:

i. Unique neutrals If $a + x = a$ for all a, then $x = 0$. If $ax = a$ for all a, then $x = 1$.

ii. Unique inverses If $a + x = 0$, then $x = -a$. If $ax = 1$, then $x = a^{-1}$.

iii. No divisors of zero If $xy = 0$, then $x = 0$ or $y = 0$ (or both).

iv. Cancellation laws for addition If $a + x = b + x$, then $a = b$. If $a + x \leqslant b + x$, then $a \leqslant b$.

v. Cancellation laws for multiplication If $ax = bx$ and $x \neq 0$, then $a = b$. If $ax \geqslant bx$, then $a \geqslant b$ if $x > 0$ and $a \leqslant b$ if $x < 0$.

vi. $0x = 0$ for all x.

vii. $-(-x) = x$ for all x.

viii. $-x = (-1)x$ for all x.

ix. If $x \neq 0$, the $x^{-1} \neq 0$ and $(x^{-1})^{-1} = x$.

x. If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$ and $(xy)^{-1} = x^{-1}y^{-1}$.

xi. If $x \leqslant y$ and $0 \leqslant z$, then $xz \leqslant yz$. If $x \leqslant y$ and $z \leqslant 0$, then $xz \geqslant yz$.

xii. If $x \leqslant 0$ and $y \leqslant 0$, then $xy \geqslant 0$. If $x \leqslant 0$ and $y \geqslant 0$, then $xy \leqslant 0$.

xiii. $0 < 1$.
xiv. $x^2 \geqslant 0$ for all x.

■ **Example 1.1** Using property **xiv** above it follows that $(a-b)^2 \geqslant 0$ for any two numbers a and b. Using twice the distributive law, $(a-b)^2 = a^2 - 2ab + b^2 \geqslant 0$. Thus, adding $2ab$ to both terms of the inequality we get $a^2 + b^2 \geqslant 2ab$. Dividing by $2(>0)$ and using property **v**,

$$ab \leqslant \frac{a^2+b^2}{2}.$$

■

■ **Example 1.2** Applying twice the distributive law we get, for any pair of numbers a and b,

$$(a-b)(a+b) = a^2 - b^2.$$

(This identity is more useful applied right to left.) ■

■ **Example 1.3** If $0 \leqslant a < b$, then it follows that $a^2 < b^2$. To see that, applying property **xi** above we obtain $a^2 \leqslant ab$ (multiplying by $a \geqslant 0$) and also $ab \leqslant b^2$ (multiplying by $b \geqslant 0$). So $a^2 \leqslant b^2$. But we can exclude that $a^2 = b^2$, for if that were true, then $b^2 - a^2 = (b-a)(b+a) = 0$. But $b+a > 0$, so the only possibility we are left with is $b = a$, which is not true. Therefore we must conclude that $a^2 < b^2$. ■

1.2 Number Systems

Let us have a look at the different number systems that have been constructed, the kind of problems that they are meant to solve and their peculiarities, up to the appearance of real numbers. There is a further set of numbers, *complex numbers,* which to a certain extent close the need for different number systems —at least from an algebraic point of view. In a way they "complete" real numbers and are necessary in linear algebra and in advanced applications of analysis. They will not be covered in this course though, so we will just mention them briefly.

1.2.1 Natural Numbers

This is the most basic set of numbers. Their meaning is intuitive and its main use is counting. All human cultures have a name for at least the first few natural numbers, and most have a way to give a name to natural numbers of arbitrary (or al least very large) size. Most cultures can add natural numbers, and many can multiply them.

In mathematics, they are introduced as the set $\mathbb{N} = \{1, 2, 3, \dots\}$. In more abstract terms, natural numbers are constructed with two axioms:

1. There is one (and only one) first element $1 \in \mathbb{N}$.
2. Every element $n \in \mathbb{N}$ has a *successor* $n+1$.

Notice that 1, n, or $n+1$ are just arbitrary numbers. We commonly use the arabic numerals with positional notation to name them, but their names change in different languages. Often we even denote them with roman numerals (I, II, III, IV, etc.). As a matter of fact, this is the reason why sometimes 0 is considered the first element of $\mathbb{N}$, instead of 1.

Natural numbers can be added and multiplied (multiplication is the abbreviation of a repeated addition). However, addition is conmutative and associative (and may even have a neutral element if we start in 0 rather than 1), but there is no inverse of a number. In other words, the equation $n + x = 0$ cannot be solved in $\mathbb{N}$ (there is no $x \in \mathbb{N}$ that satisfies the equation). This means that we cannot define a substraction operation that works for all pairs of natural numbers.

Multiplication is also problematic. It is conmutative, associative, there is a neutral element (1), and it is distributive. But there is not inverse either (equation $nx = 1$, in general, has no solution in $\mathbb{N}$). This means that we cannot define a division in $\mathbb{N}$ (this is another way of saying that divisions in $\mathbb{N}$ have, in general, a nonzero remainder).

Natural numbers do satisfy all axioms of order, so $\mathbb{N}$ is a totally ordered set. Not just that: since every subset of $\mathbb{N}$ contains a first element, we say that the order is *perfect* (or that $\mathbb{N}$ is *well-ordered*).

1.2.2 Induction principle

The recursive construction of natural numbers leads to an important type of mathematical proof: proofs by *induction*. The idea is that we have a proposition, $p(n)$, which is supposed to be valid for all $n \in \mathbb{N}$, and we want to prove it. Then all we need to do is to prove these two alternative propositions:

1. Proposition $p(1)$ is true.
2. If proposition $p(n)$ is true then necessarily proposition $p(n+1)$ is also true.

■ **Example 1.4** We will prove by induction this famous formula (first obtained by Gauss):

$$1+2+\cdots+n = \frac{n(n+1)}{2}, \quad n \in \mathbb{N}. \tag{1.1}$$

This is a proposition that is supposed to hold for all natural numbers.

So let us have a look at $p(1)$:

$$1 = \frac{1(1+1)}{2},$$

which is evindently true. This proves the first induction step. As for the second, let us assume that (1.1) holds for a given $n \in \mathbb{N}$ and let us add $n+1$ to both sides of the equation:

$$1+2+\cdots+n+(n+1) = \frac{n(n+1)}{2} + (n+1) = (n+1)\left(\frac{n}{2}+1\right) = \frac{(n+1)(n+2)}{2}.$$

This is equation (1.1) with n replaced by $n+1$. Thus we have *derived* proposition $p(n+1)$ out of proposition $p(n)$, and therefore, if $p(n)$ is true so must be $p(n+1)$.

The formula is proven for all $n \in \mathbb{N}$. ■

R It is important to realise the difference between the proposition

$p(n)$ is true for all $n \in \mathbb{N}$,

and the proposition

$p(n)$ is true.

The former is what we want to prove. The latter is valid for *that* particular n and no other, and is what we are *assuming* as the second induction step in order to actually prove $p(n+1)$.

Exercise 1.1 Prove by induction the formulas

$$1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6},$$
$$1^3+2^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}.$$

■

1.2.3 Integer numbers

The set of integer numbers is introduced to satisfy all the addition axioms of the definition of field. Without being rigorous, the set of integers numbers is $\mathbb{Z} = \{\dots,-3,-2,-1,0,1,2,3,\dots\}$. In other words, we complete the set $\mathbb{N}$ with 0 and with another copy of $\mathbb{N}$, which we label with a minus sign $(-\mathbb{N})$ and call *negative numbers*. Thus, $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup (-\mathbb{N})$.

Addition can be defined casewise:

1. Natural numbers follow the standard addition rule.
2. To add two negative numbers we add the numbers without sign and put a minus sign to the result.
3. To add $x \in \mathbb{N}$ and $y \in -\mathbb{N}$ we ignore signs, substract the smallest from the largest, and put the original sign of the largest to the result (with the caveat that 0 has no sign).

If you find these rules bizarre and lacking common sense, think of an alternative, more practical definition. Assume that $\mathbb{Z}$ is an infinite storey building, 0 being the ground floor, 1 the first floor, 2 the second floor, etc, and likewise -1 is the first basement (underneath the ground floor), -2 the second basement (underneath the first basement), etc. Now think of addition as the result of moving through floors in a lift. The first number of the addition will be the floor we start off from; the second number is the number of floors that the lift will move up (if sign is positive) or down (if sign is negative). Thus $3+(-2)=1$ because going down two floors from the 3rd floor we end up in the 1st one. Similarly, $2+(-3)=-1$ because going down three floors from the 2nd one we end up in the 1st basement. And $(-1)+(-2)=-3$ means that starting from the 1st basement and going down two floors we end up in the 3rd basement.

Multiplication of integers follows the rule of multiplication of natural numbers, ignoring signs, and then the sign of the result follows the sign rules

×/÷	+	−
+	+	−
−	−	+

Again it is not difficult to understand why we have to adopt these rules. We do not need to justify the $+\cdot+=+$ rule, because that is the standard multiplication in $\mathbb{N}$. As for $-\cdot+=-$ (or $+\cdot-=-$), consider the product $(-1)\cdot 3$. The standard way to interpret this multiplication is 'add -1 three times', and if we do that we obtain

$$(-1)\cdot 3 = (-1)+(-1)+(-1) = -3.$$

Now think of this sequence of operations:

$$\begin{aligned}
&(-1)\cdot 3 = -3,\\
&(-1)\cdot 2 = -2,\\
&(-1)\cdot 1 = -1,\\
&(-1)\cdot 0 = 0,\\
&(-1)\cdot(-1) = ?
\end{aligned}$$

What number should we put in place of the question mark? Well, if we continue the sequence we observe on the right-hand sides of these equations, logically it would be 1. Thus $(-1)\cdot(-1)=1$, and this justifies the rule $-\cdot-=+$.

It is not difficult to prove that all addition axioms are satified by $\mathbb{Z}$, so substraction is well defined. However, we have not added any new axiom to the multiplication, for $zx=1$ is still an equation without solution for most $z \in \mathbb{Z}$.

$\mathbb{Z}$ is also a totally ordered set, although it has lost the perfect order of $\mathbb{N}$ because it lacks a first element.

1.2.4 Rational numbers

Rational numbers are the response to the search for a solution of the equation $ax = 1$ for any number a, and the first ordered field. They are defined as $\mathbb{Q} = \{n/m : n, m \in \mathbb{Z}, m \neq 0\} = \mathbb{Z} \times (\mathbb{Z} - \{0\})$. The fraction n/m is the solution of the equation $mx = n$.

There are infinitely many fractions that can solve the equation $mx = n$ ($m \neq 0$), namely n/m, $2n/2m$, $(-n)/(-m)$, etc., or if $n = pk$ and $m = pl$, also k/l. Thus all these fractions are "equivalent" in the sense that they represent the same rational number. So rational numbers are actually equivalence classes of fractions, where two fractions n_1/m_1 and n_2/m_2 are said to be equivalent if $n_1m_2 = n_2m_1$.

Rational numbers can also be ordered. If $r_1, r_2 \in \mathbb{Q}$ and we take representative fractions of them $r_1 = n_1/m_1$ and $r_2 = n_1/m_2$ (where $m_1 > 0$ and $m_2 > 0$), then $r_1 \leqslant r_2$ if $n_1m_2 \leqslant n_2m_1$, and $r_1 > r_2$ otherwise. As it can be seen, the order is linear.

$\mathbb{Q}$ is dense in itself, meaning that there is a rational number between any two distinct rational numbers. Clearly, if $r_1 < r_2$ are two rational numbers, $r = (r_1 + r_2)/2$, also a rational number, is between them two.

We might impose a well-ordering on $\mathbb{Q}$ if we wanted to, by the following procedure: start with 0, 1, -1; then append all *irreducible* fractions p/q such that $-2 \leqslant p, q, \leqslant 2$; then append those such that $-3 \leqslant p, q, \leqslant 3$; and so on and so forth. The result is that $\mathbb{Q}$ can be explicitly displayed as

$$\mathbb{Q} = \left\{0, 1, -1, 2, -2, \frac{1}{2}, -\frac{1}{2}, 3, -3, \frac{3}{2}, -\frac{3}{2}, \frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}, 4, -4, \frac{4}{3}, -\frac{4}{3}, \frac{1}{4}, -\frac{1}{4}, \frac{3}{4}, -\frac{3}{4}, \dots\right\}.$$

A very important consequence of this fact is that there are as many numbers in $\mathbb{N}$ as in $\mathbb{Q}$ (because ordering $\mathbb{Q}$ as above means that a one-to-one correspondence can be established between both sets).

1.2.5 Real numbers

Despite the density of rational numbers, there are "holes" in between. For instance, the length of the diagonal of a square of unit side is $\sqrt{2}$, not a rational number. One can find rational numbers larger or smaller, and arbitrarily close to $\sqrt{2}$, but never a rational number that is exactly $\sqrt{2}$.

R The proof of this fact is an elegant example of *reductio ad absurdum*. Suppose $\sqrt{2}$ is rational and let n/m be its irreducible fraction (n and m have no common factors that can be simplified). Squaring the expression we obtain $2m^2 = n^2$, so n^2 is even, and therefore so is n (because if n were odd, n^2, being the product of two odd numbers, would be odd as well). Thus $n = 2k$. Substituting in this expression $2m^2 = 4k^2$, which we can simplify to $m^2 = 2k^2$. This implies that m^2 is even and therefore so is m. But that is impossible because m and n should not have any common factor.

So, from the assumption that $\sqrt{2}$ is rational we arrive at a contradiction, therefore the assumption is false and $\sqrt{2}$ is an *irrational* number.

Further insight on irrational numbers can be gained introducing the decimal representation of rational numbers. We can represent rationals as decimal expressions which contain an integer number, a decimal point, and a infinite sequence of digits. For example

$$\frac{1}{2} = 0.5\underline{0}0000\dots, \qquad \frac{1}{3} = 0.\underline{3}33333\dots, \qquad \frac{7}{6} = 1.1\underline{6}6666\dots, \qquad \frac{23}{13} = 1.\underline{769230}769230\dots$$

What all these expressions have in common is that the infinite sequence of digits on the right of the decimal point is eventually periodic. The period may be longer or shorter, but there is always a definite period. Irrational numbers are all sequences of *aperiodic* decimal expressions. For instance,

$$\sqrt{2} = 1.41421356237309504880168872420969807856967187537694807317\ldots$$

We could continue calculating decimals of this number but the sequence would never become periodic.

We can construct the following bracketing of any irrational number by sequences of pairs of rational numbers:

$$\begin{array}{rcl} 1.41 < & \sqrt{2} & < 1.42 \\ 1.414 < & \sqrt{2} & < 1.415 \\ 1.4142 < & \sqrt{2} & < 1.4143 \\ 1.41421 < & \sqrt{2} & < 1.41422 \\ \vdots & \vdots & \vdots \end{array}$$

This procedure allows us to have a mental representation of any irrational number as that number that would be bracketed by the two infinite sequences. This is how rational numbers can be "completed" by filling in the holes they leave. This is how *real numbers* are constructed.

Real numbers inherit all properties from their rational "approximants" and thus form an ordered field. But on top of that, the set they form, $\mathbb{R}$, has a significant difference when compared to $\mathbb{Q}$: it is complete. We will later come back to this important property of real numbers, but we want to emphasise here that the main implication of this property is that there is a one-to-one correspondence between real numbers and points in a straight line: there is a real number (and only one) to represent every scalar magnitude of any type. This is what makes real numbers so useful. It is also the reason why we often refer to $\mathbb{R}$ as *the real line.*

R A remarkable observation is that $\mathbb{R}$ is a set much bigger than $\mathbb{Q}$ —which, as we kown, is as "large" as $\mathbb{N}$ and no more. As a matter of fact, we will show that there are more real numbers between 0 and 1 than there are in $\mathbb{N}$. To prove it we also proceed through *reductio ad absurdum.* Let us us assume that we can order all those numbers in a sequence (as we did for $\mathbb{Q}$, for instance). To avoid being too abstract, just imagine that this is the beginning of the list:

$$\begin{aligned} r_1 &= 0.\underline{0}1004872657892653490023\ldots \\ r_2 &= 0.9\underline{8}2964800108264022289 46\ldots \\ r_3 &= 0.61\underline{1}5555100010298892220 0\ldots \\ r_4 &= 0.111\underline{1}1989887811110101010\ldots \\ &\vdots \end{aligned}$$

We have underlined the first decimal digit of the first number, the second decimal of the second numbers, and so on. We will construct a real number between 0 and 1 as follows: take for the first decimal a number different from the number underlined in r_1; for the second decimal a number different from the number underlined in r_2; and so on. For instance, $r = 0.2790\ldots$. This number is not in the list, because it is different —by construction— to every one of the elements of the list. However, it is a real number between 0 and 1 and therefore should be in the list!

The conclusion from this contradiction is that our assumption is false: real numbers cannot be listed. In other words, there is not a one-to-one correspondence between $\mathbb{R}$ and $\mathbb{N}$ because there are more real numbers than natural numbers. Real numbers *cannot be counted.* Clearly this means that irrational numbers are the ones that cannot be counted, so there are many more irrational numbers that rational numbers!

1.3 Absolute value, bounds, and intervals

In order to quantify how big or small are real numbers we have to get rid of their sign. This leads to introduce their *absolute value* or *magnitude,* which we denote $|x|$. In particular, $|x-y|$ quantifies the difference between x and y, and therefore their *distance* in the real line.

Definition 1.3.1 — Absolute value. For any $x \in \mathbb{R}$ we define its *absolute value* as

$$|x| = \max\{x, -x\} = \begin{cases} x & \text{if } x \geqslant 0, \\ -x & \text{if } x < 0. \end{cases}$$

The absolute value has some important properties:

Proposition 1.3.1 For all $x, y \in \mathbb{R}$,

1. $|x| \geqslant 0$.
2. $|x| = 0$ if and only if $x = 0$.
3. $|xy| = |x||y|$.
4. $|x+y| \leqslant |x| + |y|$.
5. $\big||x| - |y|\big| \leqslant |x-y|$.

All of them are easy to check except 5. But this one follows from 4. To prove it we observe that

$$|x| = |x-y+y| \leqslant |x-y| + |y| \quad \Rightarrow \quad |x| - |y| \leqslant |x-y|.$$

But on the other hand,

$$|y| = |y-x+x| \leqslant |y-x| + |x| \quad \Rightarrow \quad |y| - |x| \leqslant |y-x| = |x-y|.$$

If both numbers $|x| - |y|$ and $|y| - |x| = -(|x| - |y|)$ are not larger than $|x-y|$, then $\big||x| - |y|\big| \leqslant |x-y|$.

Definition 1.3.2 — Bounds. Let A be a nonempty subset of $\mathbb{R}$, and let $x \in \mathbb{R}$.

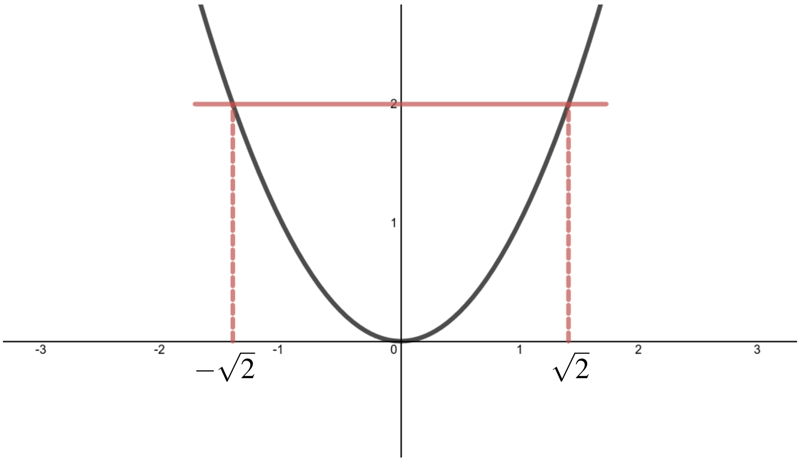
1. x is an **upper bound** of A if $x \geqslant a$ for all $a \in A$ (A is then **bounded from above**).
2. x is an **lower bound** of A if $x \leqslant a$ for all $a \in A$ (A is then **bounded from below**).
3. A is **bounded** if it is bounded from above and from below.
4. x is the **supremum** of A ($x = \sup A$) if it is its *least upper bound* ($\sup A = \infty$ if A is not bounded from above).
5. x is the **infimum** of A ($x = \inf A$) if it is its *greatest lower bound* ($\inf A = -\infty$ if A is not bounded from below).
6. If $\sup A \in A$ it is called **maximum** ($\max A$).
7. If $\inf A \in A$ it is called **minimum** ($\min A$).

We have not properly defined real numbers. In particular, we have not provided a rigorous statement of the property of *completeness* that characterises $\mathbb{R}$ ($\mathbb{R}$ is the only complete ordered field). There are many ways of introducing this axiom, and one in particular relies on the idea of bracketing intervals intuitively introduced in the previous section. There is an important result that turns out to be equivalent to the completeness axiom, and which calculus uses profusely.

Theorem 1.3.2 — Completeness theorem. Let $A \subset \mathbb{R}$ be nonempty ($A \neq \varnothing$). The following properties hold:

(i) Supremum property: If A is bounded from above then it has a supremum.
(ii) Infimum property: If A is bounded from below then it has an infimum.

■ **Example 1.5** Consider the set $A = \{x \in \mathbb{R} : x^2 < 2\}$. This set is the portion of the real line for which the parabola $y = x^2$ lies strictly below the horizontal line $y = 2$, as illustrated by this picture:



From the figure it is clear that $A = \{x \in \mathbb{R} : -\sqrt{2} < x < \sqrt{2}\}$, because $x = \pm\sqrt{2}$ are the two points where $y = x^2$ meets $y = 2$. So $\sqrt{2}$ is an upper bound, and there can be no smaller bound, so $\sup A = \sqrt{2}$. Likewise, $\inf A = -\sqrt{2}$. (They are not maximum and minimum because they do not belong to A.)

Notice that if we consider instead the subset of $\mathbb{Q}$ defined as $A' = \{x \in \mathbb{Q} : x^2 < 2\}$, even though the definition looks similar to that of A, this set A' has neither a supremum nor an infimum in $\mathbb{Q}$. Any upper bound $r \in \mathbb{Q}$ of A' must necessarily be $r > \sqrt{2}$, and we can always find another $r' \in \mathbb{Q}$ such that $\sqrt{2} < r' < r$. Therefore no least upper bound can exist. ■

Definition 1.3.3 — Intervals. The following subsets of $\mathbb{R}$ are globally referred to as intervals:
Open interval: $(a,b) = \{x \in \mathbb{R} : a < x < b\}$.
Closed interval: $[a,b] = \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$.
Semiopen intervals:

$(a,b] = \{x \in \mathbb{R} : a < x \leqslant b\}$,
$[a,b) = \{x \in \mathbb{R} : a \leqslant x < b\}$.

Infinite intervals:

$(-\infty,b] = \{x \in \mathbb{R} : x \leqslant b\}$,
$(-\infty,b) = \{x \in \mathbb{R} : x < b\}$,
$[a,\infty) = \{x \in \mathbb{R} : a \leqslant x\}$,
$(a,\infty) = \{x \in \mathbb{R} : a < x\}$,
$(-\infty,\infty) = \mathbb{R}$.

■ **Example 1.6** Consider the set $A = \{x \in \mathbb{R} : |x| \leqslant 3\}$. Let us discuss which real numbers belong to A. To do that we need to distinguish the cases (a) $x \geqslant 0$ and (b) $x < 0$.

(a) If $x \geqslant 0$ then condition $|x| \leqslant 3$ reads $x \leqslant 3$. Thus the interval $[0,3]$ is part of A.
(b) If $x < 0$ the condition $|x| \leqslant 3$ reads $-x \leqslant 3$, or equivalently, $x \geqslant -3$. Therefore $[-3,0)$ is also part of A.

Summarising, $A = [-3,0) \cup [0,3] = [-3,3]$. ■

■ **Example 1.7** Consider the set $B = \{x \in \mathbb{R} : (x-1)(x-2)(x-3) < 0\}$. The condition that defines B depends on the sign of the product three factors, so we need to know the sign of each of them. These signs depend on whether (a) $x < 1$, (b) $1 < x < 2$, (c) $2 < x < 3$, or (d) $3 < x$. In cases (a) and (c) there is an odd number of negative factors, so the product is negative, whereas in cases (b) and (d) there is an even number of them, so the product is positive. Thus $B = (-\infty,1) \cup (2,3)$. ■

Problems

Problem 1.1 Given the real numbers $0 < a < b$ and $c > 0$, prove the inequalities

$$\text{(a)}\ \ a < \sqrt{ab} < \frac{a+b}{2} < b, \qquad \text{(b)}\ \ \frac{a}{b} < \frac{a+c}{b+c}.$$

Problem 1.2 Prove that $|a+b| = |a| + |b|$ if and only if $ab \geqslant 0$.

Problem 1.3 Prove that

$$\text{(a)}\ \ \max\{x,y\} = \frac{x+y+|x-y|}{2}, \qquad \text{(b)}\ \ \min\{x,y\} = \frac{x+y-|x-y|}{2}.$$

Problem 1.4 Find, using the absolute value, a formula to express the function

$$\varphi(x) = \begin{cases} x & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Problem 1.5 Factor out the following expressions of $n \in \mathbb{N}$, so that the corresponding statements become self-evident:

(a) $n^2 - n$ is even,
(b) $n^3 - n$ is a multiple of 6,
(c) $n^2 - 1$ is a multiple of 8 when n is odd.

Problem 1.6 Prove by induction the following statements valid for all $n \in \mathbb{N}$:

(a) $a^n - b^n = (a-b)\displaystyle\sum_{k=1}^{n} a^{n-k}b^{k-1}$ for all $n \in \mathbb{N}$,

(b) $n^5 - n$ is a multiple of 5 for all $n \in \mathbb{N}$,

(c) $(1+x)^n \geqslant 1 + nx$ if $x \geqslant -1$.

Problem 1.7 Prove by induction the following statements valid for all natural numbers $n > 1$:

(a) $n! < \left(\dfrac{n+1}{2}\right)^n$,

(b) $2!4!\cdots(2n)! > \big[(n+1)!\big]^n$,

(c) $1 + \dfrac{1}{\sqrt{2}} + \cdots + \dfrac{1}{\sqrt{n}} > \sqrt{n}$.

HINT: In (a) use the inequality $\left(1 + \dfrac{1}{n+1}\right)^{n+1} > 2$, valid for all $n \in \mathbb{N}$. In (b) prove first that $(2n+2)! > (n+2)^n(n+2)!$.

Problem 1.8

(a) Show, with an example, that the sum of two irrational numbers can be rational.
(b) Show, with an example, that the product of two irrational numbers can be rational.
(c) Is it possible to find irrational numbers x and y such that $x^y \in \mathbb{Q}$?

Problem 1.9 Prove that

(a) $\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$,
(b) $\sqrt{n} \notin \mathbb{Q}$ if n is not a perfect square (HINT: write $n = k^2 r$, where r does not contain any square factor),
(c) $\sqrt{n-1} + \sqrt{n+1} \notin \mathbb{Q}$ for all $n \in \mathbb{N}$.

Problem 1.10 Prove the identity, valid for all $x \in \mathbb{R}$,

$$\left(\frac{x+|x|}{2}\right)^2 + \left(\frac{x-|x|}{2}\right)^2 = x^2.$$

Problem 1.11 Identify the following sets:

(i) $A = \{x \in \mathbb{R} : |x-3| \leqslant 8\}$,
(ii) $B = \{x \in \mathbb{R} : 0 < |x-2| < 1/2\}$,
(iii) $C = \{x \in \mathbb{R} : x^2 - 5x + 6 \geqslant 0\}$,
(iv) $D = \{x \in \mathbb{R} : x^3(x+3)(x-5) < 0\}$,
(v) $E = \left\{x \in \mathbb{R} : \dfrac{2x+8}{x^2+8x+7} > 0\right\}$,
(vi) $F = \left\{x \in \mathbb{R} : \dfrac{4}{x} < x\right\}$,
(vii) $G = \{x \in \mathbb{R} : 4x < 2x+1 \leqslant 3x+2\}$,
(viii) $H = \{x \in \mathbb{R} : |x^2 - 2x| < 1\}$,
(ix) $I = \{x \in \mathbb{R} : |x-1||x+2| = 10\}$,
(x) $J = \{x \in \mathbb{R} : |x-1| + |x+2| > 1\}$.

Problem 1.12 Given real numbers $a < b$ we define, for each $t \in \mathbb{R}$, the real number $x(t) = (1-t)a + tb$. Identify the following sets:

(i) $A = \{x(t) : t = 0, 1, 1/2\}$,
(ii) $B = \{x(t) : t \in (0,1)\}$,
(iii) $C = \{x(t) : t < 0\}$,
(iv) $D = \{x(t) : t > 1\}$.

Problem 1.13 Find supremum and infimum (deciding whether they are maximum and minimum respectively) of the following sets:

(i) $A = \{-1\} \cup [2,3)$,
(ii) $B = \{3\} \cup \{2\} \cup \{-1\} \cup [0,1]$,
(iii) $C = \{2 + 1/n : n \in \mathbb{N}\}$,
(iv) $D = \{(n^2+1)/n : n \in \mathbb{N}\}$,
(v) $E = \{x \in \mathbb{R} : 3x^2 - 10x + 3 < 0\}$,
(vi) $F = \{x \in \mathbb{R} : (x-a)(x-b)(x-c)(x-d) < 0\}$, with $a < b < c < d$ given real numbers,
(vii) $G = \{2^{-p} + 5^{-q} : p, q \in \mathbb{N}\}$,
(viii) $H = \{(-1)^n + 1/m : n, m \in \mathbb{N}\}$,

2. Real Functions

2.1 Definition and basic concepts

Formally, a **real function** is a map from a set $A \subset \mathbb{R}$ to $\mathbb{R}$. In practical terms, it is a "rule" that "assigns" one —and only one!— real number to each element $x \in A$. A basic notation for a function is

$$\begin{aligned} f : A &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = f(x) \end{aligned} \tag{2.1}$$

Functions are also referred to as *maps* or *mappings*. They are usually denoted $y = f(x)$, where f represents the rule that assigns y to x.

■ **Example 2.1**

(a) $y = x^2$ represents the rule $f(x) = x^2$ that maps each number x to its square.
(b) $y = |x|$ represents the rule $f(x) = |x|$ that maps each number x to its absolute value.
(c) The function

$$f(x) = \begin{cases} x^2 & x \leqslant 2, \\ x^3 - 3 & x > 2, \end{cases}$$

maps all real numbers smaller than or equal to 2 to their square, and those larger than 2 to their cube minus 3.
(d) The function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}, \end{cases}$$

maps all rational numbers to 1 and all irrational numbers to 0.

■

The **domain** of a function is the set A. This domain is maximal if the function cannot be defined for numbers $x \notin A$.

The **image** or **range** of a function is the set $f(A) \equiv \{f(x) : x \in A\}$.

Likewise, we call *image of the set* $C \subset A$ to the set $f(C) \equiv \{f(x) : x \in C\}$.

We call *inverse image of a set* $B \subset \mathbb{R}$ to the set $f^{-1}(B) \equiv \{x \in A : f(x) \in B\}$. Note that $f^{-1}(B) \subset A$.

The **graph** of a function $f(x)$ is the subset of $\mathbb{R}^2$ defined by the points $\{(x, f(x)) : x \in A\}$. Plotting this set is how we represent functions.

A function is **injective,** or **one-to-one,** if for every pair of number $x_1 \neq x_2$ we have $f(x_1) \neq f(x_2)$. If a function is injective, the equation $y = f(x)$ has either no solution or a unique solution.

A function is **surjective,** or **onto,** if $f(A) = \mathbb{R}$. If a function is surjective, the equation $y = f(x)$ always has at least one solution.

A function is **bijective** if it is both injective and surjective. If a function is bijective, the equation $y = f(x)$ always has one, and only one, solution for each $y \in \mathbb{R}$.

A function is **periodic** if there exists $c > 0$ such that $f(x+c) = f(x)$. The smallest such c is referred to as the *period* of the function.

A function is **even** if $f(-x) = f(x)$, and **odd** if $f(-x) = -f(x)$.

A function is **bounded** if there exists $M > 0$ such that $|f(x)| \leqslant M$ for all x in its domain.

A function is **monotonically increasing** if for every x, y in its domain such that $x < y$ it satisfies $f(x) \leqslant f(y)$, and is **monotonically decreasing** if $f(x) \geqslant f(y)$. We say it is **monotonic strictly increasing/decreasing** if inequalities are strict. (Note that a constant is both monotonically increasing and decreasing, but not strictly.)

■ **Example 2.2**

(a) The domain of $f(x) = x^2$ is $\mathbb{R}$ and its image is $f(\mathbb{R}) = [0, \infty)$. This function is not injective because x and $-x$ have the same square. It is not surjective either because $f(\mathbb{R}) \neq \mathbb{R}$. The inverse image of the interval $[4,9]$ is $f^{-1}([4,9]) = [-3,-2] \cup [2,3]$.

(b) The domain of $f(x) = \log x$ is $(0, \infty)$ and its image is $\mathbb{R}$. It is injective because two different numbers have different logarithms. It is also surjective because any number y is always the logarithm of a number, namely e^y. So it is bijective.

(c) $F(x) = e^x - e^{-x}$ is an odd function because $f(-x) = e^{-x} - e^x = -f(x)$.

(d) $f(x) = \cos x$ is even because $\cos(-x) = \cos x$.

(e) $f(x) = \sin^2 x$ is periodic of period π because $\sin^2(x+\pi) = \sin^2 x$.

■

2.2 Elementary functions

There is a wide range of elementary functions that we will work with. They include polynomials, rational functions, trigonometric functions, the exponential and the logarithm.

2.2.1 Polynomials

These are functions of the form

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \tag{2.2}$$

where $a_k \in \mathbb{R}$ for all $k = 0, 1, \ldots, n$. The largest power, n, is called the *degree* of the polynomial. Constants are polynomials of degree 0. Given the operations that define them, the domain of any polynomial is $\mathbb{R}$.

2.2.2 Rational functions

They are defined as quotients of two polynomials, namely

$$f(x) = \frac{P_n(x)}{Q_m(x)}. \tag{2.3}$$

The domain of both polynomials is $\mathbb{R}$, but $Q_m(x)$ may be zero at some points, where the quotient will thus not be defined. Hence the domain of $f(x)$ is $\{x \in \mathbb{R} : Q_m(x) \neq 0\}$.

2.2.3 Trigonometric functions

The two basic trigonometric functions are the sine ($\sin x$) and the cosine ($\cos x$). In terms of them we can define also the tangent and cotangent:

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}. \tag{2.4}$$

The geometric definition of these functions, based on the unit circle, is described in Figure 2.1.

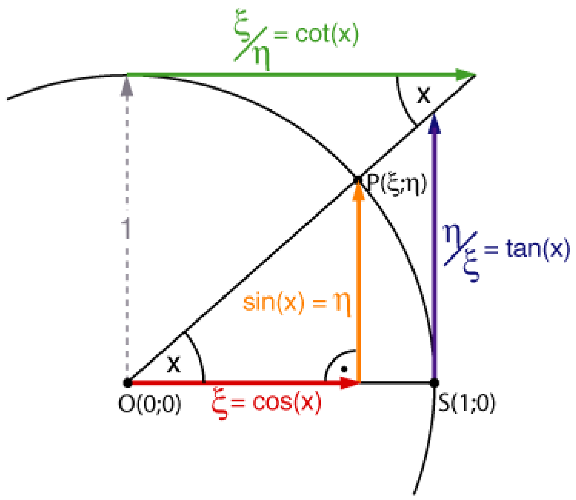


Figure 2.1: Geometric definition of $\sin x$, $\cos x$, $\tan x$, and $\cot x$.

There are two more trigonometric functions, although less common than the previous one, namely the secant ($\sec x$) and the cosecant ($\csc x$):

$$\sec x = \frac{1}{\cos x}, \qquad \csc x = \frac{1}{\sin x}. \tag{2.5}$$

The graphs of $\sin x$ and $\cos x$ are plotted in Figure 2.2. Those of $\tan x$ and $\cot x$ in Figure 2.3.

Trigonometric identities	
$\cos^2 x + \sin^2 x = 1$	$1 + \cot^2 x = \csc^2 x$
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$	$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$
$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$	$\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$
$\cos 2x = \cos^2 x - \sin^2 x$	$\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$
$\sin 2x = 2\sin x \cos x$	$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$
$1 + \tan^2 x = \sec^2 x$	$\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$

Table 2.1: Some important trigonometric identities.

Given their geometric definitions, all these functions are related by geometric identities. The main one are listed in Table 2.1.

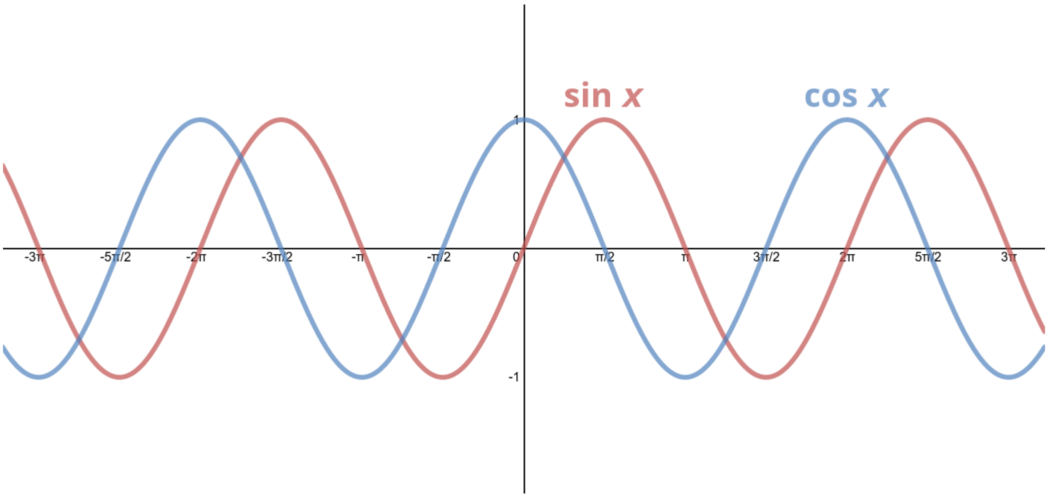


Figure 2.2: Plot of $\sin x$ and $\cos x$.

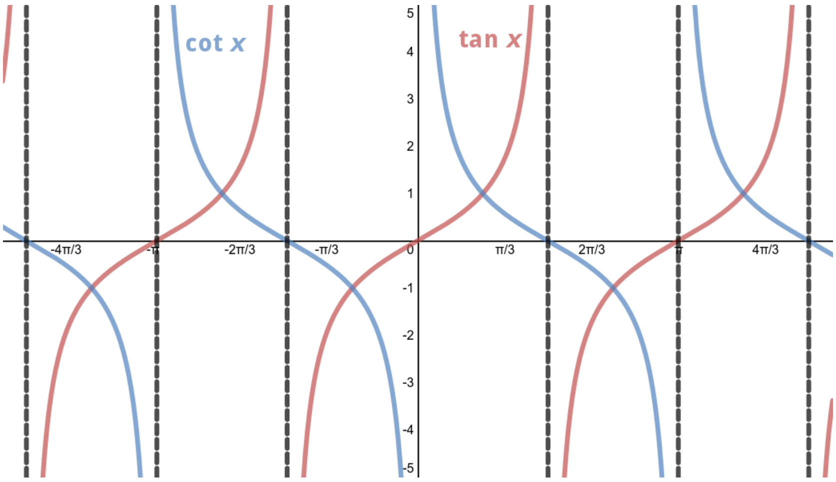


Figure 2.3: Plot of $\tan x$ and $\cot x$.

2.2.4 Exponential

This is the function defined as $f(x) = e^x$. The constant e appearing in this definition is the irrational number introduced by Euler

$$e = 2.71828182845904523536028747135266249775724709369995957496 6\ldots$$

We will see a proper definition of this constant later on. Apart from that, the definition of the exponential involves raising a real number to a real power. This requires some clarifications.

Integer powers of real numbers are easily defined through the concept of repeated product. Thus $e^3 = e \cdot e \cdot e$. With this definition, for any $n, m \in \mathbb{N}$ it is straightforward that

$$e^{n+m} = e^n e^m, \tag{2.6}$$

from which it follows

$$(e^m)^n = \underbrace{e^m \cdot e^m \cdots e^m}_{n \text{ times}} = e^{m+m+\cdots+m} = e^{nm}. \tag{2.7}$$

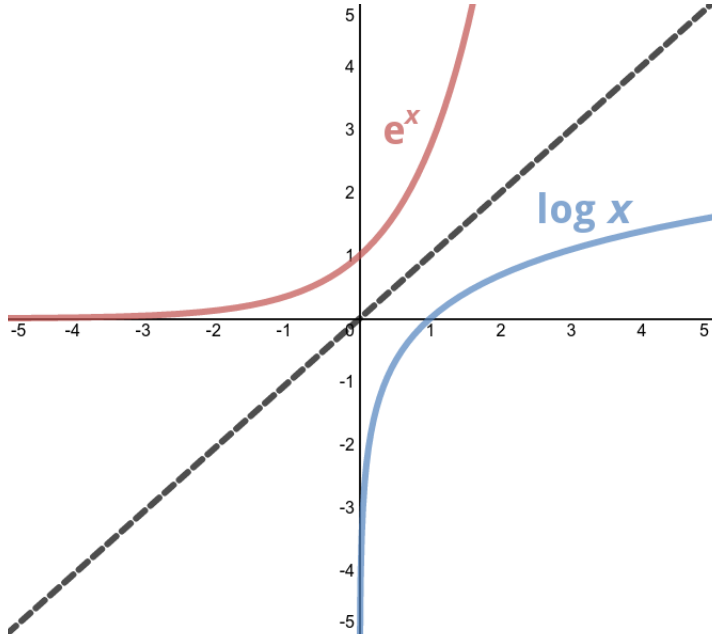


Figure 2.4: Plot of e^x and $\log x$.

We will take these formulas as a basic definition. Extending them will provide meaning to powers other than natural numbers. For instance, applying (2.6),

$$e^{n-m}e^m = e^{n-m+m} = e^n \quad \Rightarrow \quad e^{n-m} = \frac{e^n}{e^m}.$$

But extending (2.6) means assuming $e^{n-m} = e^n e^{-m}$. Cancelling a factor e^n in both sides leads to

$$e^{-m} = \frac{1}{e^m},$$

which provides a meaning to negative powers. And from this definition it follows

$$e^0 = e^{n-n} = \frac{e^n}{e^n} = 1.$$

As for fractional powers, equation (2.7) implies

$$(e^{1/n})^n = e^{n/n} = e \quad \Rightarrow \quad e^{1/n} = \sqrt[n]{e}.$$

Thus, $e^{m/n} = \sqrt[n]{e^m}$. This extension of the basic multiplicative rule provides a definition of the exponential valid for all rational powers. It only remains to define it for irrational powers. But irrational numbers can be approximated as much as we like by rational numbers. If fact, as we have seen, irrational numbers can be bracketed by sequences of rational approximants; i.e., if x is an irrational number, we can find two sequences of rational numbers such that

$$p_1 < p_2 < p_3 < \cdots < p_n < \cdots < x < \cdots < q_n < \cdots < q_3 < q_2 < q_1.$$

Thus we can define e^x as the number bracketed by

$$e^{p_1} < e^{p_2} < e^{p_3} < \cdots < e^{p_n} < \cdots < e^x < \cdots < e^{q_n} < \cdots < e^{p_3} < e^{p_2} < e^{p_1}.$$

Using this definition we can summarise the properties of the exponential as follows:

1. Its domain is $\mathbb{R}$.
2. $e^x > 0$ for all $x \in \mathbb{R}$.
3. It is monotonic strictly increasing —hence injective.
4. $e^0 = 1$.
5. $(e^x)^a = e^{ax}$ for any $a \in \mathbb{R}$.
6. $e^{x+y} = e^x e^y$.
7. $e^{-x} = 1/e^x$.

A plot of the exponential function is shown in Figure 2.4.

Some important functions defined in terms of exponentials define what is known as the *hyperbolic trigonometry*. The main one are the hyperbolic cosine ($\cosh x$) and sine ($\sinh x$), defined as

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}. \tag{2.8}$$

Their plots are shown in Figure 2.5. It can be seen that $\cosh x$ is even whereas $\sinh x$ is odd.

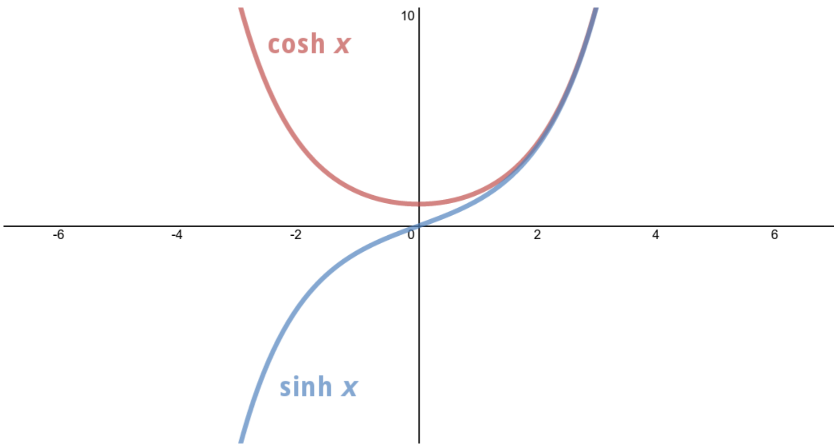


Figure 2.5: Plot of $\cosh x$ and $\sinh x$.

Hyperbolic tangent ($\tanh x$) and cotangent ($\coth x$) can also be defined (see Figure 2.6 for their plots):

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{1}{\tanh x}, \tag{2.9}$$

and similarly $\operatorname{sech} x = 1/\cosh x$ and $\operatorname{csch} x = 1/\sinh x$.

There is a list of identities relating these functions similar to that of the ordinary trigonometry, as illustrated in Table 2.2.

2.2.5 Logarithm

This is the inverse of the exponential. If $y = \log x$ it means that $x = e^y$. Its plot can be seen in Figure 2.4 to mirror that of the exponential with respect to the line $y = x$.

R Along these notes, whenever we write $x = \log y$ we mean that x is the solution of the equation $e^x = y$, in other words, log of a number is the exponent to which we need to rise e in order to obtain that number. In particular $\log 1 = 0$ and $\log e = 1$.

The main properties of the logarithm (derived from those of the exponential) are the following:

1. Its domain is $(0, \infty)$.

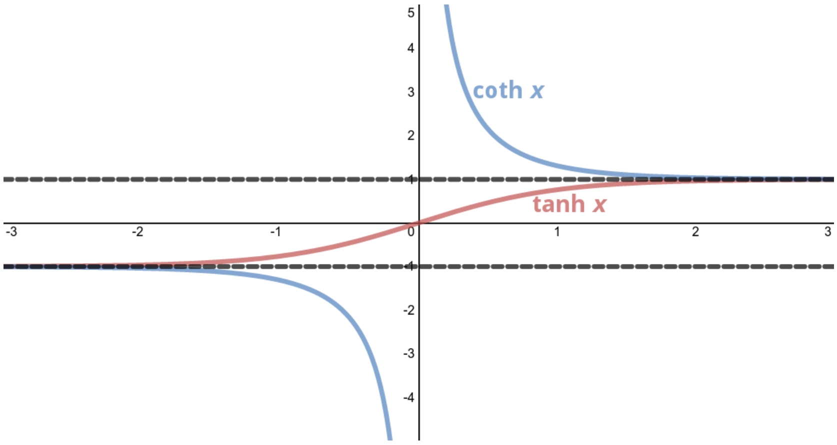


Figure 2.6: Plot of $\tanh x$ and $\coth x$.

Hyperbolic trigonometric identities	
$\cosh^2 x - \sinh^2 x = 1$	$\coth^2 x - 1 = \operatorname{csch}^2 x$
$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$	$\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$	$\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$
$\cosh 2x = \cosh^2 x + \sinh^2 x$	$\cosh x \cosh y = \frac{1}{2}[\cosh(x+y) + \cosh(x-y)]$
$\sinh 2x = 2\sinh x \cosh x$	$\sinh x \sinh y = \frac{1}{2}[\cosh(x+y) - \cosh(x-y)]$
$1 - \tanh^2 x = \operatorname{sech}^2 x$	$\sinh x \cosh y = \frac{1}{2}[\sinh(x+y) + \sinh(x-y)]$

Table 2.2: Some important trigonometric identities.

2. Its image is $\mathbb{R}$ —hence it is surjective.
3. It is monotonic strictly increasing —hence injective.
4. $\log 1 = 0$.
5. $\log(x^a) = a \log x$.
6. $\log(xy) = \log x + \log y$.
7. $\log(x/y) = \log x - \log y$.

2.3 Operations with functions

2.3.1 Algebraic operations

Let $A, B \subset \mathbb{R}$ and consider the two real functions

$$\begin{aligned} f : A &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = f(x) \end{aligned} \qquad \begin{aligned} g : B &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = g(x) \end{aligned} \tag{2.10}$$

With these two functions we can perform the following algebraic operations:

(i) **Addition:** If $C = A \cap B$ —where both functions are defined—,

$$\begin{aligned} f+g : C &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = f(x) + g(x) \end{aligned} \tag{2.11}$$

(ii) **Product:** If $C = A \cap B$,

$$\begin{aligned} fg : C &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = f(x)g(x) \end{aligned} \tag{2.12}$$

(iii) **Quotient:** If $C = A \cap B'$, where $B' \equiv \{x \in B \,:\, g(x) \neq 0\}$,

$$\begin{aligned} f/g : C &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = f(x)/g(x) \end{aligned} \tag{2.13}$$

For example, a polynomial is actually a sum of monomials, each a different function; or a rational function is the quotient of two polynomials.

2.3.2 Compositions

A more involved operation is the composition of two functions. It is defined as

$$\begin{aligned} f \circ g : C &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = f\big(g(x)\big) \end{aligned} \tag{2.14}$$

The problem is to find the domain of this function, given the domains A and B of the composed functions. For $f \circ g$ to be defined x must belong to B, for $g(x)$ to be well defined, so $C \subset B$. But in order to evaluate $f\big(g(x)\big)$, the number $g(x) \in A$. Therefore

$$C = \{x \in B \,:\, g(x) \in A\}. \tag{2.15}$$

Even if A and B are simple sets, C may be much more involved.

■ **Example 2.3** Consider the functions $f(x) = 1/(x-1)$ and $g(x) = \sin x$. Clearly $A = \mathbb{R} - \{1\}$ and $B = \mathbb{R}$, two very simple sets. However, the domain of their composition $f \circ g$ is the domain of the function

$$(f \circ g)(x) = \frac{1}{\sin x - 1},$$

i.e., $\mathbb{R}$ excluding those values for which $\sin x = 1$ (because the denominator vanishes). This set is

$$C = \mathbb{R} - \left\{ \left(2n + \frac{1}{2}\right)\pi \,:\, n \in \mathbb{Z} \right\}.$$

■

Composition is a noncommutative operation, i.e., $f \circ g \neq g \circ f$. In the example above, $(f \circ g)(x) = 1/(\sin x - 1)$ is very different from $(g \circ f)(x) = \sin\left(\frac{1}{x-1}\right)$.

It is, however, associative, i.e., $f \circ (g \circ h) = (f \circ g) \circ h$. We can thus define multiple compositions, like $f \circ g \circ h \circ w = f(g(h(w(x))))$, without ambiguity.

2.3.3 Inverses

We can introduce the identity function $\mathrm{Id}(x) = x$. Given a function $f : A \longrightarrow \mathbb{R}$, its **inverse** would be a function $f^{-1} : f(A) \longrightarrow \mathbb{R}$ such that $f \circ f^{-1} = f^{-1} \circ f = \mathrm{Id}$. The idea is that if f maps x to y, its inverse f^{-1} maps y back to x.

Not all functions have an inverse that is defined all over their image $f(A)$. For an inverse to exist the equation $x = f(y)$, for a given $x \in f(A)$, must have a unique solution;[1] in other words, f

[1] It already has at least one solution because $x \in f(A)$.

must be injective. Monotonic strictly increasing or decreasing functions are injective. This is why the exponential has an inverse —the logarithm.

For those functions that are not injective in their domain A, we might be able to define several inverses by constraining the domain to any subset where they are made injective. Thus, noninjective functions may have several inverses.

Example 2.4 Let $f(x) = x^2$. Its domain is $\mathbb{R}$, but this function is not injective in its domain. However, we can constraint the domain to be $[0,\infty)$. In that case $f(x)$ is injective and we can obtain the inverse function by finding the unique solution of the equation $x = f(y) = y^2$, where $0 \leqslant y$. Clearly this solution is $y = \sqrt{x}$, therefore, within $[0,\infty)$, the inverse of f is $f^{-1}(x) = \sqrt{x}$.

Note that we might alternatively chosen the domain to be $(-\infty,0]$, where the function f is again injective. However now the solution of $x = y^2$ with $y \leqslant 0$ is $y = -\sqrt{x}$. So another inverse of f is $f^{-1}(x) = -\sqrt{x}$. ■

Example 2.5 Periodic functions are clearly not injective. Take $\sin x$, for instance. An interval where it is injective is $[-\pi/2,\pi/2]$. The inverse of this function within this interval is usually called the *arc sine:* $\sin^{-1} x = \arcsin x$. But we might have taken the interval $[\pi/2,3\pi/2]$, for instance. In that case the inverse would be different: $\sin^{-1} x = \pi - \arcsin x$. Or in the interval $[3\pi/2,5\pi/2]$ the inverse would be $\sin^{-1} x = 2\pi + \arcsin x$.

Similarly, $\arccos x = \cos^{-1} x$ when the domain of $\cos x$ is taken to be $[0,\pi]$, and $\arctan x = \tan^{-1} x$ when the domain of $\tan x$ is taken to be $(-\pi/2,\pi/2)$.

Note that $\arccos x$ (or $\operatorname{arccot} x$ for that matter) is redundant, because

$$\arccos x = \frac{\pi}{2} - \arcsin x, \qquad \operatorname{arccot} x = \frac{\pi}{2} - \arctan x.$$

■

The graph of $f^{-1}(x)$ can be obtained from that of $f(x)$ as the mirror image with respect to the line $y = x$ (see Figure 2.4).

R BEWARE!! Never confuse $f^{-1}(x)$ with $f(x)^{-1} = 1/f(x)$. In the case $f(x) = \sin x$, is inverse $\sin^{-1} x = \arcsin x$, whereas $(\sin x)^{-1} = \csc x$.

Exercise 2.1 Argue that $\sinh x$ has a unique inverse over $\mathbb{R}$ and that it can be obtained as

$$\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right). \tag{2.16}$$

The function $\cosh x$ has two inverses (why?) that can be obtained as

$$\cosh^{-1} x = \pm\log\left(x + \sqrt{x^2-1}\right). \tag{2.17}$$

HINT: For $\sinh x$, solve $2x = e^y - e^{-y}$ by transforming it into $e^{2y} - 2xe^y - 1 = 0$ and reading it as a quadratic equation in e^y. Use a similar procedure for $\cosh x$.

Find an expression for $\tanh^{-1} x$. ■

Problems

Problem 2.1 Determine the domain of the following functions:

(i) $f(x) = \dfrac{1}{x^2-5x+6}$;

(ii) $f(x) = \sqrt{1-x^2}+\sqrt{x^2-1}$;

(iii) $f(x) = \dfrac{1}{x-\sqrt{1-x^2}}$;

(iv) $f(x) = \sqrt{1-\sqrt{4-x^2}}$;

(v) $f(x) = \dfrac{1}{1-\log x}$;

(vi) $f(x) = \log(x-x^2)$;

(vii) $f(x) = \dfrac{\sqrt{5-x}}{\log x}$;

(viii) $f(x) = \arcsin(\log x)$.

Problem 2.2

(a) If f and g are both odd functions, what are $f+g$, fg, and $f\circ g$?

(b) And what are the same functions if now f is even and g is odd?

Problem 2.3 Check whether the following functions are even or odd:

(i) $f(x) = \dfrac{x}{x^2+1}$;

(ii) $f(x) = \dfrac{x^2-x}{x^2+1}$;

(iii) $f(x) = \dfrac{\sin x}{x}$;

(iv) $f(x) = \cos(x^3)\sin(x^2)e^{-x^4}$;

(v) $f(x) = \dfrac{1}{\sqrt{x^2+1}-x}$;

(vi) $f(x) = \log\left(\sqrt{x^2+1}-x\right)$.

Problem 2.4 For which numbers $a,b,c,d\in\mathbb{R}$ the function $f(x) = \dfrac{ax+b}{cx+d}$ is its own inverse (i.e., $f\circ f = \mathrm{Id}$) in the domain of f?

Problem 2.5 Check that the function $f(x) = \dfrac{x+3}{1+2x}$ is bijective and maps its domain $\mathbb{R}-\{-1/2\}$ to $\mathbb{R}-\{1/2\}$.

Problem 2.6

(a) Determine which of these functions are injective. For those that are obtain their inverse. For those that are not, find two points with the same image.

(i) $f(x) = 7x-4$;

(ii) $f(x) = \sin(7x-4)$;

(iii) $f(x) = (x+1)^3+2$;

(iv) $f(x) = \dfrac{x+2}{x+1}$;

(v) $f(x) = x^2-3x+2$;

(vi) $f(x) = \dfrac{x}{x^2+1}$;

(vii) $f(x) = e^{-x}$;

(viii) $f(x) = \log(x+1)$.

(b) Prove that $f(x) = x^2-3x+2$ is injective in $(3/2,\infty)$.

(c) Prove that $f(x) = \dfrac{x}{x^2+1}$ is injective in $(1,\infty)$ and find $f^{-1}(\sqrt{2}/3)$.

(d) Determine if those same functions are surjective and bijective in their domains.

Problem 2.7 Calculate:

(i) $\arctan\frac{1}{2}+\arctan\frac{1}{3}$;   (ii) $\arctan 2+\arctan 3$;   (iii) $\arctan\frac{1}{2}+\arctan\frac{1}{5}+\arctan\frac{1}{8}$.

HINT: Calculate the tangent of those expressions using the formula for the tangent of the sum and paying attention to the signs.

Problem 2.8 Simplify the following expressions:

(i) $f(x) = \sin(\arccos x)$;
(ii) $f(x) = \sin(2\arcsin x)$;
(iii) $f(x) = \tan(\arccos x)$;
(iv) $f(x) = \sin(2\arctan x)$;
(v) $f(x) = \cos(2\arctan x)$;
(vi) $f(x) = e^{4\log x}$.

Problem 2.9 Solve, for $x, y > 0$, the system of equations

$$\begin{cases} x^y = y^x, \\ y = 3x. \end{cases}$$

Problem 2.10

(a) Describe the function g in terms of f in the following cases ($c \in \mathbb{R}$ is a constant):

(i) $g(x) = f(x) + c$;
(ii) $g(x) = f(x + c)$;
(iii) $g(x) = f(cx)$;
(iv) $g(x) = f(1/x)$;
(v) $g(x) = f(|x|)$;
(vi) $g(x) = |f(x)|$;
(vii) $g(x) = 1/f(x)$;
(viii) $g(x) = [f(x)]_+ \equiv \max\{f(x), 0\}$.

(b) Plot the functions when $f(x) = x^2$.
(c) Plot the functions when $f(x) = \sin x$.

Problem 2.11 Sketch, using the fewest possible calculations, the graph of the following functions:

(i) $f(x) = (x+2)^2 - 1$;
(ii) $f(x) = \sqrt{4 - x}$;
(iii) $f(x) = x^2 + \dfrac{1}{x}$;
(iv) $f(x) = \dfrac{1}{1 + x^2}$;
(v) $f(x) = \min\{x, x^2\}$;
(vi) $f(x) = |e^x - 1|$;
(vii) $f(x) = \sqrt{|x| - x}$;
(viii) $f(x) = \dfrac{1}{\lfloor 1/x \rfloor}$;
(ix) $f(x) = |x^2 - 1|$;
(x) $f(x) = 1 - e^{-x}$;
(xi) $f(x) = \log(x^2 - 1)$;
(xii) $f(x) = x\sin(1/x)$.

HINT: In (viii) $\lfloor x \rfloor$ denotes the integer part of x, i.e., the largest integer $n \leqslant x$.

Problem 2.12

(a) Prove that $\cosh x$ is even and $\sinh x$ is odd.
(b) Prove the identities $\cosh^2 x - \sinh^2 x = 1$ and $\sinh(2x) = 2\sinh x \cosh x$.

II

Sequences and Series

3. Sequences

3.1 Sequences of real numbers

A *sequence* is simply an infinite ordered list of real numbers

$$\{a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots\}.$$

(We normally use the convention of delimiting sequences by curly brackets.) Often the first terms of the sequence self-explain the rest of the sequence. This is the case of sequences such as

$$\{1,2,3,4,\ldots\},\qquad \{1,1,1,1,\ldots\},\qquad \{1,0,1,0,1,0,\ldots\}.$$

In most cases though, they are given by a formula as a function of n, e.g.,

$$a_n = n,\qquad b_n = (-1)^n,\qquad c_n = \frac{1}{n},\qquad d_n = \left(1+\frac{1}{n}\right)^n.$$

Another possibility is to obtain a sequence through a recurrence. A recurrence is a formula that obtains the nth term in the sequence given the previous k terms (most often just one or two). For instance,

$$a_n = \sqrt{a_{n-1}+1},\quad a_1 = 1.$$

Sometimes the solution of a recurrence is given by a formula, but most of the times it is not possible to find such a formula. Anyway, the recurrence provides a constructive way of describing the full sequence.

R We usually represent a sequence with the symbol $\{a_n\}_{n=1}^{\infty}$, where a_n is referred to as the *general term*, regardless of whether there is an explicit formula that provides a_n as a function of n or not.

In more rigorous terms we have the following definition of sequence:

Definition 3.1.1 A sequence is a map $f : \mathbb{N} \to \mathbb{R}$.

As a matter of fact, this is what the symbol a_n denotes: to each $n \in \mathbb{N}$ there correspond a real number a_n.

R Sequences can begin at $n = 0$ instead of $n = 1$, or in general at any index $n = k$, with $k = 0, 1, 2, \ldots$ Note that $\{a_n\}_{n=k}^{\infty} = \{a_{n+k-1}\}_{n=1}^{\infty}$, so that any sequence can be rewritten as a true map $f : \mathbb{N} \to \mathbb{R}$.

■ **Example 3.1 — Fibonacci sequence.** Leonardo di Pisa (c. 1170 – c. 1250), known as *Fibonacci* (son of Bonacci), was an Italian mathematician, considered to be "the most talented Western mathematician of the Middle Ages". Fibonacci popularized the Hindu-Arabic numeral system to the Western World mainly through his book, *Liber Abaci (Book of Calculation),* published in 1202.

An example of *Liber Abaci* is the well-known sequence of Fibonacci numbers. Fibonacci proposed the sequence as the solution to a problem of how a population of idealised rabbits grows generation after generation. His assumptions were:

1. Begin with one male-female couple of rabbits that have just been born.
2. Rabbits reach sexual maturity after one month.
3. The gestation period of a rabbit is one month.
4. After reaching sexual maturity, female rabbits give birth every month.
5. A female rabbit gives birth to one male rabbit and one female rabbit.
6. Rabbits do not die.

If F_n is the number of rabbit couples at month n, then $F_0 = 1$ (the initial couple has just been born), $F_1 = 1$ (rabbits need a month to reach sexual maturity) and then

$$F_{n+1} = F_n + F_{n-1}, \quad n > 0, \tag{3.1}$$

in other words, the number of couples next month (F_{n+1}) is the number of couples the current month (F_n) plus all new couples. There is a new couple for every sexually mature couple, and the sexually mature couples are those couples that existed last month (F_{n-1}), because it takes a month to reach sexual maturity.

Iterating equation (3.1) we get the *Fibonacci sequence*

$$\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}.$$

■

Definition 3.1.2 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that this sequence

(a) **increases monotonically** if $a_n \leqslant a_{n+1}$ for all $n \in \mathbb{N}$;
(b) **decreases monotonically** if $a_n \geqslant a_{n+1}$ for all $n \in \mathbb{N}$;
(c) is **alternating** if $(a_{n+1} - a_n)(a_n - a_{n-1}) < 0$ for all $n \in \mathbb{N}$ (i.e., it goes up and down alternatively);
(d) is **bounded from above** if there exists $c \in \mathbb{R}$ such that $a_n \leqslant c$ for all $n \in \mathbb{N}$;
(e) is **bounded from below** if there exists $c \in \mathbb{R}$ such that $a_n \geqslant c$ for all $n \in \mathbb{N}$.

■ **Example 3.2** Let us consider the sequence $a_n = \dfrac{n}{n+1}$. It is easy to see that it is bounded from above by 1, because the denominator is always larger than the numerator. Another way to see it is by looking for a c such that $a_n < c$, i.e.,

$$\frac{n}{n+1} < c \quad \Leftrightarrow \quad n < cn + c.$$

Clearly this second inequality holds if we take $c = 1$.

Let us now prove that $\{a_n\}_{n=1}^{\infty}$ increases monotonically. To check for monotonicity it is normally easier to check for the sign of $a_n - a_{n+1}$ or whether a_n/a_{n+1} is larger or smaller than 1. Let us try the former:

$$a_n - a_{n+1} = \frac{n}{n+1} - \frac{n+1}{n+2} = \frac{n(n+2)-(n+1)^2}{(n+1)(n+2)} = \frac{n^2+2n-n^2-2n-1}{(n+1)(n+2)} = \frac{-1}{(n+1)(n+2)} < 0,$$

therefore $a_n < a_{n+1}$ and so the sequence increases monotonically. ■

Definition 3.1.3 — Subsequence. A **subsequence** of a sequence $\{a_n\}_{n=1}^{\infty}$ is any choice $\{a_{n_k}\}_{k=1}^{\infty}$ of infinitely many elements of the sequence. (n_k is the rule that tells what is the kth element of the subsequence.)

■ **Example 3.3** The sequence $\left\{\frac{1}{2k-1}\right\}_{k=1}^{\infty}$ is a subsequence of the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$. The rule $n_k = 2k-1$ tells that we are selecting only the odd terms of the sequence. ■

■ **Example 3.4** The sequence $\left\{\frac{1}{2^{k-1}}\right\}_{k=1}^{\infty}$ is a subsequence of the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$. The rule $n_k = 2^{k-1}$ tells that we are selecting only the terms of the sequence whose index is a power of 2. ■

Proposition 3.1.1 Every sequence has at least one monotonic subsequence (either increasing or decreasing).

3.2 Limit of a sequence

Consider the sequence $a_n = \dfrac{n}{n+1}$. Let us explicitly display some of its terms:

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{1000}{1001}, \ldots\right\} = \{0.5000,\ 0.6667,\ 0.7500,\ 0.8000, \ldots,\ 0.9990, \ldots\}$$

where we have calculated the numbers to an accuracy of four decimal places. These numbers as well as Figure 3.2 both illustrate the fact that the more we increase n the closer is a_n to the value 1. Sequences exhibiting this behavior are said to have a limit. In this case, the limit of a_n is 1.

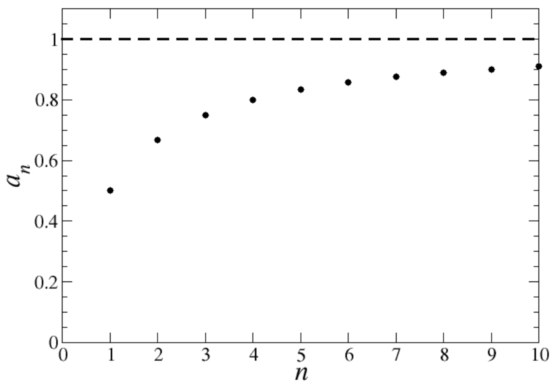


Figure 3.1: Plot of the first ten terms of the sequence $a_n = \dfrac{n}{n+1}$.

We need a more precise definition of limit that captures this idea in all its flavour. To this purpose we have the following definition:

Definition 3.2.1 — Limit of a sequence. The real number a is said to be the **limit** of the sequence $\{a_n\}$ if for any real number $\varepsilon > 0$ —no matter how small— there exits an index N —which may depend on ε— such that for every $n > N$ the sequence satisfies the inequality

$$|a_n - a| < \varepsilon. \tag{3.2}$$

The sequence is then said to be **convergent.**

■ **Example 3.5** Let us apply the definition to actually prove that the limit of $a_n = \dfrac{n}{n+1}$ is 1. Let $\varepsilon > 0$ be a given arbitrary number. We need to find for which indices n the inequality

$$\left|\frac{n}{n+1} - 1\right| < \varepsilon \tag{3.3}$$

holds. Clearly,

$$\frac{n}{n+1} - 1 = \frac{n-(n+1)}{n+1} = \frac{-1}{n+1} \quad \Rightarrow \quad \left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1}.$$

Thus inequality (3.3) is equivalent to

$$\frac{1}{n+1} < \varepsilon \quad \Leftrightarrow \quad n+1 > \frac{1}{\varepsilon} \quad \Leftrightarrow \quad n > \frac{1}{\varepsilon} - 1.$$

We have the proof we wanted. Suppose $\varepsilon = 0.1$. Then

$$\frac{1}{\varepsilon} - 1 = \frac{1}{0.1} - 1 = 10 - 1 = 9,$$

so we can take $N = 9$ and the definition applies. Suppose $\varepsilon = 0.01$. Then

$$\frac{1}{\varepsilon} - 1 = \frac{1}{0.01} - 1 = 100 - 1 = 99,$$

so we can take $N = 99$ and again the definition applies.

It is clear that we can take ε smaller and smaller, and that will imply that N is larger and larger, but nevertheless, no matter how small ε is taken, there always exists N satisfying the definition. ■

We can also characterise sequences like $a_n = n$, which not only do not have a limit, but they grow without bound as n increases.

Definition 3.2.2 — Divergent sequence. The sequence $\{a_n\}$ is said to be **divergent to** $+\infty$ if for any real number $C > 0$ —no matter how large— there exits an index N —which may depend on C— such that for every $n > N$ the sequence satisfies the inequality

$$a_n > C. \tag{3.4}$$

Likewise, it is said to be **divergent to** $-\infty$ if for any real number $C < 0$ there exits an index N such that for every $n > N$ the sequence satisfies the inequality

$$a_n < C. \tag{3.5}$$

R We denote the limit of a sequence with the symbol $\lim_{n\to\infty} a_n = a$ if it converges, or $\lim_{n\to\infty} a_n = \pm\infty$ if it diverges to $+\infty$ or $-\infty$.

■ **Example 3.6** The sequence $a_n = n^p$ diverges to $+\infty$ if $p > 0$, and converges if $p \leqslant 0$. We will prove it by applying the definition.

Let $p > 0$ and $C > 0$ and consider the inequality

$$a_n = n^p > C \qquad \Leftrightarrow \qquad n > C^{1/p}.$$

In other words, if $N > C^{1/p}$, then for every $n > N$ we will have $a_n > C$.

Let now $p = 0$. Then $a_n = 1$ for all $n \in \mathbb{N}$, and therefore

$$\lim_{n\to\infty} a_n = 1.$$

Finally, let $p = -q < 0$ and $\varepsilon > 0$. Then,

$$|a_n - 0| = \left|\frac{1}{n^q} - 0\right| = \frac{1}{n^q} < \varepsilon \qquad \Leftrightarrow \qquad n^q > \frac{1}{\varepsilon} \qquad \Leftrightarrow \qquad n > \left(\frac{1}{\varepsilon}\right)^{1/q}.$$

So if we take $N > (1/\varepsilon)^{1/q}$ then for every $n > N$ we will have $|a_n - 0| < \varepsilon$. Therefore

$$\lim_{n\to\infty} a_n = 0.$$

■

Aside from convergent and divergent sequences there are sequences that neither converge nor diverge. This are simply said to be **nonconvergent** sequences.

■ **Example 3.7** The sequence $a_n = (-1)^n n$ is nonconvergent, because $|a_n| = n$ is divergent, but a_n alternates sign. ■

Proposition 3.2.1 If the limit of $\{a_n\}_{n=1}^{\infty}$ exists, it is unique.

Proof. Suppose that a and b ($a < b$) are two limits of the same sequence. Then, according to the definition, for every $\varepsilon > 0$

$$|a_n - a| < \varepsilon, \qquad |a_n - b| < \varepsilon.$$

In other words,

$$a_n < a + \varepsilon, \qquad b - \varepsilon < a_n.$$

But if ε is such that $a + \varepsilon < b - \varepsilon$ this two inequalities cannot hold at the same time. And this can be accomplished by any $\varepsilon < (b-a)/2$. Hence b and a cannot be two different numbers. ■

Proposition 3.2.2 Every subsequence of a convergent sequence has the same limit as the sequence.

Applying the definition to prove that a sequence has a limit may be a daunting task. For that reason we normally apply some properties that all convergent sequences satisfy in order to simplify the problem. These are some algebraic properties of the limits:

Proposition 3.2.3 Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two convergent sequences with limits a and b respectively. Then the following properties hold:

1. $\lim_{n\to\infty} (a_n \pm b_n) = a \pm b$;
2. $\lim_{n\to\infty} a_n b_n = ab$;
3. if $b_n \neq 0$ for all $n \in \mathbb{N}$ and $b \neq 0$, then $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{a}{b}$;
4. $\lim_{n\to\infty} a_n^{b_n} = a^b$;
5. $\lim_{n\to\infty} \log a_n = \log a$.

Two further theorems turn out to be very practical for calculating limits.

Theorem 3.2.4 — Sandwich rule. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = l$, and for all $n > N$, for some $N \in \mathbb{N}$, we have $a_n \leqslant b_n \leqslant c_n$, then also $\lim_{n\to\infty} b_n = l$.

In particular this implies that if $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$ too because $-|a_n| \leqslant a_n \leqslant |a_n|$.

Theorem 3.2.5 Every monotonically increasing (respectively decreasing) sequence bounded from above (respectively below) converges to some limit.

Proof. Suppose $\{a_n\}_{n=1}^{\infty}$ increases monotonically and is bounded above. The *supremum property* (Theorem 1.3.2(i)) guarantees that $\{a_n\}_{n=1}^{\infty}$ has a supremum a. So $a_n \leqslant a$ for all $n \in \mathbb{N}$ and no other real number $c < a$ is an upper bound. Let us take $\varepsilon > 0$ and set $c = a - \varepsilon$. This is not an upper bound, therefore there must be some $N \in \mathbb{N}$ such that $a_N > a - \varepsilon$. But since the sequence is increasing we then have

$$a - \varepsilon < a_N \leqslant a_{N+1} \leqslant a_{N+2} \leqslant \cdots \leqslant a < a + \varepsilon.$$

In other words, $|a_n - a| < \varepsilon$ for all $n > N$. ■

As the proof shows, this theorem is a consequence of the completeness of real numbers. Intuitively it makes perfect sense, for if a sequence keeps on increasing but cannot trespass a certain bound, it must converge to something. And as a matter of fact, the proof shows that this "something" is precisely the supremum of the sequence, if it increases, or its infimum, if it decreases.

■ **Example 3.8** Consider the sequence

$$a_n = \frac{\sin n}{n}.$$

In principle a_n is the quotient of two sequences, namely $b_n = \sin n$ and $c_n = n$. However, none of them has a limit: c_n diverges to $+\infty$ and b_n exhibits a random behavior, as illustrated by Figure 3.3.

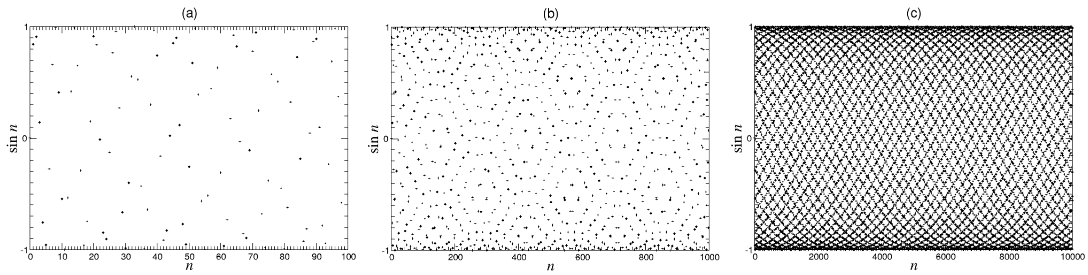


Figure 3.2: Plot of $\sin n$ at three different scales: (a) for $1 \leqslant n \leqslant 100$, (b) for $1 \leqslant n \leqslant 1000$, and (c) for $1 \leqslant n \leqslant 10000$.

However, it is always true that, irrespective of n, $-1 \leqslant \sin n \leqslant 1$, so

$$-\frac{1}{n} \leqslant a_n \leqslant \frac{1}{n}.$$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, applying the sandwich rule we conclude that $\lim_{n\to\infty} a_n = 0$. ■

■ **Example 3.9** Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined by the recurrence

$$a_{n+1} = \frac{1}{2}(a_n + 6), \qquad a_1 = 2.$$

The first few terms look like

$$\left\{2,4,5,\frac{11}{2},\frac{23}{4},\frac{47}{8},\frac{95}{16},\ldots\right\} = \{2,\ 4,\ 5,\ 5.5,\ 5.75,\ 5.875,\ 5.9375,\ldots\},$$

so it seems that it does converge toward 6 and is monotonically increasing.

Let us first show, by induction, that $\{a_n\}_{n=1}^{\infty}$ is bounded above by 6. To begin with, $a_1 = 2 < 6$. Now suppose $a_n < 6$. Then

$$a_{n+1} = \frac{1}{2}(a_n+6) < \frac{1}{2}(6+6) = 6,$$

so if $a_n < 6$ then also $a_{n+1} < 6$.

Finally let us show that the sequence is monotonically increasing. For that, let us calculate

$$a_{n+1} - a_n = \frac{1}{2}(a_n+6) - a_n = \frac{1}{2}(6-a_n) > 0$$

because we have shown that $a_n < 6$. Therefore $a_{n+1} > a_n$, and because of the theorem, we have proven that the sequence converges to some number a. To actually determine a we must take the limit in the recurrence as

$$\lim_{n\to\infty} a_{n+1} = \frac{1}{2}\left(\lim_{n\to\infty} a_n + 6\right) \quad\Leftrightarrow\quad a = \frac{1}{2}(a+6) \quad\Leftrightarrow\quad 2a = a+6 \quad\Leftrightarrow\quad a = 6.$$

■

■ **Example 3.10** Let us consider the sequence

$$a_n = \frac{3n^2+2n-1}{5n^4-2n+7}.$$

Calculating its limit through the definition is very difficult. However we can play the following trick: we will factor out the largest power both in the numerator and in the denominator,

$$a_n = \frac{n^2\left(3+\dfrac{2}{n}-\dfrac{1}{n^2}\right)}{n^4\left(5-\dfrac{2}{n^3}+\dfrac{7}{n^4}\right)} = \frac{n^2}{n^4}\cdot\frac{3+\dfrac{2}{n}-\dfrac{1}{n^2}}{5-\dfrac{2}{n^3}+\dfrac{7}{n^4}} = \frac{1}{n^2}\cdot\frac{3+\dfrac{2}{n}-\dfrac{1}{n^2}}{5-\dfrac{2}{n^3}+\dfrac{7}{n^4}},$$

and now we can apply the algebraic rules for the limits. Since for any $p > 0$

$$\lim_{n\to\infty}\frac{1}{n^p} = 0,$$

then

$$\lim_{n\to\infty}\left(3+\frac{2}{n}-\frac{1}{n^2}\right) = 3, \qquad \lim_{n\to\infty}\left(5-\frac{2}{n^3}+\frac{7}{n^4}\right) = 5,$$

and therefore

$$\lim_{n\to\infty} a_n = 0\cdot\frac{3}{5} = 0.$$

■

From the example above we can infer the following proposition:

Proposition 3.2.6 Let

$$a_n = \frac{\alpha_p n^p + \alpha_{p-1} n^{p-1} + \cdots + \alpha_1 n + \alpha_0}{\beta_q n^q + \beta_{q-1} n^{q-1} + \cdots + \beta_1 n + \beta_0}.$$

Then

(a) if $q = p$, $\lim_{n\to\infty} a_n = \frac{\alpha_p}{\beta_q}$;

(b) if $q > p$, $\lim_{n\to\infty} a_n = 0$;

(c) if $q < p$, $\lim_{n\to\infty} a_n = +\infty$ if $\alpha_p \beta_q > 0$ and $\lim_{n\to\infty} a_n = -\infty$ if $\alpha_p \beta_q < 0$.

Proof. (a) and (b) are proven as in the example. For (c) we can prove equivalently that $\lim_{n\to\infty} \frac{1}{a_n} = 0$ because it reduces to (b). The sign is the sign of the quotient α_p/β_q —which is the same as that of the product $\alpha_p \beta_q$. ■

Here is a list of some important limits that often occur in calculations:

1. $\lim_{n\to\infty} \sqrt[n]{a} = 1$ for all $a > 0$.
2. $\lim_{n\to\infty} \sqrt[n]{n^p} = 1$ for all $p \in \mathbb{R}$.
3. $\lim_{n\to\infty} a^n = 0$ for all $|a| < 1$ and $\lim_{n\to\infty} a^n = +\infty$ for all $a > 1$.
4. $\lim_{n\to\infty} \frac{n^r}{a^n} = 0$ for all $|a| > 1$ and $r \in \mathbb{R}$.

Finally, there is a very important result that at this point will be very easy to prove:

Theorem 3.2.7 — Bolzano-Weierstrass theorem. Every bounded sequence has at least one convergent subsequence.

Proof. The proof is very simple. From Proposition 3.1.1 we know that every sequence has at least one monotonic subsequence. This subsequence will be bounded because the whole sequence is bounded. Therefore this subsequence must converge to some limit by virtue of Theorem 3.2.5. ■

R Note that this result holds even for sequences without limit.

■ **Example 3.11** The sequence $\{(-1)^n\}_{n=1}^{\infty}$ does not converge; however it is bounded, and the subsquence $\{1\}_{n=1}^{\infty}$ containing only the even terms clearly converges to 1. ■

3.3 Number *e*

There is a special convergent sequence whose limit defines an irrational number of great importance in mathematics. It is the following:

Definition 3.3.1

$$e \equiv \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = 2.71828182845904523536028747135266249775 72\ldots \tag{3.6}$$

The sequence is monotonically increasing and bounded above, but converges very slowly to its limit, as Table 3.1 shows. Both the prove of the convergence of this sequence and that of the irrationality of its limit appear in Apppendix C.

n	$\left(1+\frac{1}{n}\right)^n$
1	**2**
2	**2**.25
5	**2**.48832
10	**2**.59374246
100	**2.7**04813829
1000	**2.71**6923932
10000	**2.718**145927
100000	**2.7182**68237

Table 3.1: Some values of the sequence $\left(1+\frac{1}{n}\right)^n$. Note that n must increase by an order of magnitude in order to obtain a new decimal figure of e.

Number e is not only an irrational number, but a transcendental one. This means that there is no algebraic equation whose solution is e (in particular, e cannot be expressed in terms of radicals).

Many limits involve e. Here are a few examples:

■ **Example 3.12** Calculate

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}.$$

Since we can factor out

$$\left(1+\frac{1}{n}\right)^{n+1}=\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right),$$

then

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\left(1+\underbrace{\lim_{n\to\infty}\frac{1}{n}}_{=0}\right)=e.$$

■

■ **Example 3.13** Calculate

$$\lim_{n\to\infty}\left(1+\frac{1}{n^2+1}\right)^{n^2+1}.$$

The sequence

$$\left(1+\frac{1}{k^2+1}\right)^{k^2+1},\quad k\in\mathbb{N},$$

is a subsequence of

$$\left(1+\frac{1}{n}\right)^{n}$$

(that corresponding to those n such that $n=k^2+1$). Any subsequence of a convergent sequence has the same limit as the original sequence, therefore

$$\lim_{n\to\infty}\left(1+\frac{1}{n^2+1}\right)^{n^2+1}=e.$$

■

■ **Example 3.14** Calculate

$$\lim_{n\to\infty}\left(1+\frac{1}{3n^2+1}\right)^{2n^2-3}.$$

We can rewrite

$$\left(1+\frac{1}{3n^2+1}\right)^{2n^2-3} = \left[\left(1+\frac{1}{3n^2+1}\right)^{3n^2+1}\right]^{\frac{2n^2-3}{3n^2+1}},$$

so that

$$\lim_{n\to\infty}\left(1+\frac{1}{3n^2+1}\right)^{2n^2-3} = \left[\lim_{n\to\infty}\left(1+\frac{1}{3n^2+1}\right)^{3n^2+1}\right]^{\lim_{n\to\infty}\frac{2n^2-3}{3n^2+1}}.$$

Now, since

$$\lim_{n\to\infty}\left(1+\frac{1}{3n^2+1}\right)^{3n^2+1} = e, \qquad \lim_{n\to\infty}\frac{2n^2-3}{3n^2+1} = \frac{2}{3},$$

we finally have

$$\lim_{n\to\infty}\left(1+\frac{1}{3n^2+1}\right)^{2n^2-3} = e^{2/3}.$$

■

It is easy to generalise this example and show that any limit of the form

$$\lim_{n\to\infty}(1+b_n)^{c_n},$$

where

$$\lim_{n\to\infty} b_n = 0, \qquad \lim_{n\to\infty} c_n = \infty,$$

can be calculated as

$$\lim_{n\to\infty}(1+b_n)^{c_n} = e^{\left(\lim_{n\to\infty} b_n c_n\right)}.$$

■ **Example 3.15** Calculate

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n}.$$

We can rewrite

$$\left(1-\frac{1}{n}\right)^{n} = \left(\frac{n-1}{n}\right)^{n} = \left(\frac{n-1}{1+(n-1)}\right)^{n} = \frac{1}{\left(\frac{1+(n-1)}{n-1}\right)^{n}} = \frac{1}{\left(1+\frac{1}{n-1}\right)^{n}}.$$

Therefore

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n} = \frac{1}{e^{\left(\lim_{n\to\infty}\frac{n}{n-1}\right)}} = \frac{1}{e} = e^{-1}.$$

Incidentally, this example shows that the above argument is valid regardless of the sign of b_n. ■

As a matter of fact, number e is involved in any limit corresponding to what is referred to as an *indeterminacy* of type 1^∞. This is a short way to refer to the limit of a sequence of the form $a_n^{c_n}$, where $\lim_{n\to\infty} a_n = 1$ and $\lim_{n\to\infty} c_n = \infty$. Here is an example:

■ **Example 3.16** Calculate

$$\lim_{n\to\infty}\left(\frac{3n^2+1}{(3n+2)(n-3)}\right)^{2n+1}.$$

First we check that

$$\lim_{n\to\infty}\frac{3n^2+1}{(3n+2)(n-3)} = \lim_{n\to\infty}\frac{3n^2+1}{3n^2-7n-6} = 1, \qquad \lim_{n\to\infty}(2n+1) = \infty,$$

so we are dealing with a 1^∞ indeterminacy. The way to proceed is always the same. We rewrite

$$\frac{3n^2+1}{(3n+2)(n-3)} = \frac{3n^2+1}{3n^2-7n-6} = 1+\left(\frac{3n^2+1}{3n^2-7n-6}-1\right) = 1+\frac{7n+7}{3n^2-7n-6},$$

so that we can transform the original limit into

$$\lim_{n\to\infty}\left(\frac{3n^2+1}{(3n+2)(n-3)}\right)^{2n+1} = \lim_{n\to\infty}\left(1+\frac{7n+7}{3n^2-7n-6}\right)^{2n+1}.$$

Now

$$\lim_{n\to\infty}\frac{7n+7}{3n^2-7n-6} = 0,$$

therefore

$$\lim_{n\to\infty}\left(\frac{3n^2+1}{(3n+2)(n-3)}\right)^{2n+1} = e^c,$$

where

$$c = \lim_{n\to\infty}\frac{(7n+7)(2n+1)}{3n^2-7n-6} = \lim_{n\to\infty}\frac{14n^2+21n+7}{3n^2-7n-6} = \frac{14}{3}.$$

■

3.4 Indeterminacies

Apart from the 1^∞ indeterminacy we have just encountered and which is related to the number e, there are other indeterminacies that often appear when we calculate limits. They are basically these:

$$\frac{0}{0}, \qquad \frac{\infty}{\infty}, \qquad 0\cdot\infty, \qquad 0^0, \qquad \infty^0, \qquad \infty-\infty.$$

Their meaning is similar to that of the 1^∞ indeterminacy. For instance, $0\cdot\infty$ just denotes the case where we find a limit such as $\lim_{n\to\infty} a_n b_n$, where $\lim_{n\to\infty} a_n = 0$, and $\lim_{n\to\infty} b_n = \infty$ (and similarly for the other cases).

R Note that the following expressions are not indeterminacies, but well defined limits:

$$\infty^\infty = \infty, \qquad 0^\infty = 0, \qquad \infty+\infty = \infty, \qquad \frac{0}{\infty} = 0, \qquad \frac{\infty}{0} = \pm\infty.$$

Indeterminacies are related to one another. For instance, if $\lim_{n\to\infty} a_n = \infty$, and $\lim_{n\to\infty} b_n = \infty$, then

$$\underbrace{\lim_{n\to\infty} \frac{a_n}{b_n}}_{\frac{\infty}{\infty}} = \underbrace{\lim_{n\to\infty} a_n b_n^{-1}}_{\infty\cdot 0} = \underbrace{\lim_{n\to\infty} \frac{b_n^{-1}}{a_n^{-1}}}_{\frac{0}{0}}.$$

Indeterminacies must be solved case by case —there is no general rule to apply. We have already found the $\frac{\infty}{\infty}$ indeterminacy in the case of rational sequences, and as Proposition 3.2.6 shows, the solution depens on the specific sequence whose limit we want to calculate.

Other cases of this indeterminacy as well as the $\frac{0}{0}$ indeterminacy require the following result:

Theorem 3.4.1 — Stolz theorem. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers. Suppose that $\{b_n\}_{n=1}^{\infty}$ is strictly monotonic (increasing or decreasing) and either:

(a) $\lim_{n\to\infty} b_n = \pm\infty$, or

(b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$.

Then

$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \ell \qquad \Rightarrow \qquad \lim_{n\to\infty} \frac{a_n}{b_n} = \ell. \tag{3.7}$$

From this theorem we can derive three important corollaries:

Corollary 3.4.2

$$\lim_{n\to\infty} c_n = c \qquad \Rightarrow \qquad \lim_{n\to\infty} \frac{1}{n}(c_1 + c_2 + \cdots + c_n) = c.$$

Proof. Take the sequences $a_n = c_1 + c_2 + \cdots + c_n$ and $b_n = n$, and note that b_n is strictly increasing and diverges. Then

$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \lim_{n\to\infty} \frac{(c_1 + c_2 + \cdots + c_{n-1} + c_n) - (c_1 + c_2 + \cdots + c_{n-1})}{n - (n-1)} = \lim_{n\to\infty} c_n = c,$$

therefore

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c.$$

■

Corollary 3.4.3

$$\lim_{n\to\infty} c_n = c \qquad \Rightarrow \qquad \lim_{n\to\infty} \sqrt[n]{c_1 c_2 \cdots c_n} = c.$$

Proof. Taking logarithms

$$\log \sqrt[n]{c_1 c_2 \cdots c_n} = \frac{1}{n}(\log c_1 + \log c_2 + \cdots + \log c_n).$$

Since $\lim_{n\to\infty} \log c_n = \log c$, applying the previous corollary we obtain

$$\lim_{n\to\infty} \log \sqrt[n]{c_1 c_2 \cdots c_n} = \lim_{n\to\infty} \frac{1}{n}(\log c_1 + \log c_2 + \cdots + \log c_n) = \log c,$$

and therefore

$$\lim_{n\to\infty} \sqrt[n]{c_1 c_2 \cdots c_n} = c.$$

■

Corollary 3.4.4

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = a \qquad \Rightarrow \qquad \lim_{n\to\infty} \sqrt[n]{a_n} = a.$$

Proof. We can write

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_2}{a_1} a_1,$$

so if we define $c_n \equiv a_n/a_{n-1}$ for $n > 1$ and $c_1 = a_1$ we have

$$a_n = c_1 c_2 \cdots c_n.$$

But $\lim_{n\to\infty} c_n = a$, therefore the previous corollary implies

$$\lim_{n\to\infty} \sqrt[n]{c_1 c_2 \cdots c_n} = \lim_{n\to\infty} \sqrt[n]{a_n} = a.$$

■

■ **Example 3.17** We can easily show that $\lim_{n\to\infty} \sqrt[n]{a} = 1$, for all $a > 0$, as an application of the last corollary, because

$$\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} \frac{a}{a} = 1.$$

Similarly we can prove that $\lim_{n\to\infty} \sqrt[n]{n^p} = 1$, for all $p \in \mathbb{R}$, because

$$\lim_{n\to\infty} \sqrt[n]{n^p} = \lim_{n\to\infty} \frac{n^p}{(n-1)^p} = \left(\lim_{n\to\infty} \frac{n}{n-1}\right)^p = 1^p = 1.$$

■

■ **Example 3.18** We can calculate[1]

$$\lim_{n\to\infty} \sqrt[n]{\binom{2n}{n}}$$

using the third corollary as

$$\lim_{n\to\infty} \frac{\binom{2n}{n}}{\binom{2n-2}{n-1}} = \lim_{n\to\infty} \frac{(2n)!}{n!n!} \frac{(n-1)!(n-1)!}{(2n-2)!} = \lim_{n\to\infty} \frac{2n(2n-1)}{n^2} = \lim_{n\to\infty} \frac{4n-2}{n} = 4,$$

where we have used the fact that $(2n)! = 2n(2n-1)\cdot(2n-2)!$ and $n! = n\cdot(n-1)!$. ■

Stolz theorem is particularly useful when one of the sequences involved is a sum of n terms, as in this example:

[1] The symbols $\binom{n}{k}$ represent combinatorial coefficients. Their definition and properties can be found in Appendix B.

■ **Example 3.19** Calculate

$$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{\log n}.$$

An application of the theorem transforms this limit into

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\log n - \log(n-1)} = \lim_{n\to\infty} \frac{1}{n\log\left(\frac{n}{n-1}\right)} = \lim_{n\to\infty} \frac{1}{\log\left(\frac{(n-1)+1}{n-1}\right)^n} = \lim_{n\to\infty} \frac{1}{\log\left(1+\frac{1}{n-1}\right)^n}.$$

But we know that

$$\lim_{n\to\infty}\left(1+\frac{1}{n-1}\right)^n = e,$$

and that $\log e = 1$, therefore

$$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{\log n} = 1.$$

■

The indeterminacy $\infty-\infty$ is a particularly difficult one, but often can be solved by an algebraic manipulation of the expressions.

■ **Example 3.20** Calculate

$$\lim_{n\to\infty}\left(\sqrt{n^2+2n-1}-\sqrt{n^2-3}\right).$$

Both terms in this expression are divergent sequences, so this is an $\infty-\infty$ indeterminacy. The trick to calculate this limit is to make use of the identity

$$x^2-y^2=(x-y)(x+y) \qquad \Rightarrow \qquad x-y=\frac{x^2-y^2}{x+y}.$$

In this case

$$\begin{aligned}\sqrt{n^2+2n-1}-\sqrt{n^2-3} &= \frac{n^2+2n-1-(n^2-3)}{\sqrt{n^2+2n-1}+\sqrt{n^2-3}} = \frac{2n+2}{n\sqrt{1+\frac{2}{n}-\frac{1}{n^2}}+n\sqrt{1-\frac{3}{n^2}}} \\ &= \frac{2+\frac{2}{n}}{\sqrt{1+\frac{2}{n}-\frac{1}{n^2}}+\sqrt{1-\frac{3}{n^2}}} \xrightarrow[n\to\infty]{} \frac{2}{1+1}=1.\end{aligned}$$

In the case that there are higher order roots involved, in may be useful to apply the generalised identity

$$x^{p+1}-y^{p+1}=(x-y)(x^p+x^{p-1}y+x^{p-2}y^2+\cdots+xy^{p-1}+y^p)$$

that leads to

$$x-y=\frac{x^{p+1}-y^{p+1}}{x^p+x^{p-1}y+x^{p-2}y^2+\cdots+xy^{p-1}+y^p}.$$

■

3.5 Asymptotic comparison of sequences

In order to solve indeterminacies it is handy to compare sequences for large values of n. If they behave similarly they can often be replaced by each other. If one is negligible with respect to the other, it is the second one that decides the trend. Let us formalise this notion:

Definition 3.5.1 — Asymptotic comparison. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ two sequences which either both diverge or both converge to 0. We say that

(a) a_n and b_n are **equivalent** (we denote it $a_n \sim b_n$) if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1;$$

(b) a_n is **negligible** compared to b_n (we denote it $a_n \ll b_n$) if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

■ **Example 3.21** Example 3.19 proves that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \sim \log n, \tag{3.8}$$

(which incidentally proves that

$$\lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) = \infty, \tag{3.9}$$

a fact that will be of utmost relevance in the next chapter). ■

■ **Example 3.22** Suppose that $\{\varepsilon_n\}_{n=1}^{\infty}$ is a sequence that converges to 0. We have seen that

$$\lim_{n\to\infty} (1+\varepsilon_n)^{1/\varepsilon_n} = e.$$

Taking logarithms

$$\lim_{n\to\infty} \frac{\log(1+\varepsilon_n)}{\varepsilon_n} = 1;$$

in other words, $\log(1+\varepsilon_n) \sim \varepsilon_n$.

We can now define the new sequence $\delta_n \equiv \log(1+\varepsilon_n)$, whose limit is 0. Then $\varepsilon_n = e^{\delta_n} - 1$, so we transform the previous limit into

$$1 = \lim_{n\to\infty} \frac{\log(1+\varepsilon_n)}{\varepsilon_n} = \lim_{n\to\infty} \frac{\delta_n}{e^{\delta_n} - 1}.$$

Therefore $e^{\delta_n} - 1 \sim \delta_n$. ■

We can infer further equivalences from geometric arguments. Figure 3.4 shows that[2] $|\sin x| \leqslant |x| \leqslant |\tan x|$ for all $-\pi/2 \leqslant x \leqslant \pi/2$ (we take absolute values because all quantities become negative for negative x, but the relation between the lengths remains true). From the first inequality we conclude that, within this interval,

$$\left|\frac{\sin x}{x}\right| \leqslant 1.$$

[2] The first inequality, $|\sin x| \leqslant |x|$, is obvious from the figure. The second one, $|x| \leqslant |\tan x|$ follows because the area of the sector is obviously smaller than that of the big triangle. Now, the former is $\frac{1}{2}|x|$ whereas the latter is $\frac{1}{2}|\tan x|$, hence the inequality.

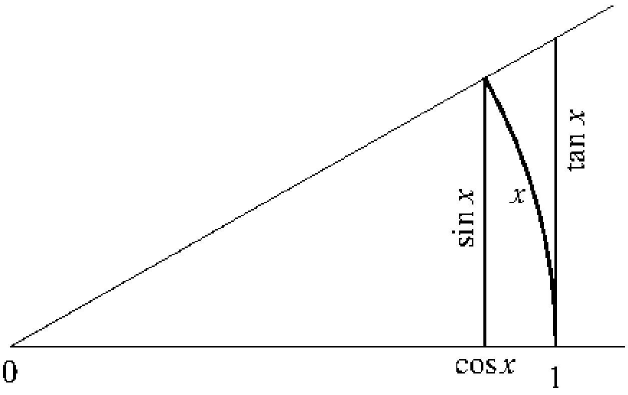


Figure 3.3: Comparison of $\sin x$, x, and $\tan x$ for small $0 \leqslant x \leqslant \pi/2$.

But $\dfrac{\sin x}{x} \geqslant 0$ for all $-\pi/2 \leqslant x \leqslant \pi/2$ (because both $\sin x$ and x have the same sign), so

$$0 \leqslant \frac{\sin x}{x} \leqslant 1.$$

From the second inequality,

$$|x| \leqslant \frac{|\sin x|}{|\cos x|} \qquad \Leftrightarrow \qquad |\cos x| \leqslant \left|\frac{\sin x}{x}\right|.$$

But since $\dfrac{\sin x}{x} \geqslant 0$ and $\cos x \geqslant 0$ for all $-\pi/2 \leqslant x \leqslant \pi/2$,

$$\cos x \leqslant \frac{\sin x}{x} \leqslant 1.$$

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence that converges to 0. From the inequality $|\sin \varepsilon_n| \leqslant |\varepsilon_n|$ (which is equivalent to $-|\varepsilon_n| \leqslant \sin \varepsilon_n \leqslant |\varepsilon_n|$) and the sandwich rule we conclude that

$$\lim_{n\to\infty} \sin \varepsilon_n = 0. \tag{3.10}$$

On the other hand, $\cos \varepsilon_n = \sqrt{1 - \sin^2 \varepsilon_n}$, so

$$\lim_{n\to\infty} \cos \varepsilon_n = 1. \tag{3.11}$$

Since we have

$$\cos \varepsilon_n \leqslant \frac{\sin \varepsilon_n}{\varepsilon_n} \leqslant 1,$$

again using the sandwich rule we conclude that

$$\lim_{n\to\infty} \frac{\sin \varepsilon_n}{\varepsilon_n} = 1, \tag{3.12}$$

or equivalently that $\sin \varepsilon_n \sim \varepsilon_n$.

Also,

$$\lim_{n\to\infty} \frac{\tan \varepsilon_n}{\varepsilon_n} = \lim_{n\to\infty} \frac{\sin \varepsilon_n}{\varepsilon_n \cos \varepsilon_n} = 1,$$

therefore $\tan \varepsilon_n \sim \varepsilon_n$.

In summary, all these sequences are equivalent:

$$\log(1+\varepsilon_n) \sim (e^{\varepsilon_n}-1) \sim \sin\varepsilon_n \sim \tan\varepsilon_n \sim \varepsilon_n. \tag{3.13}$$

Exercise 3.1 Using the identities

$$\cos^2 x + \sin^2 x = 1, \qquad \cos^2 x - \sin^2 x = \cos 2x,$$

prove that

$$1-\cos\varepsilon_n \sim \frac{\varepsilon_n^2}{2}. \tag{3.14}$$

■

A very important equivalence is given in the following theorem:

Theorem 3.5.1 — Stirling formula.

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n}. \tag{3.15}$$

The correct use of equivalences is as follows. Suppose $a_n \sim c_n$ and $b_n \sim d_n$. Then

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} \frac{a_n}{c_n}\frac{b_n}{d_n} c_n d_n = \underbrace{\lim_{n\to\infty}\frac{a_n}{c_n}}_{=1}\underbrace{\lim_{n\to\infty}\frac{b_n}{d_n}}_{=1}\lim_{n\to\infty} c_n d_n = \lim_{n\to\infty} c_n d_n.$$

So sequences can be replaced by equivalent sequences in products (and it can easily be shown that also in quotients).

Exercise 3.2 Calculate

$$\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\log\left(\frac{n+1}{n}\right)}.$$

■

However, using equivalences in differences can lead to incorrect results. This example is illustrative of the sort of problems one can meet.

■ **Example 3.23** We want to calculate

$$\lim_{n\to\infty}\left(\sqrt{n^4+n^2}-n^2-1\right).$$

Proceeding as in Example 3.20,

$$\sqrt{n^4+n^2}-n^2-1 = \frac{n^4+n^2-(n^2+1)^2}{\sqrt{n^4+n^2}+n^2+1} = \frac{n^4+n^2-n^4-2n^2-1}{n^2\sqrt{1+\frac{1}{n^2}}+n^2\left(1+\frac{1}{n^2}\right)} = \frac{-1-\frac{1}{n^2}}{\sqrt{1+\frac{1}{n^2}}+1+\frac{1}{n^2}}$$

therefore

$$\lim_{n\to\infty}\left(\sqrt{n^4+n^2}-n^2-1\right) = -\frac{1}{2}.$$

However, $\sqrt{n^4+n^2} \sim n^2$ because

$$\frac{\sqrt{n^4+n^2}}{n^2} = \sqrt{1+\frac{1}{n^2}} \xrightarrow[n\to\infty]{} 1,$$

so we might be tempted to reason as follows:

$$\sqrt{n^4+n^2}-n^2-1 \sim n^2-n^2-1=-1,$$

in which case we would conclude *incorrectly* that

$$\lim_{n\to\infty}\left(\sqrt{n^4+n^2}-n^2-1\right)=-1.$$

■

The problem that this example illustrates is that in replacing a sequence by an equivalent one we are ignoring smaller terms, which may become relevant if the dominant terms cancel out —as it usually happens in $\infty-\infty$ indeterminacies.

Negligible sequences, on the contrary, are relevant in sums and differences. For suppose $b_n \ll a_n$ and we want to calculate

$$\lim_{n\to\infty}\frac{c_n}{a_n+b_n}=\lim_{n\to\infty}\frac{c_n}{a_n\left(1+\frac{b_n}{a_n}\right)}=\lim_{n\to\infty}\frac{c_n}{a_n},$$

because

$$\lim_{n\to\infty}\frac{b_n}{a_n}=0.$$

In other words, in an expression like a_n+b_n we can simply eliminate the negligible sequence.

■ **Example 3.24** Let us show that

$$\log(3n^6-5n^2+2)\sim 6\log n.$$

Note that $2 \ll 5n^2 \ll 3n^6$, therefore

$$\log(3n^6-5n^2+2)\sim\log(3n^6)=\log 3+6\log n.$$

But $\log 3 \ll 6\log n$, hence the equivalence. ■

There is a hierarchy of negligible sequences which turns out to be very useful in calculations:

For any $a,b>0$, and $c>1$, it holds

$$(\log n)^a \ll n^b \ll c^n \ll n! \ll n^n. \tag{3.16}$$

Problems

Problem 3.1

(a) Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence and let $\{y_n\}_{n=1}^{\infty}$ be a divergent sequence. What can be said of the produt sequence $\{x_n y_n\}_{n=1}^{\infty}$.

(b) If a sequence of integer numbers is convergent, what is this sequence like?

(c) Prove that every convergent sequence is bounded.

Problem 3.2 Given the following recurrent sequences, find the general term and compute their limit:

(i) $a_{n+1} = \dfrac{a_n + 1}{2}$, with $a_0 = 0$;

(ii) $b_{n+1} = \sqrt{2b_n}$, with $b_0 = 1$.

Problem 3.3 Calculate the following limits:

(i) $\lim\limits_{n\to\infty} \sqrt[n]{a^n + b^n}$, with $a, b > 0$;

(ii) $\lim\limits_{n\to\infty} \left(\dfrac{\sqrt[n]{a} + \sqrt[n]{b}}{2}\right)^n$, with $a, b > 0$;

(iii) $\lim\limits_{n\to\infty} n\left(\sqrt{n^2+1} - n\right)$;

(iv) $\lim\limits_{n\to\infty} \sqrt{n}\left(\sqrt[4]{n^2+1} - \sqrt{n+1}\right)$;

(v) $\lim\limits_{n\to\infty} \dfrac{2^{n+1} + 3^{n+1}}{2^n + 3^n}$;

(vi) $\lim\limits_{n\to\infty} \left(\dfrac{n^2+1}{n^2-3n}\right)^{\frac{n^2-1}{2n}}$.

Problem 3.4 Calculate the following limits:

(i) $\lim\limits_{n\to\infty} \dfrac{n}{\pi} \sin n\pi$;

(ii) $\lim\limits_{n\to\infty} \dfrac{n\left(e^{1/n} - e^{\sin(1/n)}\right)}{1 - n\sin(1/n)}$;

(iii) $\lim\limits_{n\to\infty} \dfrac{n}{\sqrt[n]{n!}}$;

(iv) $\lim\limits_{n\to\infty} n^{-3/n}$;

(v) $\lim\limits_{n\to\infty} \dfrac{2^n}{n!}$;

(vi) $\lim\limits_{n\to\infty} \dfrac{n^2}{2^n}$;

(vii) $\lim\limits_{n\to\infty} \dfrac{n^{n-1}}{(n-1)^n}$;

(viii) $\lim\limits_{n\to\infty} \dfrac{1 + 2\sqrt{2} + 3\sqrt[3]{3} + \cdots + n\sqrt[n]{n}}{n^2}$.

Problem 3.5 If $a > 0$ and $\lim\limits_{n\to\infty} u_n = 0$, calculate the following limits:

(i) $\lim\limits_{n\to\infty} \left(\cos\dfrac{b}{n} + a\sin\dfrac{b}{n}\right)^n$;

(ii) $\lim\limits_{n\to\infty} \sqrt[u_n]{\dfrac{a - bu_n}{a + bu_n}}$.

Problem 3.6 Calculate the following limits:

(i) $\lim\limits_{n\to\infty} \dfrac{\sum_{k=1}^{n} \sin(\pi/k)}{\log n}$;

(ii) $\lim\limits_{n\to\infty} \prod_{k=1}^{n} (2k-1)^{1/n^2}$;

(iii) $\lim\limits_{n\to\infty} \sum_{k=1}^{n} \dfrac{k^2}{n^2} \sin\dfrac{1}{k}$.

Problem 3.7 Given that $\lim\limits_{n\to\infty} a_n = a$, calculate

$$\lim_{n\to\infty} \frac{a_1 + \frac{a_2}{2} + \cdots + \frac{a_n}{n}}{\log(n+1)}.$$

Problem 3.8 Calculate the limit

$$\lim_{n\to\infty}\sum_{k=1}^{3n}\frac{1}{\sqrt{n^2+k}}$$

using the sandwich rule.
HINT: Use the largest and smallest terms in the sum to bound the sum from above and from below, respectively.

Problem 3.9 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive terms such that $\lim_{n\to\infty}(a_n-n)=\ell$.

(a) Prove that $\lim_{n\to\infty}\frac{a_n}{n}=1$.

(b) Prove that $\lim_{n\to\infty} n\log(a_n/n)=\ell$.

Problem 3.10 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive terms such that $\lim_{n\to\infty}(a_{n+1}/a_n)=\ell$. Apply Stolz theorem to calculate the limit

$$\lim_{n\to\infty}\sqrt[n^2]{\frac{a_n^n}{a_1a_2\cdots a_n}}$$

Problem 3.11 Prove that the following sequences are monotonic, determine whether they are bounded, and find the limit in case they are:

(i) $\sqrt{2},\sqrt{2\sqrt{2}},\sqrt{2\sqrt{2\sqrt{2}}},\dots;$

(ii) $\sqrt{2},\sqrt{2+\sqrt{2}},\sqrt{2+\sqrt{2+\sqrt{2}}},\dots;$

(iii) $u_{n+1}=3+\frac{u_n}{2}$, with $u_0=0$;

(iv) $u_{n+1}=3+2u_n$, with $u_0=0$;

(v) $u_{n+1}=\frac{u_n^3+6}{7}$, with (a) $u_0=1/2$, (b) $u_0=3/2$, and (c) $u_0=3$.

Problem 3.12 Consider the sequence defined by $a_{n+1}=\sqrt{1+3a_n}-1$, with $a_0=1/2$.

(a) Prove that the sequence has a limit and find it.

(b) Compute $\lim_{n\to\infty}\frac{a_{n+1}-1}{a_n-1}$.

Problem 3.13 Consider the sequence defined by $b_{n+1}=1-b_n/2$, with $b_0=0$.

(a) Prove that the sequence is alternating, i.e., $(b_{n+1}-b_n)(b_n-b_{n-1})<0$.

(b) Assuming that it has a limit ℓ, find it.

(c) Prove that $|b_{n+1}-\ell|=\frac{1}{2}|b_n-\ell|$.

(d) Prove that the sequence has indeed a limit.

Problem 3.14 Consider the sequence defined by

$$x_{n+1}=\frac{x_n(1+x_n)}{1+2x_n},\qquad x_1=1.$$

(a) Prove that $x_n>0$ for all $n\in\mathbb{N}$.

(b) Prove that the sequence is monotonically decreasing.

(c) Calculate its limit.

4. Series

4.1 Series of real numbers

Series are a special kind of sequences —those made of sums of terms of other sequences. As sequences, they share all properties of sequences of real numbers studied in the previous chapter. However, studying the convergence of a series is normally a difficult task —not to mention to actually calculate its limit when it exists. That is why they are studied separately, and special techniques to address their convergence have to be developed.

Suppose we have a sequence of real numbers $\{a_n\}_{n=1}^{\infty}$. With this sequence we can construct another sequence, that we will denote $\{S_k\}_{k=1}^{\infty}$, whose general term is the sum of all terms of the original sequence up to the nth, namely

$$S_k = a_1 + a_2 + \cdots + a_k = \sum_{n=1}^{k} a_n. \tag{4.1}$$

(In what follows we will make common use of symbolic sums, whose properties can be found in Appendix A.) The general term S_k of this new sequence is often referred to as the kth **partial sum** of the original sequence.

The limit of the sequence of partial sums has a special notation:

$$\lim_{k\to\infty} S_k = \sum_{n=1}^{\infty} a_n. \tag{4.2}$$

This "infinite sum" (actually a limit) is what we normally refer to as a **series.**

■ **Example 4.1** As a first illustrative example we will introduce the **geometric series**

$$\sum_{n=0}^{\infty} x^n, \qquad x \in \mathbb{R}, \tag{4.3}$$

in other words, out of the sequence $\{x^n\}_{n=0}^{\infty}$ we construct the sequence of partial sums

$$S_k = \sum_{n=0}^{k} x^n = 1 + x + x^2 + \cdots + x^k.$$

The geometric series is the limit of this sequence.

The geometric series is one of the very few cases in which not only its convergence can be fully characterised, but also the sum can be explicitly computed when it converges. The reason is that an alternative expression for S_n can be obtained.

This is achieved by realising that if we multiply S_n by x we almost recover S_n again:

$$xS_k = \underbrace{x + x^2 + \cdots + x^k}_{=S_k - 1} + x^{k+1} = S_k - 1 + x^{k+1} \qquad \Rightarrow \qquad (x-1)S_k = x^{k+1} - 1.$$

So if $x \neq 1$, we obtain

$$S_k = \frac{x^{k+1} - 1}{x - 1} = \frac{1 - x^{k+1}}{1 - x},$$

and if $x = 1$ clearly $S_k = k + 1$. In summary,

$$S_k = \begin{cases} \dfrac{1 - x^{k+1}}{1 - x}, & x \neq 1, \\ k + 1 & x = 1. \end{cases} \tag{4.4}$$

In terms of x we can distinguish these cases:

(a) If $x = 1$ then $S_k = k + 1$ diverges to $+\infty$.

(b) If $|x| < 1$ then $\lim\limits_{k\to\infty} x^{k+1} = 0$

$$\lim_{k\to\infty} S_k = \frac{1}{1 - x}.$$

(c) If $x > 1$ then $\lim\limits_{k\to\infty} x^{k+1} = +\infty$ and S_k diverges to $+\infty$.

(d) If $x < -1$ then x^k is an alternating sequence with no limit.

All this information is usually summarised in the formula

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}, \qquad |x| < 1. \tag{4.5}$$

■

This example also illustrate the three cases we can meet when we address the convergence of a series:

(a) S_n converges; then we say that $\sum\limits_{n=1}^{\infty} a_n$ is **convergent.**

(b) S_n diverges to $\pm\infty$; then we say that $\sum\limits_{n=1}^{\infty} a_n$ is **divergent.**

(c) S_n has no limit (e.g., is alternating); then we say that $\sum\limits_{n=1}^{\infty} a_n$ is **not convergent.**

R The convergence of a series is not affected by altering (adding, removing, changing...) a *finite* number of its terms. However, if it converges, the sum does change.

■ **Example 4.2** The convergence of the series

$$\sum_{n=r}^{\infty} x^n$$

in terms of x is exactly the same as that of the geometric series. The removal of the first r terms in the latter does not affect its character. In this case it is particularly evident because

$$\sum_{n=r}^{\infty} x^n = \sum_{n=0}^{\infty} x^{r+n} = x^r \sum_{n=0}^{\infty} x^n,$$

so both series are proportional to each other. In particular this implies

$$\sum_{n=r}^{\infty} x^n = \frac{x^r}{1-x}, \qquad |x| < 1, \tag{4.6}$$

so the sum is different. ■

■ **Example 4.3** A small variation of the geometric series is the arithmetic-geometric series. It is defined as

$$\sum_{n=1}^{\infty} nx^n.$$

(Starting at $n = 1$ or $n = 0$ is irrelevant because for $n = 0$ the corresponding term $nx^n = 0$.) We proceed through a similar argument. Let

$$S_k = \sum_{n=1}^{k} nx^n = x + 2x^2 + 3x^3 + \cdots + (k-1)x^{k-1} + kx^k.$$

To begin with, if $x = 1$ then

$$S_k = 1 + 2 + 3 + \cdots + (k-1) + k = \frac{k(k+1)}{2}.$$

In this case $\lim_{k\to\infty} S_k = \infty$.

Suppose now that $x \neq 1$ and multiply S_k by x. Then

$$xS_k = x^2 + 2x^3 + 3x^4 + \cdots + (k-1)x^k + kx^{k+1}.$$

Then, substracting

$$\begin{aligned} S_k - xS_k &= \left(x + 2x^2 + 3x^3 + \cdots + kx^k\right) - \left(x^2 + 2x^3 + \cdots + (k-1)x^k + kx^{k+1}\right) \\ &= x + x^2 + x^3 + \cdots + x^k - kx^{k+1}. \end{aligned}$$

The positive terms in the right-hand side form the partial sum of the geometric —except for the first term, which is missing. Then

$$(1-x)S_k = \frac{x^{k+1}-1}{x-1} - 1 - kx^{k+1} = \frac{x^{k+1} - x - kx^{k+2} + kx^{k+1}}{x-1} = \frac{x - (k+1)x^{k+1} + kx^{k+2}}{1-x}$$

and therefore

$$S_k = \frac{x - (k+1)x^{k+1} + kx^{k+2}}{(1-x)^2}.$$

Now, if $x > 1$ the partial sum S_k diverges when $k \to \infty$ because of the term kx^{k+2}. If $x < -1$ the partial sum S_k does not have a limit when $k \to \infty$ because kx^{k+2} alternates sign and grows indefinitely in size. Finally, if $|x| < 1$ then

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{x - (k+1)x^{k+1} + kx^{k+2}}{(1-x)^2} = \frac{x}{(1-x)^2}.$$

All this is summarized in the equation

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}, \qquad |x| < 1. \tag{4.7}$$

■

Exercise 4.1 Using a similar procedure, prove that

$$(1-x)\sum_{n=1}^{k} n^2x^n = 2\sum_{n=1}^{k} nx^n - \sum_{n=1}^{k} x^n - k^2x^{k+1}.$$

By taking the limit in this expression, finally prove that

$$\sum_{n=1}^{\infty} n^2x^n = \frac{x+x^2}{(1-x)^3}, \qquad |x|<1. \tag{4.8}$$

Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series. This means that $\lim_{n\to\infty} S_n$ exists. But clearly $a_n = S_n - S_{n-1}$, therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = 0.$$

In other words:

Proposition 4.1.1 If $\sum_{n=1}^{\infty} a_n$ is a convergent series then $\lim_{n\to\infty} a_n = 0$.

Or in a form that turns out to be more useful: if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ does not converge (it either diverges or not converges).

■ **Example 4.4** The series

$$1-1+1-1+1-1+\cdots = \sum_{n=0}^{\infty}(-1)^n$$

does not converge because $a_n = (-1)^n$ does not tend to zero as $n\to\infty$ (as a matter of fact, it does not even converge because it is alternating).

This is another case in which all can be told from the expresion of the partial sum, because

$$S_k = \begin{cases} 1 & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

So S_k itself is alternating, and therefore not convergent. ■

■ **Example 4.5** The series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$$

does not converge (in fact it diverges to $+\infty$) because

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1 \neq 0.$$

■

Unfortunately the converse of Proposition 4.1.1 is not true (otherwise telling the convergence of a series would be a trivial matter), as the following example illustrates:

■ **Example 4.6**

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n}=\infty.$$

The reason is that $S_k \sim \log k$, as we know from Example 3.19. However, as we can see,

$$\lim_{n\to\infty}\frac{1}{n}=0.$$

■

4.2 Series of nonnegative terms

A **series of nonnegative terms** is a series $\sum_{n=1}^{\infty} a_n$ such that $a_n \geqslant 0$ for all $n \in \mathbb{N}$.

What is most relevant of this kind of series is that *the sequence of partial sums is monotically increasing,* because $S_k - S_{k-1} = a_k \geqslant 0$. Therefore either the partial sums are bounded above —and then the series converges— or they are unbounded —and then the series diverges to $+\infty$.

Because of this property, every subsequence of $\{S_k\}_{k=1}^{\infty}$ will also be monotically increasing and convergent to the same limit (or divergent it the series diverges). In other words: in this kind of series *convergence can be decided on any subsequence of partial sums.* Often this simplifies the problem.

A further consequence is that series, interpreted as "infinite sums", satisfy the conmutative and associative properties. This is sucintly captured by the term *inconditional convergence.* Series of nonnegative terms are inconditionally convergent.

Based on these facts there is a set of tests to check convergence of a series of nonnegative terms. Of the very many that can be found in the literature, we will simply list here the most common ones.

Theorem 4.2.1 — Comparison test. Let $0 \leqslant a_n \leqslant b_n$ for all $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} b_n < \infty \qquad \Rightarrow \qquad \sum_{n=1}^{\infty} a_n < \infty.$$

Alternatively,

$$\sum_{n=1}^{\infty} a_n = \infty \qquad \Rightarrow \qquad \sum_{n=1}^{\infty} b_n = \infty.$$

Proof. Since $\sum_{n=1}^{k} a_n \leqslant \sum_{n=1}^{k} b_n \leqslant \sum_{n=1}^{\infty} b_n < \infty$, the sequence of partial sums of $\{a_n\}_{n=1}^{\infty}$ is bounded above. ■

The requirement $0 \leqslant a_n \leqslant b_n$ for all $n \in \mathbb{N}$ is too strict. We can relax it to $0 \leqslant a_n \leqslant b_n$ for all $n > N$ (for some $N \in \mathbb{N}$). The reason is that what happens to a finite number of terms (the first N ones) is irrelevant for the convergence of the series.

Exercise 4.2 Show that $\sum_{n=0}^{\infty}\frac{1}{n!} < \infty$ by comparing this series with the geometric series. (Note that $n! > 2^n$ for $n > 3$.) ■

Exercise 4.3 Show that $\sum_{n=2}^{\infty} \frac{1}{\log n} = \infty$ by comparing this series with the harmonic series. ■

Theorem 4.2.2 — Condensation test. Let $\{a_n\}_{n=1}^{\infty}$ be a monotonically decreasing sequence of nonnegative terms, and let $q_0 < q_1 < \cdots < q_k < \cdots$ be a strictly increasing sequence of natural numbers. Then

$$\sum_{k=0}^{\infty}(q_{k+1}-q_k)a_{q_k} < \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n < \infty \quad \text{and} \quad \sum_{k=0}^{\infty}(q_{k+1}-q_k)a_{q_{k+1}} = \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n = \infty.$$

Proof. We can split the partial sum $S_{q_{p+1}} = \sum_{n=1}^{q_{p+1}} a_n$ in blocks as

$$S_{q_{p+1}} = S_{q_0} + (S_{q_1} - S_{q_0}) + (S_{q_2} - S_{q_1}) + \cdots + (S_{q_{p+1}} - S_{q_p}).$$

Then

$$S_{q_{k+1}} - S_{q_k} = a_{q_k+1} + a_{q_k+2} + \cdots + a_{q_{k+1}} \leqslant (q_{k+1} - q_k)a_{q_k}$$

because the sequence a_n is monotonically decreasing and thus a_{q_k} is an upper bound to all terms in the sum. Therefore

$$S_{q_{p+1}} \leqslant S_{q_0} + \sum_{k=0}^{p}(q_{k+1} - q_k)a_{q_k},$$

so $\sum_{k=0}^{\infty}(q_{k+1} - q_k)a_{q_k} < \infty$ implies $\sum_{n=1}^{\infty} a_n < \infty$ because of the comparison test. Likewise, as $a_{q_{k+1}}$ is a lower bound to all terms in the sum, $S_{q_{k+1}} - S_{q_k} \geqslant (q_{k+1} - q_k)a_{q_{k+1}}$, so

$$S_{q_0} + \sum_{k=0}^{p}(q_{k+1} - q_k)a_{q_{k+1}} \leqslant S_{q_{p+1}}$$

and hence $\sum_{k=0}^{\infty}(q_{k+1} - q_k)a_{q_{k+1}} = \infty$ implies $\sum_{n=1}^{\infty} a_n = \infty$, again as a consequence of the comparison test. ■

As a corollary to this theorem we can obtain a simpler version that commonly appears in textbooks as *Cauchy's condensation test*. It just amounts to taking the sequence of integers $q_k = 2^k$ in the previous theorem. As $q_{k+1} - q_k = 2^k$, then $(q_{k+1} - q_k)a_{q_k} = 2^k a_{2^k}$ and $(q_{k+1} - q_k)a_{q_{k+1}} = 2^k a_{2^{k+1}} = \frac{1}{2}2^{k+1}a_{2^{k+1}}$. Hence both "condensed" series are one and the same. As a matter of fact, the choice of 2 is made for purely historical reasons because any other natural number $m > 1$ will do—notice that $m^{k+1} - m^k = (m-1)m^k = m^{-1}(m-1)m^{k+1}$.

Corollary 4.2.3 — Cauchy's condensation test. Let $\{a_n\}_{n=1}^{\infty}$ be a monotonically decreasing sequence of nonnegative terms. Then

$$\sum_{k=0}^{\infty} 2^k a_{2^k} < \infty \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} a_n < \infty.$$

■ **Example 4.7** We will apply Cauchy's condensation test to decide the convergence of Riemann's (or the generalised harmonic) series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}.$$

To this purpose we compute

$$\sum_{k=0}^{\infty} \frac{2^k}{2^{k\alpha}} = \sum_{k=0}^{\infty} \frac{1}{2^{k(\alpha-1)}} = \sum_{k=0}^{\infty} \left(\frac{1}{2^{\alpha-1}}\right)^k .$$

This is the geometric series, which we know is convergent if and only if

$$\frac{1}{2^{\alpha-1}} < 1 \qquad \Leftrightarrow \qquad 2^{\alpha-1} > 1 \qquad \Leftrightarrow \qquad \alpha > 1.$$

Therefore Riemann's series converges if and only if $\alpha > 1$. ■

■ **Example 4.8** Cauchy's condensation test is particularly useful when logarithms are involved. For instance, consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^\alpha}$$

and compute

$$\sum_{k=1}^{\infty} \frac{2^k}{2^k(k\log 2)^\alpha} = \frac{1}{(\log 2)^\alpha} \sum_{k=1}^{\infty} \frac{1}{k^\alpha},$$

which, up to a constant, is Riemann's series. Therefore the tested series converges if and only if $\alpha > 1$. ■

One of the most powerful tests is the *limit comparison test*. The idea behind it is that two series that behave similarly as $n \to \infty$ have the same convergence properties. So the notions of equivalent and negligible sequences adquire a special relevance here.

Theorem 4.2.4 — Limit comparison test. Given the series of nonnegative terms $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} b_n$:

(a) If $a_n \sim b_n$ then

$$\sum_{n=1}^{\infty} a_n < \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} b_n < \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} a_n = \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} b_n = \infty.$$

(b) If $a_n \ll b_n$ then

$$\sum_{n=1}^{\infty} b_n < \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n < \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} a_n = \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} b_n = \infty.$$

Proof.

(a) $a_n \sim b_n$ means $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 1$. Thus, according to the definition of limit, given $\varepsilon > 0$,

$$-\varepsilon < \frac{a_n}{b_n} - 1 < \varepsilon \quad \Leftrightarrow \quad 1-\varepsilon < \frac{a_n}{b_n} < 1+\varepsilon \quad \Leftrightarrow \quad (1-\varepsilon)b_n < a_n < (1+\varepsilon)b_n$$

for large enough n. If we take $\varepsilon < 1$, (a) follows from the last inequality by the comparison test.

(b) $a_n \ll b_n$ means $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$. Thus, given $\varepsilon > 0$,

$$-\varepsilon < \frac{a_n}{b_n} < \varepsilon \quad \Leftrightarrow \quad -\varepsilon b_n < a_n < \varepsilon b_n$$

for large enough n. Hence (b) follows from $a_n < \varepsilon b_n$ by the comparison test (the inequality $-\varepsilon b_n < a_n$ is true but useless).

■

■ **Example 4.9** In order to know if the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n^2+2n+7}}$$

converges or diverges, all we need to know is that $3n^2+2n+7 \sim 3n^2$, so $\sqrt{3n^2+2n+7} \sim \sqrt{3}n$. Since the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3}n} = \frac{1}{\sqrt{3}} \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

because it is the harmonic series, then the series we are testing also diverges. ■

Exercise 4.4 Prove, using appropriately the limit comparison test, that if $a_n \geqslant 0$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} n^{\alpha} a_n = \ell \geqslant 0,$$

then $\sum_{n=1}^{\infty} a_n < \infty$ if $\alpha > 1$ and $\sum_{n=1}^{\infty} a_n = \infty$ if $\alpha \leqslant 1$. ■

Theorem 4.2.5 — Root test. If $a_n \geqslant 0$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \ell \geqslant 0,$$

then $\sum_{n=1}^{\infty} a_n < \infty$ if $\ell < 1$ and $\sum_{n=1}^{\infty} a_n = \infty$ if $\ell > 1$. (The case $\ell = 1$ remains undecided.)

Proof. Given $\ell > \varepsilon > 0$,

$$-\varepsilon < \sqrt[n]{a_n} - \ell < \varepsilon \quad \Leftrightarrow \quad \ell - \varepsilon < \sqrt[n]{a_n} < \ell + \varepsilon \quad \Leftrightarrow \quad (\ell-\varepsilon)^n < a_n < (\ell+\varepsilon)^n$$

for large enough n. Now, if $\ell < 1$ we can take ε so that $\ell + \varepsilon < 1$ (e.g., $\varepsilon = (1-\ell)/2$). Therefore the geometric series

$$\sum_{n=1}^{\infty} (\ell+\varepsilon)^n < \infty$$

and $\sum_{n=1}^{\infty} a_n < \infty$ by the comparison test. On the contrary, if $\ell > 1$ we can take ε so that $\ell - \varepsilon > 1$ (e.g., $\varepsilon = (\ell-1)/2$). Hence

$$\sum_{n=1}^{\infty} (\ell+\varepsilon)^n = \infty$$

and $\sum_{n=1}^{\infty} a_n = \infty$ by the comparison test.

If $\ell = 1$ neither inequality is useful and we can conclude nothing. ■

Corollary 3.4.4 to Stolz's theorem transforms this test into another one:

Corollary 4.2.6 — Quotient test. If $a_n \geqslant 0$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \ell \geqslant 0,$$

then $\sum_{n=1}^{\infty} a_n < \infty$ if $\ell < 1$ and $\sum_{n=1}^{\infty} a_n = \infty$ if $\ell > 1$. (The case $\ell = 1$ remains undecided.)

4.3 Alternating series

When the terms of a series can be either positive or negative, the test developed in the previous section are no longer valid. However, these series can be classified into two main groups according to whether they are *absolutely convergent* or not.

Definition 4.3.1 — Absolutely converget series. $\sum_{n=1}^{\infty} a_n$ is said to be an absolutely convergent series if $\sum_{n=1}^{\infty} |a_n| < \infty$.

It is easy to prove that absolutely convergent series are also convergent in the usual sense. Therefore all tests for series of nonnegative terms can be applied to the series of absolute values.

Dirichlet proved that absolutely convergent series converge inconditionally. On the contrary, series that do not converge absolutely are conditionally convergent. This means that a permutation and/or association of their terms can change their sum (this result was proven by Riemann). Their interpretation as "infinite sums" is thus weaker than that of absolutely convergent series, because an "order of sum" must be specified in advance (as a matter of fact, this is what the definition of a convergent series does).

■ **Example 4.10** Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

We will show later, in Example 4.11, that this series converges, but clearly does not do it absolutely (because the series of absolute values is the harmonic series). Let us denote S its sum. Then

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Let us reorder its terms by choosing one positive followed by two negative terms, in order, and then associate each positive with the first subsequent negative. We obtain

$$\begin{aligned} S' &= \underbrace{\left(1 - \frac{1}{2}\right)}_{=1/2} - \frac{1}{4} + \underbrace{\left(\frac{1}{3} - \frac{1}{6}\right)}_{=1/6} - \frac{1}{8} + \underbrace{\left(\frac{1}{5} - \frac{1}{10}\right)}_{=1/10} - \frac{1}{12} + \cdots \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right) = \frac{S}{2}, \end{aligned}$$

so with this manipulation the series sums half its initial value. ■

Among the series with arbitrary sign patterns the most frequently met are those with alternating signs. These are referred to as **alternating series.** They can have either of the two forms

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n, \qquad \sum_{n=1}^{\infty} (-1)^n a_n, \tag{4.9}$$

where $a_n \geqslant 0$.

R Note that $\sum_{n=1}^{\infty}(-1)^n a_n = -\sum_{n=1}^{\infty}(-1)^{n+1}a_n$, so the two forms are actually equivalent.

There is only one test for alternating series:

Theorem 4.3.1 — Leibniz test. If $\{a_n\}_{n=1}^{\infty}$ decreases monotonically and $\lim_{n\to\infty} a_n = 0$, then the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n \tag{4.10}$$

converges.

■ **Example 4.11** The series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

converges according to Leibniz's test because $a_n = 1/n$ is a monotonically decreasing sequence that approaches 0 as $n \to \infty$. ■

4.4 Telescoping series

A series $\sum_{n=1}^{\infty} a_n$ is said to be **telescoping** if there exists a sequence $\{u_n\}_{n=1}^{\infty}$ such that $a_n = u_n - u_{n+1}$ for all $n \in \mathbb{N}$.

The importance of telescoping series is that they can be easily summed. The reason is that the partial sum

$$S_k = \sum_{n=1}^{k} a_n = \sum_{n=1}^{k}(u_n - u_{n+1}) = u_1 - u_{k+1},$$

therefore we have the simple formula

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(u_n - u_{n+1}) = u_1 - \lim_{n\to\infty} u_n. \tag{4.11}$$

■ **Example 4.12** Consider the series

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}.$$

Since we can expand

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

the series telescopes identifying $u_n = 1/n$, so

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \lim_{n\to\infty}\frac{1}{n} = 1.$$

■

■ **Example 4.13** Consider the series

$$\sum_{n=1}^{\infty} \log\left(1+\frac{1}{n}\right).$$

Since

$$\log\left(1+\frac{1}{n}\right) = \log\left(\frac{n+1}{n}\right) = \log(n+1) - \log n$$

the series telescopes identifying $u_n = -\log n$, so

$$\sum_{n=1}^{\infty} \log\left(1+\frac{1}{n}\right) = 0 + \lim_{n\to\infty} \log n = \infty.$$

The series diverges to $+\infty$. ■

■ **Example 4.14** Consider the series

$$\sum_{n=1}^{\infty} \arctan \frac{1}{1+n+n^2}.$$

The key to show that this series telescopes is to realise that

$$\frac{1}{1+n+n^2} = \frac{1}{1+n(n+1)} = \frac{(n+1)-n}{1+n(n+1)},$$

and to compare this formula with the trigonometric identity

$$\tan(x-y) = \frac{\tan x - \tan y}{1+\tan x \tan y},$$

which allows us to indentify $x = \arctan(n+1)$ and $y = \arctan n$. In other words,

$$\arctan \frac{1}{1+n+n^2} = \arctan\left(\frac{(n+1)-n}{1+n(n+1)}\right) = \arctan(n+1) - \arctan n$$

and the series telescopes identifying $u_n = -\arctan n$. Therefore

$$\sum_{n=1}^{\infty} \arctan \frac{1}{1+n+n^2} = -\arctan 1 + \lim_{n\to\infty} \arctan n = -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}.$$

■

To be honest, every series is telescoping because

$$S_k = \sum_{n=1}^{k} a_n = \sum_{n=1}^{k} (u_n - u_{n+1}) = u_1 - u_{k+1},$$

so we can identify $u_k \equiv u_1 - S_{k-1}$, and choose u_1 arbitrarily. The problem is that this is tantamount to being able to calculate S_k —usually a difficult problem. So we really call telescoping those series for which the identification $a_n = u_n - u_{n+1}$ is more or less explicit —as in the previous examples.

Problems

Problem 4.1 Determine the convergent character of the following series of nonnegative terms:

(i) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n+1}{2n-1}\right)^n$;

(ii) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(3n-1)^2}$;

(iii) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n^4+1}}$;

(iv) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)}}$;

(v) $\displaystyle\sum_{n=1}^{\infty}\frac{|\sin n|}{n^2+n}$;

(vi) $\displaystyle\sum_{n=1}^{\infty}\sin\frac{1}{n^2}$;

(vii) $\displaystyle\sum_{n=1}^{\infty}\arcsin\frac{1}{\sqrt{n}}$;

(viii) $\displaystyle\sum_{n=1}^{\infty}\frac{3n-1}{(\sqrt{2})^n}$;

(ix) $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{3^n n!}$;

(x) $\displaystyle\sum_{n=1}^{\infty}(\sqrt[n]{n}-1)^n$;

(xi) $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}3^{-n}$;

(xii) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{(\log n)^n}$;

(xiii) $\displaystyle\sum_{n=2}^{\infty}\frac{n^2}{(\log n)^n}$;

(xiv) $\displaystyle\sum_{n=1}^{\infty}\left(\sqrt{n^2+1}-n\right)$;

(xv) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^{\log n}}$;

(xvi) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{(\log n)^{\log n}}$;

(xvii) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\sqrt[n]{n}}$;

(xviii) $\displaystyle\sum_{n=2}^{\infty}\left(\frac{n}{n-1}\right)^n$.

Problem 4.2 Prove that the series

$$\sum_{n=1}^{\infty}\left(\frac{a}{2n-1}-\frac{b}{2n+1}\right)$$

converges if, and only if, $a=b$, and in that case calculate its sum.

Problem 4.3 Discuss, depending on the value of the parameter a in the given range, whether the following series converge or diverge:

(i) $\displaystyle\sum_{n=1}^{\infty}n(1+a)^n e^{-na}$, for $a>-1$;

(ii) $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{a^n n!}$, for $a>0$;

(iii) $\displaystyle\sum_{n=1}^{\infty}\frac{n!e^n}{n^{n+a}}$, for any $a\in\mathbb{R}$;

(iv) $\displaystyle\sum_{n=1}^{\infty}\frac{a^n}{(1+a)(1+a^2)\cdots(1+a^n)}$, for $a\geqslant 0$.

Problem 4.4 Determine whether the following series are absolutely convergent, and if not, whether they converge conditionally:

(i) $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{\log n}$;

(ii) $\displaystyle\sum_{n=1}^{\infty}\sin\left(\pi n+\frac{1}{n}\right)$;

(iii) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\arctan\frac{1}{n}\right)^2$;

(iv) $\displaystyle\sum_{n=1}^{\infty}(-1)^n(\arctan n)^2$;

(v) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{n^2-1}-n\right)$;

(vi) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\log\left(\frac{n}{n+1}\right)$;

(vii) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(1-\cos\frac{1}{n}\right)$;

(viii) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\log(e^n+e^{-n})}$.

Problem 4.5 Sum the following series:

(i) $\sum_{n=0}^{\infty} \frac{3^{n+1}-2^{n-3}}{4^n}$;

(ii) $\sum_{n=1}^{\infty} \frac{n}{2^n}$;

(iii) $\sum_{n=0}^{\infty} \frac{4n+1}{3^n}$;

(iv) $\sum_{n=1}^{\infty} \left(\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n}\right)$;

(v) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n(n+1)}}$;

(vi) $\sum_{n=1}^{\infty} \log\left[\frac{n(n+2)}{(n+1)^2}\right]$;

(vii) $\sum_{n=0}^{\infty} x^{\lfloor \frac{n}{2} \rfloor} y^{\lfloor \frac{n+1}{2} \rfloor}$, with $|xy|<1$;

(viii) $\sum_{n=0}^{\infty} \frac{1}{2^n} \cos\frac{2\pi n}{3}$.

Problem 4.6 Obtain the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3+3n^2+2n}$ by rewriting it as a telescoping series.
HINT: Expand the general term in elementary fractions.

Problem 4.7 Let $\mathscr{C}_0$ be a circle of radius r. Let $\mathscr{Q}_0$ be a square inscribed in $\mathscr{C}_0$. Let $\mathscr{C}_1$ be the circle inscribed in $\mathscr{Q}_0$, and $\mathscr{Q}_1$ a square inscribed in $\mathscr{C}_1$. Continue the process this way and obtain the sequence of circles $\{\mathscr{C}_n\}_{n=0}^{\infty}$ with radii $\{r_n\}_{n=0}^{\infty}$. What is the sum of the areas of these infinitely many circles?

Problem 4.8 Calculate $\lim_{n\to\infty} a_n$, where $a_n = \sqrt{2}\sqrt[4]{2}\sqrt[8]{2}\cdots\sqrt[2^n]{2}$.
HINT: Calculate the limit of $\log a_n$ first.

Problem 4.9 Let $b_0 \in \mathbb{Z}$, $b_n \in \{0,1,2,\dots,9\}$, for $n=1,2,\dots$, and form the series

$$\sum_{n=0}^{\infty} \frac{b_n}{10^n}.$$

(a) Prove that this series converges.

(b) Discuss the meaning of this series and why it is so important.

(c) Calculate its sum for $b_n = 9$ for all $n \geqslant 0$.

(d) Calculate its sum if $b_n = 1$ for n even and $b_n = 2$ for n odd.

Problem 4.10

(a) Prove (graphically or otherwise) that the equation $\tan x = x$ has a solution $(2n-1)\frac{\pi}{2} < \lambda_n < (2n+1)\frac{\pi}{2}$ for every $n \in \mathbb{N}$.

(b) Prove that $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty$.

Problem 4.11

(a) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ two convergent series of nonnegative terms. Prove that $\sum_{n=1}^{\infty} \sqrt{a_n b_n} < \infty$.
HINT: Use the inequality $xy \leqslant (x^2+y^2)/2$.

(b) As an application prove that if the series of nonnegative terms $\sum_{n=1}^{\infty} a_n < \infty$ then $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n} < \infty$.

Problem 4.12 Let $\{u_n\}_{n=1}^{\infty}$ be the sequence of all positive integers containing *no zeros* in their decimal expression.

(a) Prove that $\sum_{n=1}^{\infty} \frac{1}{u_n} < 90$.
HINT: Group all terms u_n with the same number of decimal digits.

(b) What can you say about the series $\sum_{n=1}^{\infty} \frac{1}{w_n}$, where $\{w_n\}_{n=1}^{\infty}$ is the sequence of all positive integers containing *at least one zero* in their decimal expression?

Problem 4.13 In a real *tour-de-force* we are going to calculate the —apparently impossible— sum of the conditionally convergent series

$$\sum_{n=1}^{\infty}(-1)^n \log\left(\frac{n}{n+1}\right).$$

We will do that in steps:

(a) Show that

$$2\cdot 4\cdot 6\cdots(2n) = n!2^n, \qquad 1\cdot 3\cdot 5\cdots(2n-1) = \frac{(2n)!}{n!2^n}.$$

(b) Use Stirling to prove

$$2\cdot 4\cdot 6\cdots(2n) \sim \sqrt{2\pi n}e^{-n}(2n)^n, \qquad 1\cdot 3\cdot 5\cdots(2n-1) \sim \sqrt{2}e^{-n}(2n)^n.$$

(c) Show that the partial sum S_{2k} of the series above can be written

$$S_{2k} = \log\left(\frac{2^2\cdot 4^2\cdot 6^2\cdots(2k)^2}{1\cdot 3^2\cdot 5^2\cdots(2k-1)^2(2k+1)}\right).$$

(d) Use the Stirling formulas derived above to calculate the limit of S_{2k} when $k\to\infty$. Why does this provide the answer to the problem?

Problem 4.14 Suppose a certain series can be written as

$$\sum_{n=1}^{\infty}(\alpha_0 u_n + \alpha_1 u_{n+1} + \alpha_2 u_{n+2}), \qquad \alpha_0+\alpha_1+\alpha_2 = 0.$$

(a) Rewrite the general term as an ordinary telescoping series and provide a formula for the sum.

(b) Apply this result to calculate the sum

$$\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)(n+2)}.$$

(c) Do the same for the general case

$$\sum_{n=1}^{\infty}(\alpha_0 u_n + \alpha_1 u_{n+1} + \cdots + \alpha_k u_{n+k}), \qquad \sum_{j=0}^{k}\alpha_j = 0.$$

HINT: In (a), add and substract $\alpha_0 u_{n+1}$ and replace $\alpha_2 = -(\alpha_0+\alpha_1)$. Use a similar procedure in (c).

III Differential Calculus

5. Limit of a Function

5.1 Concept and definition

Functions are defined for every single point of their domains. However, differential calculus has to do with the behaviour of functions "around" points, not just at them. The limit of a function is a way to characterise that behavior. The idea is to know what value the function is approaching as we get closer and closer to a certain point a (not necessarily in the domain of the function).

Our first definition will base this knowledge in the well-known limit of sequences.

Definition 5.1.1 We say that the **limit of a function** $f : A \to \mathbb{R}$ when x approaches a is ℓ, and denote it

$$\lim_{x \to a} f(x) = \ell, \tag{5.1}$$

if, *for every sequence* $\{x_n\}_{n=1}^{\infty} \subset A$ such that $x_n \neq a$ for all $n \in \mathbb{N}$ and

$$\lim_{n \to \infty} x_n = a,$$

it holds

$$\lim_{n \to \infty} f(x_n) = \ell.$$

Having a sequence in the domain that tends to a as $n \to \infty$ is a way to approach a. The condition $x_n \neq a$ is there to account for those cases in which $a \notin A$. Note the remark "for every sequence" in the definition. It is very important because if it holds, then what $f(x)$ tends to does not depend on how we approach a.

■ **Example 5.1** Consider the function $f(x) = x^2$ and the point $a = 2$ (in the domain). Take the sequence $x_n = 2 + \varepsilon_n$. As long as $\varepsilon_n \neq 0$ for every $n \in \mathbb{N}$ and $\varepsilon_n \to 0$ as $n \to \infty$, this sequence satisfies the conditions of the definition. Then

$$f(x_n) = (2 + \varepsilon_n)^2 = 4 + 2\varepsilon_n + \varepsilon_n^2 \xrightarrow[n \to \infty]{} 4.$$

This proves that

$$\lim_{x\to 2} x^2 = 4.$$

■

■ **Example 5.2** The previous example might suggest that calculating a limit could be as simple as evaluating $f(a)$. To show that this is not always the case consider the function

$$f(x) = \frac{x-1}{x^2-1},$$

a rational function whose domain is $\mathbb{R} - \{1\}$. Take any sequence $x_n = 1+\varepsilon_n$, with $\varepsilon_n \neq 0$. If $\varepsilon_n \to 0$ as $n \to \infty$ then $x_n \to 1$. Now,

$$f(x_n) = \frac{\varepsilon_n}{(1+\varepsilon_n)^2-1} = \frac{\varepsilon_n}{2\varepsilon_n+\varepsilon_n^2} = \frac{1}{2+\varepsilon_n} \xrightarrow[n\to\infty]{} \frac{1}{2}.$$

This proves that

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = \frac{1}{2}$$

even though 1 is *not* in the domain of f (hence $f(1)$ does not even exists). ■

■ **Example 5.3** For a final illustrating example consider the function $f(x) = \sin(\pi/x)$, whose domain is $\mathbb{R} - \{0\}$, and take $a = 0$ (not in the domain). Consider the sequence $x_n = 1/n$, satisfying the requirements of the definition. Now,

$$f(x_n) = \sin(\pi n) = 0$$

for all $n \to \infty$. So, if the limit exists, it has to be 0.

But now consider the sequence $y_n = 2/(4n+1)$. Then,

$$f(y_n) = \sin\left(\pi\frac{4n+1}{2}\right) = \sin\left(\pi\left(2n+\frac{1}{2}\right)\right) = \sin\left(2n\pi+\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1.$$

So we have found two different sequences, x_n and y_n, such that the two limits, $f(x_n)$ and $f(y_n)$ exist, but are different. So $f(x)$ does not have a limit when $x \to 0$. ■

Checking that the limit exists for every conceivable sequence might be a daunting task. For this reason there is this alternative (but equivalent) definition of limit, which is not based on sequences and is more widely used.

Definition 5.1.2 — ε-δ definition. We say that the **limit of a function** $f : A \to \mathbb{R}$ when x approaches a is ℓ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x)-\ell| < \varepsilon$ whenever $0 < |x-a| < \delta$.

The idea of this definition is that for every $\varepsilon > 0$ you can find a $\delta > 0$ such that $(a-\delta, a) \cup (a, a+\delta) \subset f^{-1}\big((\ell-\varepsilon, \ell+\varepsilon)\big)$ (see Figure 5.1).

As with the limit of sequences (and for the same reason), if the limit exists it is unique. In other words, if the limit of $f(x)$ when $x \to a$ is both ℓ and m, then $\ell = m$.

5.1.1 One-sided limits

There is a difference between the limit when $n \to \infty$ and the limit when $x \to a$: in the former case we can only "approach ∞ from the left", whereas in the latter case we can approach a both from the left ($x < a$) or from the right ($x > a$). This motivate the definition of one-sided limit.

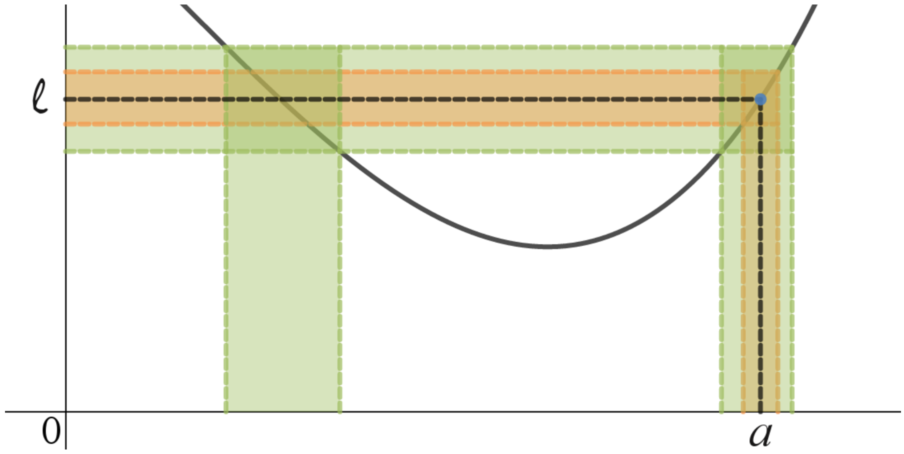


Figure 5.1: Sketch for the ε-δ definition of limit. The horizontal green stripe marks the interval $(\ell-\varepsilon, \ell+\varepsilon)$, so the two vertical stripes correspond to $f^{-1}\big((\ell-\varepsilon,\ell+\varepsilon)\big)$ —the set of points whose image is $(\ell-\varepsilon,\ell+\varepsilon)$. It is pictorially obvious that, no matter how narrow is the band, we can always construct an interval $(a-\delta, a+\delta)$ (vertical orange stripe) such that its image through f (except maybe that of a itself) (horizontal orange stripe) is contained in $(\ell-\varepsilon,\ell+\varepsilon)$.

Definition 5.1.3 — One-sided limit. We say that the **left-handed limit of a function** $f: A \to \mathbb{R}$ when x approaches a is ℓ, and denote it

$$\lim_{x\to a^-} f(x) = \ell, \tag{5.2}$$

if for every sequence $\{x_n\}_{n=1}^{\infty} \subset A$, such that $x_n < a$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = a$, we have

$$\lim_{n\to\infty} f(x_n) = \ell.$$

Similarly, we say that the **right-handed limit of a function** $f: A \to \mathbb{R}$ when x approaches a is ℓ, and denote it

$$\lim_{x\to a^+} f(x) = \ell, \tag{5.3}$$

if for every sequence $\{x_n\}_{n=1}^{\infty} \subset A$, such that $x_n > a$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = a$, we have

$$\lim_{n\to\infty} f(x_n) = \ell.$$

■ **Example 5.4** The Heaviside step function is defined as

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0, \end{cases} \tag{5.4}$$

and $H(0)$ is defined arbitrarily —if at all. Clearly for this function

$$\lim_{x\to 0^-} H(x) = 0, \qquad \lim_{x\to 0^+} H(x) = 1.$$

The two one-sided limits exist, but they are different. Clearly then $H(x)$ has no limit when $x \to 0$. ■

Proposition 5.1.1

$$\lim_{x\to a} f(x) = \ell \qquad \Leftrightarrow \qquad \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \ell.$$

5.1.2 Infinite limits

We can adapt the definitions above to describe a function that grows unbounded when $x \to a$. For instance:

Definition 5.1.4 — Infinite limits. We say that the limit of a function $f : A \to \mathbb{R}$ when x approaches a is $+\infty$ (respectively $-\infty$), and denote it

$$\lim_{x\to a} f(x) = +\infty \qquad \text{(respectively } \lim_{x\to a} f(x) = -\infty\text{)}, \tag{5.5}$$

if for every sequence $\{x_n\}_{n=1}^{\infty} \subset A$, such that $x_n \neq a$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = a$, we have

$$\lim_{n\to\infty} f(x_n) = +\infty \qquad \text{(respectively } \lim_{n\to\infty} f(x_n) = -\infty\text{)}.$$

And similarly for the one-sided limits.

Example 5.5 The function $f(x) = \frac{1}{|x|}$ tends to $+\infty$ when $x \to 0$. The reason is that for every sequence $x_n \neq 0$ such that $\lim_{n\to\infty} x_n = 0$,

$$\lim_{n\to\infty} \frac{1}{|x_n|} = \infty.$$

For the same reason, the function $f(x) = \frac{1}{x}$ has a right-handed limit $+\infty$ when $x \to 0^+$. However, its left-handed limit is $-\infty$, because taking any sequence $x_n < 0$ such that $\lim_{n\to\infty} x_n = 0$,

$$\lim_{n\to\infty} \frac{1}{x_n} = -\infty.$$

Thus, $1/x$ has no limit —not even infinite— when $x \to 0$. ■

Two particularly important one-sided limits are:

Definition 5.1.5 — Limit at the infinities. We say that the limit of a function $f : A \to \mathbb{R}$ when x approaches $+\infty$ (respectively $-\infty$) is ℓ, and denote it

$$\lim_{x\to+\infty} f(x) = \ell \qquad \text{(respectively } \lim_{x\to-\infty} f(x) = \ell\text{)}, \tag{5.6}$$

if for every sequence $\{x_n\}_{n=1}^{\infty} \subset A$, such that $\lim_{n\to\infty} x_n = +\infty$ (respectively $-\infty$), we have

$$\lim_{n\to\infty} f(x_n) = \ell.$$

Exercise 5.1 Extend the definitions above to express the cases of a function that approaches $\pm\infty$ when x approaches $\pm\infty$ (all combinations of signs). ■

A function such that $f(x) \to \pm\infty$ when $x \to a^{\pm}$ is said to *diverge* at $x = a$.

Example 5.6 The function $f(x) = \frac{1}{x} \to 0$ when $x \to \pm\infty$. Let us see it for the case $x \to +\infty$. Take any sequence x_n that diverges to $+\infty$. Then

$$\lim_{n\to\infty} \frac{1}{x_n} = 0.$$

For $x \to -\infty$ the argument is similar. ■

5.2 Algebraic properties

As in the case of sequences, computing limits through the definition is not an easy task. However —as for sequences— limits satisfy a set of properties that allow us to do algebraic manipulations with limits and simplify their calculations.

Proposition 5.2.1 Let f and g be two real functions such that $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = \ell'$. Then the following properties hold:

1. $\lim_{x\to a} \left[f(x) \pm g(x)\right] = \ell + \ell'$;
2. $\lim_{x\to a} f(x)g(x) = \ell\ell'$;
3. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\ell}{\ell'}$ provided $\ell' \neq 0$.

(These properties hold also in the case $a = \pm\infty$ or even for one-sided limits.)

Proposition 5.2.2 Let f a real function such that $\lim_{x\to a} f(x) = \ell$. Then the following properties hold:

1. f is bounded in an evironment of a;
2. if $\ell \neq 0$, $f(x)$ has the same sign as ℓ in an evironment of a;
3. if g is another function such that $\lim_{x\to b} g(x) = a$, but $g(x) \neq a$ in an environment of b, then $\lim_{x\to b} (f \circ g)(x) = \ell$; in particular, if $\ell > 0$:
   (a) $\lim_{x\to a} \log f(x) = \log \ell$;
   (b) $\lim_{x\to a} f(x)^\alpha = \ell^\alpha$, for any $\alpha \in \mathbb{R}$.

R An environment of a point $a \in \mathbb{R}$ is an interval of the form $(a-\delta, a+\delta)$, where $\delta > 0$. If a property holds "in an environment" it means that there exists some value $\delta > 0$ such that the property holds within the environment $(a-\delta, a+\delta)$.

■ **Example 5.7** Obviously $\lim_{x\to a} x = a$. But from this and property 2. above we can conclude that for all $n \in \mathbb{R}$

$$\lim_{x\to a} x^n = a^n.$$

A polynomial $P_n(x)$ is a linear combination of powers x^k, $k = 0, 1, \ldots, n$. Thus, applying properties 1. and 2. it follows that

$$\lim_{x\to a} P_n(x) = P_n(a). \tag{5.7}$$

Then, applying property 3.,

$$\lim_{x\to a} \frac{P_n(x)}{Q_m(x)} = \frac{P_n(a)}{Q_m(a)}, \tag{5.8}$$

provided a is not a root of $Q_m(x)$.

Thus, calculating limits of polynomials or rational functions is a trivial matter. ■

As for sequences, we have a Sandwich rule for functional limits:

Theorem 5.2.3 — Sandwich rule. If $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = \ell$, and in an envionment of a it holds $g(x) \leqslant f(x) \leqslant h(x)$, then $\lim_{x\to a} f(x) = \ell$. (This rule is valid even if $a = \pm\infty$ or $\ell = \pm\infty$.)

A simple consequence of this sandwich rule is that

$$\lim_{x\to a}|f(x)| = 0 \qquad \Rightarrow \qquad \lim_{x\to a} f(x) = 0, \tag{5.9}$$

simply because $-|f(x)| \leqslant f(x) \leqslant |f(x)|$.

■ **Example 5.8** Let us apply the sandwich rule to calculate

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right).$$

We cannot simply apply the algebraic rules because $\sin\left(\frac{1}{x}\right)$ has no limit when $x \to 0$ (it oscillates more and more wildly between 0 and 1 as we approach 0). However,

$$-x^2 \leqslant x^2 \sin\left(\frac{1}{x}\right) \leqslant x^2$$

because for every $x \neq 0$ we have $-1 \leqslant \sin\left(\frac{1}{x}\right) \leqslant 1$. Since $\pm x^2 \to 0$ as $x \to 0$, by the sandwich rule

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

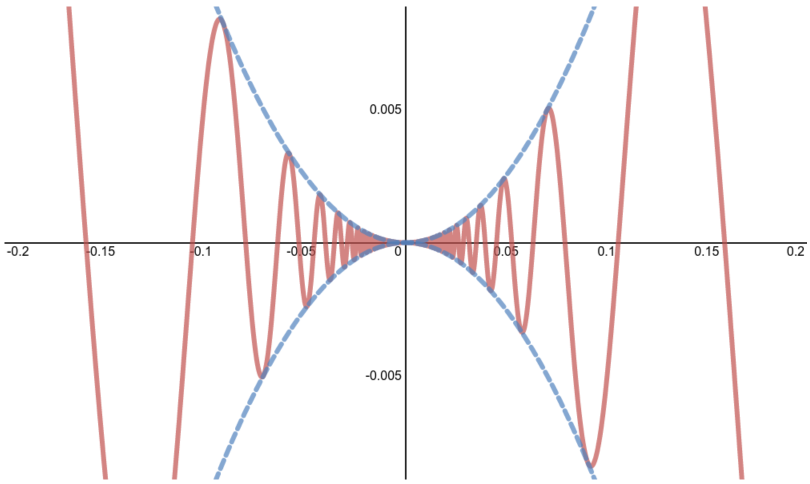


Figure 5.2: Plot of $f(x) = x^2 \sin\left(\frac{1}{x}\right)$. The dotted blue lines are plots of the two envelopes x^2 and $-x^2$.

Figure 5.2 shows a plot of $f(x)$ explaining intuitively what we have just proven analytically. ■

■ **Example 5.9** Let $a > 0$ and let us prove that

$$\lim_{x\to a} \sqrt{x} = \sqrt{a}.$$

First of all, notice that this is equivalent to proving that

$$\lim_{x\to a}\left(\sqrt{x} - \sqrt{a}\right) = 0 \qquad \Leftrightarrow \qquad \lim_{x\to a}\left|\sqrt{x} - \sqrt{a}\right| = 0$$

Now, to prove the latter we can write

$$0 \leqslant \left|\sqrt{x} - \sqrt{a}\right| = \left|\frac{x-a}{\sqrt{x}+\sqrt{a}}\right| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} \leqslant \frac{|x-a|}{\sqrt{a}} \quad \xrightarrow[x\to a]{} \quad 0,$$

and the result follows from the sandwich rule. ■

Exercise 5.2 By a similar argument prove that for all $a > 0$

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}. \tag{5.10}$$

HINT: Use the identity $a - b = (a^n - b^n)/(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$.

■ **Example 5.10** From the equivalences (3.13), (3.14), it follows that for *any* sequence $\varepsilon_n \to 0$ we have

$$\lim_{n\to\infty} \sin\varepsilon_n = 0, \qquad \lim_{n\to\infty} \cos\varepsilon_n = 1, \qquad \lim_{n\to\infty} \frac{\sin\varepsilon_n}{\varepsilon_n} = 1, \qquad \lim_{n\to\infty} \frac{1-\cos\varepsilon_n}{\varepsilon_n^2} = \frac{1}{2}.$$

By the sequential definition of limit it follows that

$$\lim_{x\to 0} \sin x = 0, \qquad \lim_{x\to 0} \cos x = 1, \qquad \lim_{x\to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}. \tag{5.11}$$

Using this limits and trigonometric identities we can easily prove that

$$\lim_{x\to a} \sin x = \sin a, \qquad \lim_{x\to a} \cos x = \cos a. \tag{5.12}$$

For instance, for the sine, setting $x = a + t$,

$$\begin{aligned} \lim_{x\to a} \sin x &= \lim_{t\to 0} \sin(a+t) = \lim_{t\to 0} (\sin a \cos t + \cos a \sin t) = \sin a \underbrace{\left(\lim_{t\to 0} \cos t\right)}_{=1} + \cos a \underbrace{\left(\lim_{t\to 0} \sin t\right)}_{=0} \\ &= \sin a, \end{aligned}$$

and similarly for the cosine. ■

5.3 Asymptotic comparison of functions

Example 5.10 makes it clear that, thanks to the sequential definition of limit, the notion of asymptotic comparison can be brought to the realm of functional limits. Hence we have:

Definition 5.3.1 — Asymptotic comparison. Let f and g be two real functions that either both diverge or both converge to 0 as $x \to a$ $(-\infty \leqslant a \leqslant \infty)$. We say that f and g are **equivalent** when $x \to a$ (and denote it $f(x) \sim g(x)$ as $x \to a$) if

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 1.$$

Accordingly, given equations (3.13) and (3.14), we can state that

$$\log(1+x) \sim (e^x - 1) \sim \sin x \sim \tan x \sim x \qquad (x \to 0), \tag{5.13}$$

$$1 - \cos x \sim \frac{x^2}{2} \qquad (x \to 0). \tag{5.14}$$

(We will later re-derive this same relations in a more systematic and natural way.)

Problems

Problem 5.1 Calculate the following limits, simplifying the common factors that numerator and denominator may contain:

(i) $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a},\ n\in\mathbb{N}$;

(ii) $\lim\limits_{x\to a}\dfrac{\sqrt{x}-\sqrt{a}}{x-a}$;

(iii) $\lim\limits_{x\to 64}\dfrac{\sqrt{x}-8}{\sqrt[3]{x}-4}$;

(iv) $\lim\limits_{x\to 0}\dfrac{1-\sqrt{1-x^2}}{x^2}$;

(v) $\lim\limits_{x\to 0}\dfrac{\frac{1}{(1-x)^3}-1}{x}$;

(vi) $\lim\limits_{x\to 1}\left(\dfrac{1}{\sqrt{x}-1}-\dfrac{2}{x-1}\right)$.

Problem 5.2 Calculate the following limits:

(i) $\lim\limits_{x\to 0}\dfrac{(\sin 2x^3)^2}{x^6}$;

(ii) $\lim\limits_{x\to 0}\dfrac{\tan x^2+2x}{x+x^2}$;

(iii) $\lim\limits_{x\to 0}\dfrac{\sin(x+a)-\sin a}{x}$;

(iv) $\lim\limits_{x\to 0}(1+x)^{1/x}$;

(v) $\lim\limits_{x\to 0}\dfrac{\log(1-2x)}{\sin x}$;

(vi) $\lim\limits_{x\to 0}(1+\sin x)^{2/x}$;

(vii) $\lim\limits_{x\to 0}\dfrac{e^x-e^{\sin x}}{x-\sin x}$;

(viii) $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$;

(ix) $\lim\limits_{x\to 0}\left(\dfrac{x}{\sin x}\right)^{\frac{\sin x}{\sin x-x}}$;

(x) $\lim\limits_{x\to 0}(\cos x)^{1/x^2}$;

(xi) $\lim\limits_{x\to \pi}\dfrac{1-\sin(x/2)}{(x-\pi)^2}$;

(xii) $\lim\limits_{x\to 0}\dfrac{a^x-b^x}{x}$.

Problem 5.3 Calculate the following limits:

(i) $\lim\limits_{x\to \infty}\dfrac{x^3+4x-7}{7x^2-\sqrt{2x^6+x^5}}$;

(ii) $\lim\limits_{x\to \infty}\dfrac{x+\sin x^3}{5x+6}$;

(iii) $\lim\limits_{x\to \infty}\dfrac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$;

(iv) $\lim\limits_{x\to \infty}\left(\sqrt{x^2+4x}-x\right)$;

(v) $\lim\limits_{x\to \pm\infty}\dfrac{e^x}{e^x-1}$;

(vi) $\lim\limits_{x\to \pm\infty}\dfrac{x-2}{\sqrt{4x^2+1}}$;

(vii) $\lim\limits_{x\to \pm\infty}\tanh x$;

(viii) $\lim\limits_{x\to \pm\infty}\dfrac{e^x}{\sinh x}$;

(ix) $\lim\limits_{x\to \pm\infty}\left(\dfrac{2x+7}{2x-6}\right)^{\sqrt{4x^2+x-3}}$.

Problem 5.4 Calculate the one-sided limits:

(i) $\lim\limits_{x\to 0^\pm}\left(\dfrac{1}{x}\right)^{\lfloor x\rfloor}$;

(ii) $\lim\limits_{x\to 0^\pm}e^{1/x}$;

(iii) $\lim\limits_{x\to 0^\pm}\dfrac{1-e^{1/x}}{1+e^{1/x}}$.

6. Continuity

6.1 Definition, properties, and continuity of elementary functions

Those functions whose limit at a point a of their domain coincides with the value of that function at that point play a very special role in calculus. They mainly coincide with those functions whose graph "can be plotted without lifting the pen from the paper" —which is the intuitive notion of a continuous function.[1] The formal definition of continuity is the following:

Definition 6.1.1 — Continuity. A real function f is said to be **continuous** at a point a of its domain if

$$\lim_{x\to a} f(x) = f(a). \tag{6.1}$$

Definition 6.1.2 — Continuity in intervals. f is said to be continuous in

(a,b) if it is continuous at every point $x \in (a,b)$;
$[a,b)$ if it is continuous in (a,b) and $\lim_{x\to a^+} f(x) = f(a)$;
$(a,b]$ if it is continuous in (a,b) and $\lim_{x\to b^-} f(x) = f(b)$;
$[a,b]$ if it is continuous in $[a,b)$ and in $(a,b]$.

■ **Example 6.1** Examples 5.7 and 5.10 prove that:

(a) polynomials $P_n(x)$ are continuous in all $\mathbb{R}$;
(b) rational functions $P_n(x)/Q_m(x)$ are continuous in all $\mathbb{R}$ except at the roots of Q_m;
(c) $\sin x$ and $\cos x$ are continuous in all $\mathbb{R}$;
(d) $\tan x$ is continuous except at the zeroes of $\cos x$;
(e) $\cot x$ is continuous except at the zeroes of $\sin x$.

■

[1] We say 'mainly' because there are very weird functions, which one would intuitively not refer to them as continuous, and nevertheless they are continuous in some subsets. But we shall not be concerned with these functions in this course. We will rather focus on practical, "sensible" functions.

■ **Example 6.2** Let us consider the exponential function. At any point $a \in \mathbb{R}$ we can write

$$e^x = e^{a+(x-a)} = e^a e^{x-a} = e^a + e^a(e^{x-a} - 1).$$

Since $e^{x-a} - 1 \sim x - a$ as $x \to a$,

$$\lim_{x\to a} e^x = e^a + e^a \lim_{x\to a}(x-a) = e^a.$$

Therefore the exponential function is continuous in $\mathbb{R}$. ■

The algebraic properties of functional limits yield the following algebraic properties for continuous functions:

Proposition 6.1.1

(i) If f and g are continuous at a, then so are $f+g$ and fg. If on top of that $g(a) \neq 0$, then f/g is also continuous at a.

(ii) If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

(iii) If an invertible function f is continuous at a, then f^{-1} is continuous at $f(a)$.

■ **Example 6.3** As a consequence of (iii) in the previous Proposition, $\log x$, $\arcsin x$, $\arccos x$, $\arctan x$, $\operatorname{arccot} x$ are continuous functions in their domains. ■

■ **Example 6.4** Function $f(x) = x^\alpha$, for $\alpha \in \mathbb{R}$, is continuous in $(0,\infty)$. The reason is that f can be written

$$f(x) = x^\alpha = e^{\alpha \log x},$$

i.e., as a composition of continuous functions.

If $\alpha > 0$ and we define $f(0) = 0$, then it is also continuous at $x = 0$ because

$$\lim_{x\to 0^+} e^{\alpha \log x} = \lim_{t\to -\infty} e^t = 0$$

(we have made the change of variable $t = \alpha \log x$).

If $\alpha = 0$ then $f(x) = 1$ in $(0,\infty)$, so it is continuous also at $x = 0$ if we define $f(0) = 1$. ■

6.2 Discontinuities

Discontinuities are points where a function is not continuous. There are several reasons why a function may not be continuous at a point, and some of them bear a specific name.

A function like $f(x) = \dfrac{\sin x}{x}$ is continuous in all $\mathbb{R}$ except $x = 0$, because the denominator vanishes at that point. However, the function has a well defined limit at that point (see equation (5.11)),

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

So we can re-define the function to be

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ 1 & x = 0, \end{cases}$$

and now it is continuous everywhere in $\mathbb{R}$. One such discontinuity is called an **avoidable discontinuity** because it can be “avoided” by properly defining the function.

The case of the Heaviside step function

$$H(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geqslant 0, \end{cases}$$

typifies a stronger case of discontinuity, which cannot be avoided. The function is continuous in $\mathbb{R}-\{0\}$ (because it is a constant for $x<0$ and for $x>0$), but at $x=0$ the left-handed limit is 0 whereas the right-handed limit is 1. So the limit when $x \to 0$ does not exist because, although the two one-sided limits exist, they are different. This is a **jump discontinuity** because the graph of the function "jumps" at that point.

R Note that, given the definition of continuity in intervals that we have made, although $H(x)$ is discontinuous at $x=0$, it is continuous e.g. in $[0,1]$ (but it not continuous in $[-1,0]$).

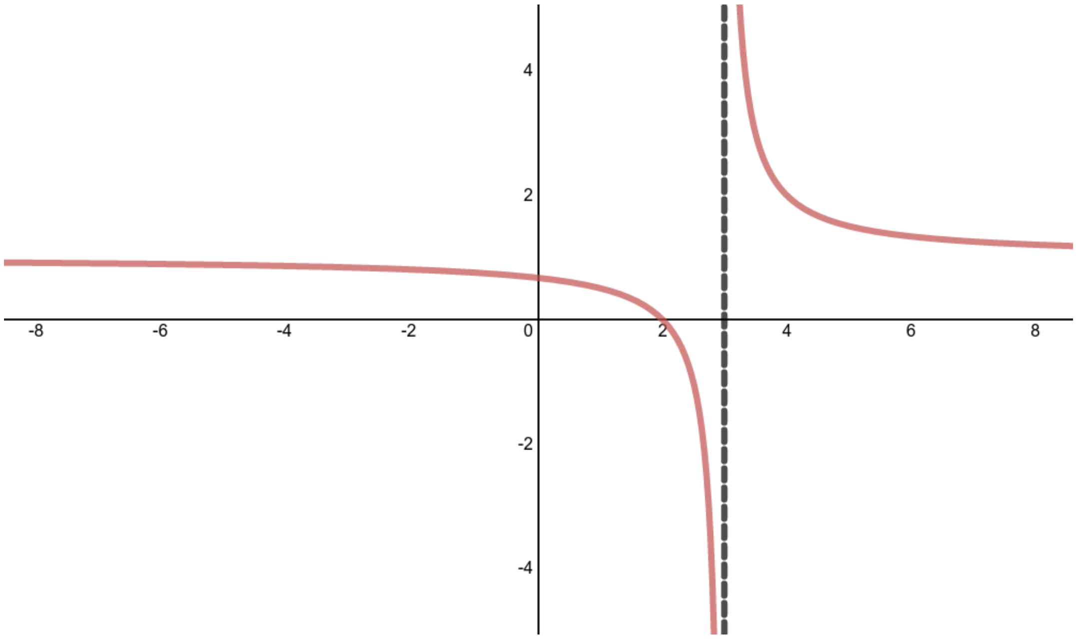


Figure 6.1: Illustration of an asymptote for the function $f(x)=\frac{x-2}{x-3}$.

In some cases the function is not continuous because the one or both of the two one-sided limits is $\pm\infty$. Such is the case of $1/x$ or $\log x$. We say that the function has a **singularity** at that point. We also call it an **asymptote** (see Figure 6.1).

Finally, a function can be discontinuous simply because it has no limit at a point. For instance, $\sin\frac{1}{x}$ is continuous in $\mathbb{R}-\{0\}$ because the limit when $x \to 0$ does not exist.

Exercise 6.1 Which kind of discontinuity has the function $f(x)=x\sin\frac{1}{x}$ at $x=0$? ■

6.3 Continuous functions in closed intervals

Continuity in a closed interval is a very restricting property. As a consequence, knowing that a function is continuous in a closed interval provides us very relevant information about the function. This information is captured in a series of theorems, the most important of which is due to Bolzano.

Theorem 6.3.1 — Bolzano's theorem. If f is a continuous function in $[a,b]$ and $f(a)f(b)<0$, then there exists $c \in (a,b)$ such that $f(c)=0$.

This theorem is the formal expression of the well-known fact that we cannot draw a continuous curve from a point above the X axis to another point below without crossing the X axis.

This theorem has an important corollary that basically expresses the same idea.

Corollary 6.3.2 — Intermediate values theorem. If f is a continuous function in $[a,b]$ and $\min\{f(a), f(b)\} < z < \max\{f(a), f(b)\}$, then there exists $c \in (a,b)$ such that $f(c) = z$.

We reword this result by stating that a continuous function in a closed interval $[a,b]$ takes all intermediate values between $f(a)$ and $f(b)$.

Proof. The proof is as simple as defining the function $g(x) = f(x) - z$, which being the sum of two continuous functions is continuous itself in $[a,b]$. But either $f(a) < z < f(b)$ (i.e., $f(a) - z < 0 < f(b) - z$) or $f(b) < z < f(a)$ (i.e., $f(b) - z < 0 < f(a) - z$); in any case, $g(a)g(b) < 0$. Then Bolzano's theorem implies that there exists $c \in (a,b)$ such that $g(c) = f(c) - z = 0$. ■

■ **Example 6.5** We shall prove, applying the intermediate value theorem, that the equation $xe^x = 1$ has a solution $x^* > 0$.

Let us define the function $f(x) = xe^x$, which is continuous in $\mathbb{R}$. Now, $f(0) = 0$ and $f(1) = e = 2.71828\ldots$ Then $f(0) < 1 < f(1)$, so 1 is an intermediate value of those f takes in the interval $[0,1]$. According to the intermediate values theorem there exists $0 < x^* < 1$ such that $f(x^*) = 1$, and that is the solution we are looking for. ■

The last important result of this sort is the following theorem:

Theorem 6.3.3 If f is a continuous function in the interval $[a,b]$ then there exists $x_m, x_M \in [a,b]$ such that $f(x_m) \leqslant f(x) \leqslant f(x_M)$ for any $x \in [a,b]$.

In other words, a continuous function in a closed interval reaches its maximum and minimum values within the interval (in particular, it is bounded).

The requirement of f to be continuous is well illustrated by the function $f(x) = 1/x$ in $[-1,1]$. It is not even bounded because is not continuous in the interval (as a matter of fact, it is not even defined at $x = 0$).

The requirement of the interval to be closed is illustrated, for instance, by the function $f(x) = x^2$ in $[0,1)$. Although the function is continuous in the whole interval it does not reach the maximum within it (the supreme of the values of f in that interval is $f(1) = 1$, but it is clearly reached outside the interval).

Problems

Problem 6.1

(a) Prove that if f is continuous then so is $|f|$. Show that the reciprocal is false by finding a counterexample.

(b) What can be said about a function that is continuous but all the values it takes are in $\mathbb{Q}$?

Problem 6.2

(a) Let $f:[0,1]\to[0,1]$ be a continuous, surjective function. Prove that there exists $c\in[0,1]$ such that $f(c)=c$.

(b) Let f be a continuous function in $[a,b]$ and let $x_1,\dots,x_n\in[a,b]$. Prove that there exists $c\in[a,b]$ such that $f(c)=\dfrac{1}{n}\sum_{k=1}^{n}f(x_k)$.

Problem 6.3 Consider the function

$$f(x)=\frac{1}{\lambda x^2-2\lambda x+1}.$$

Determine for which values $\lambda\in\mathbb{R}$ the function is continuous in (a) $\mathbb{R}$, or (b) $[0,1]$.

Problem 6.4 Study the continuity of the following functions:

(i) $f(x)=\dfrac{e^{-5x}+\cos x}{x^2-8x+12}$;

(ii) $f(x)=e^{3/x}+x^3-9$;

(iii) $f(x)=x^3\tan(3x+2)$;

(iv) $f(x)=\sqrt{x^2-5x+6}$;

(v) $f(x)=(\arcsin x)^3$;

(vi) $f(x)=(x-5)\log(8x-3)$;

(vii) $f(x)=x-\lfloor x\rfloor$;

(viii) $f(x)=\begin{cases}x^2\sin(1/x), & x\neq 0,\\ 0, & x=0;\end{cases}$

(ix) $f(x)=\begin{cases}\dfrac{\tan x}{\sqrt{x}}, & x>0,\\ 0, & x=0,\\ e^{1/x}, & x<0;\end{cases}$

(x) $f(x)=\begin{cases}x, & x\in\mathbb{Q},\\ -x, & x\notin\mathbb{Q};\end{cases}$

(xi) $f(x)=\begin{cases}\sin(\pi x), & x<-1,\\ |x|-x, & -1\leqslant x<1,\\ (x-1)^3, & x\geqslant 1;\end{cases}$

(xii) $f(x)=\begin{cases}(x+1)^2, & x\leqslant -1,\\ \operatorname{sgn}x+1, & -1<x<1,\\ 2x, & x\geqslant 1;\end{cases}$

(xiii) $f(x)=\begin{cases}x^2, & x\leqslant -2,\\ |x^2-1|, & -2<x<2,\\ 4x-5, & x\geqslant 2;\end{cases}$

(xiv) $f(x)=\begin{cases}(x-1)^2, & x>1,\\ x-\lfloor x\rfloor, & -1\leqslant x\leqslant 1,\\ x+1, & x<-1.\end{cases}$

Problem 6.5 Which of these equations have at least one solution in the specified set?:

(i) $x^2-18x+2=0$, in $[-1,1]$;

(ii) $x-\sin x=1$, in $\mathbb{R}$;

(iii) $e^x+1=0$, in $\mathbb{R}$;

(iv) $\cos x+2=0$, in $\mathbb{R}$;

(v) $f(x)=0$, in $[-2,2]$, where f is given by
$$f(x)=\begin{cases}x^2+2, & -2\leqslant x<0,\\ -(x^2+2), & 0\leqslant x\leqslant 2;\end{cases}$$

(vi) $\frac{1}{4}x^3-\sin(\pi x)+3=\frac{7}{3}$, in $[-2,2]$;

(vii) $|\sin x|=\sin x+3$, in $\mathbb{R}$.

Problem 6.6 Prove that any polynomial of odd degree has at least one real root.

7. Derivatives

7.1 Concept and definition

Derivates are introduced to characterise the *rate of variation* of a function with a number. The rate of variation measures how much the function $f(x)$ increases (positive) or decreases (negative) per unit of variation of the variable x. Thus, within the interval $[a,x]$ this rate will be

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(a)}{x - a}.$$

Figure 7.1 illustrates that the narrower the interval $[a,x]$ where the variation is measured the more accurate the estimated rate. Ideally, the measure would be perfect if this interval were infinitely narrow. This is the notion of *derivative* and the motivation of its definition:

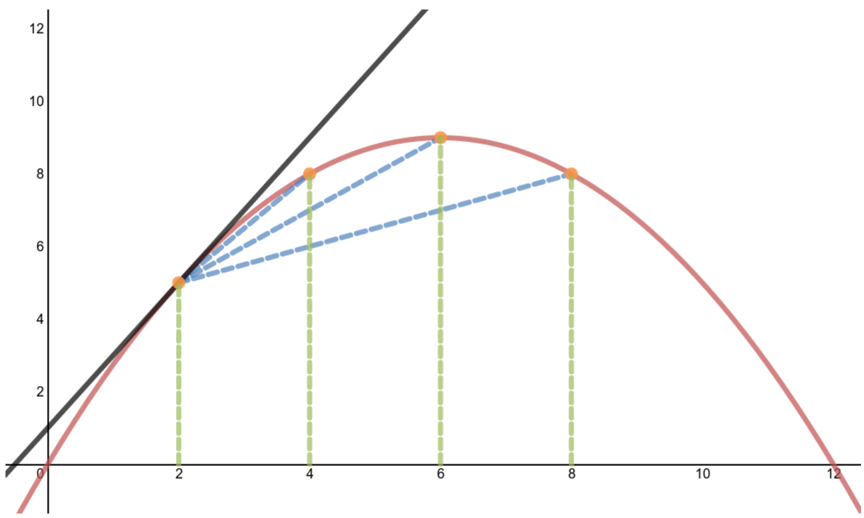


Figure 7.1: The rate of variation of $f(x)$ as obtained for different intervals.

Definition 7.1.1 — Derivative. The **derivative** of the function f at the point a of its domain is defined as

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}, \tag{7.1}$$

provided the limit exists. (When it does, we say that the function is *differentiable* at a.)

Alternatively, introducing the change of variable $x = a+h$, the limit (7.1) can be obtained as

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

Figure 7.1 also shows that $f'(a)$ —the rate of variation of $f(x)$ at $x = a$— coincides with the *slope* of the straight line *tangent* to the graph of $f(x)$ at the point $\big(a, f(a)\big)$ —which is an important geometric characterisation of the derivative concept.

R Often you will see the derivative denoted as

$$f'(a) = \frac{df}{dx}(a).$$

This is Leibniz's notation —a bit more mnemotechnical because it reminds that the derivative is, after all, a rate of change of f.

■ **Example 7.1** Consider the function $f(x) = x^2$. Its derivative at any point x would be, according to the defintion,

$$\lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0}(2x+h) = 2x.$$

Therefore $f'(x) = 2x$. ■

Exercise 7.1 Using Newton's binomial formula prove that the derivative of $f(x) = x^n$, with $n \in \mathbb{N}$ arbitrary, at any point $x \in \mathbb{R}$ is $f'(x) = nx^{n-1}$. (Note that this formula holds even if $n = 0$, for which $f(x) = 1$.) ■

■ **Example 7.2** Let $f(x) = \sin x$ and $g(x) = \cos x$. By definition

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0} \frac{\sin x\cos h+\cos x\sin h-\sin x}{h} \\ &= \sin x \lim_{h\to 0} \frac{\cos h-1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}. \end{aligned}$$

But

$$\lim_{h\to 0} \frac{\sin h}{h} = 1, \qquad \lim_{h\to 0} \frac{\cos h-1}{h} = -\lim_{h\to 0} h\frac{1-\cos h}{h^2} = -0\cdot\frac{1}{2} = 0,$$

hence $f'(x) = \cos x$.

Similarly

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0} \frac{\cos x\cos h-\sin x\sin h-\cos x}{h} \\ &= \cos x \lim_{h\to 0} \frac{\cos h-1}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h} = -\sin x. \end{aligned}$$

Thus we have the result $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$. ■

■ **Example 7.3** Let $f(x) = e^x$ and compute

$$f'(x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h} = e^x.$$

■

We say that f is differentiable in the interval (a,b) if it is differentiable at every point of the interval.

The function f', defined as

$$\begin{aligned} f' : A &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = f'(x), \end{aligned} \tag{7.2}$$

where A is the set of points where f is differentiable, is called the **derivative function** of f (or simply the *derivative* of f).

Likewise, we can introduce higher order derivatives. For instance, f'' is the *second derivative* of f, i.e., the derivative function of f'. Or f''' is the *third derivative* of f, i.e., the derivative function of f''. And so on. (Beyond the third derivative it is customary to denote higher order derivatives as $f^{(n)}$, the nth derivative of f.)

The following theorem emphasises that differentiability is a more restrictive property than continuity.

Theorem 7.1.1 If f is differentiable at a it is also continuous at a.

Proof. It is very simple. All we have to prove is that

$$\lim_{x\to a} f(x) = f(a) \qquad \Leftrightarrow \qquad \lim_{x\to a}[f(x) - f(a)] = 0.$$

For that we just need to multiply and divide by $x - a$, and apply the algebraic properties of limits:

$$\lim_{x\to a}[f(x) - f(a)] = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}(x - a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x\to a}(x - a) = f'(a)\cdot 0 = 0,$$

where we have used that the first limit exists (the hypothesis of the theorem) and it is the derivative of f at $x = a$. ■

An obvious consequence of this theorem is that discontinuous functions are not differentiable at the discontinuities.

■ **Example 7.4** Function $f(x) = |x|$ is continuous in $\mathbb{R}$, however, $f'(0)$ does not exist. The reason is that

$$\lim_{x\to 0^+} \frac{|x| - 0}{x - 0} = \lim_{x\to 0^+} \frac{x}{x} = 1$$

because $|x| = x$ for $x \geqslant 0$. However

$$\lim_{x\to 0^-} \frac{|x| - 0}{x - 0} = \lim_{x\to 0^-} \frac{-x}{x} = -1$$

because $|x| = -x$ for $x < 0$. Therefore the limit defining $f'(0)$ does not exists because the left-handed and right-handed limits are different. ■

■ **Example 7.5** Function

$$f(x)=\begin{cases} x\sin(1/x), & x\neq 0,\\ 0, & x=0\end{cases}$$

is continuous in $\mathbb{R}$, however, $f'(0)$ does not exist:

$$\lim_{x\to 0}\frac{x\sin(1/x)-0}{x-0}=\lim_{x\to 0}\sin(1/x).$$

On the contrary, function

$$g(x)=\begin{cases} x^2\sin(1/x), & x\neq 0,\\ 0, & x=0\end{cases}$$

is differentiable at $x=0$ (in fact everywhere, as we will see later) and

$$f'(0)=\lim_{x\to 0}\frac{x^2\sin(1/x)-0}{x-0}=\lim_{x\to 0}x\sin(1/x)=0.$$

■

7.2 Algebraic properties

The fact that derivatives are defined as limits leads to the following algebraic properties:

Proposition 7.2.1 Let f and g be two differentiable functions (in an appropriate set). Then:

(i) $(\lambda f+\mu g)'=\lambda f'+\mu g'$, where $\lambda,\mu\in\mathbb{R}$; *(linearity)*

(ii) $(fg)'=f'g+fg'$; *(Leibniz's rule)*

(iii) $(f\circ g)'=(f'\circ g)g'$; *(chain rule)*

(iv) $\left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}$, provided $g\neq 0$; *(quotient rule)*

(v) $(f^{-1})'=\frac{1}{f'\circ f^{-1}}$; *(inverse rule)*

Proof. Except for some technicalities —which we will omit here—, the proof of these rules is just an application of the algebraic properties of limits.

(i) From the linearity of limits,

$$\begin{aligned}(\lambda f+\mu g)'(a)&=\lim_{x\to a}\frac{\lambda f(x)+\mu g(x)-\lambda f(a)+\mu g(a)}{x-a}\\ &=\lambda\lim_{x\to a}\frac{f(x)-f(a)}{x-a}+\mu\lim_{x\to a}\frac{g(x)-g(a)}{x-a}\\ &=\lambda f'(a)+\mu g'(a).\end{aligned}$$

(ii) Now we need to add and substract $f(a)g(x)$:

$$\begin{aligned}(fg)'(a)&=\lim_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a}=\lim_{x\to a}\frac{f(x)g(x)-f(a)g(x)+f(a)g(x)-f(a)g(a)}{x-a}\\ &=\lim_{x\to a}g(x)\frac{f(x)-f(a)}{x-a}+f(a)\lim_{x\to a}\frac{g(x)-g(a)}{x-a}=g(a)f'(a)+f(a)g'(a),\end{aligned}$$

where we have used that g is continuous because it is differentiable.

(iii) Here we need to multiply and divide by $g(x)-g(a)$:

$$(f\circ g)'(a)=\lim_{x\to a}\frac{f\big(g(x)\big)-f\big(g(a)\big)}{x-a}=\lim_{x\to a}\frac{f\big(g(x)\big)-f\big(g(a)\big)}{g(x)-g(a)}\frac{g(x)-g(a)}{x-a}$$
$$=\lim_{y\to g(a)}\frac{f(y)-f\big(g(a)\big)}{y-g(a)}\lim_{x\to a}\frac{g(x)-g(a)}{x-a}=f'\big(g(a)\big)g'(a)=(f'\circ g)(a)g'(a),$$

where we have changed the variable $y=g(x)$, so that $y\to g(a)$ as $x\to a$ because g is continuous.

(iv) First we need to prove $(x^{-1})'=-x^{-2}$ for any $x\neq 0$. We do that using the definition:

$$\left(\frac{1}{x}\right)'=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\lim_{h\to 0}\frac{x-(x+h)}{hx(x+h)}=\lim_{h\to 0}\frac{-h}{hx(x+h)}=\lim_{h\to 0}\frac{-1}{x(x+h)}=-\frac{1}{x^2}.$$

Now we write the quotien as a product and apply the product rule:

$$\left(\frac{f}{g}\right)'=\left(f\cdot\frac{1}{g}\right)'=f'\left(\frac{1}{g}\right)+f\left(\frac{1}{g}\right)'.$$

But $1/g=h\circ g$, where $h(x)=x^{-1}$, so we can apply the chain rule and obtain

$$\left(\frac{1}{g}\right)'=-\frac{g'}{g^2}.$$

Thus, finally,

$$\left(\frac{f}{g}\right)'=\frac{f'}{g}-\frac{fg'}{g^2}=\frac{f'g-fg'}{g^2}.$$

(v) The equation that defines the inverse is $(f\circ f^{-1})(x)=x$. If we differentiate this equation, we get

$$\left(f\circ f^{-1}\right)'(x)=1.$$

Applying the chain rule we obtain

$$\left(f'\circ f\right)(x)\left(f^{-1}\right)'(x)=1,$$

from which we arrive at the result

$$\left(f^{-1}\right)'(x)=\frac{1}{\left(f'\circ f\right)(x)}.$$

■

The following examples illustrate how these rules can be applied to obtain new derivatives:

■ **Example 7.6** If $f(x)=e^x$ then $f^{-1}(x)=\log x$. Therefore

$$(\log x)'=\frac{1}{e^{\log x}}=\frac{1}{x}.$$

Logarithms can have a different base, say $a>0$. They are denoted $\log_a x$ and form the inverse function of a^x. Now $a^x=e^{x\log a}$, so by the chain rule

$$(a^x)'=\left(e^{x\log a}\right)'=\left(e^{x\log a}\right)\log a=a^x\log a.$$

Therefore

$$(\log_a x)'=\frac{1}{a^{\log_a x}\log a}=\frac{1}{x\log a}.$$

■

■ **Example 7.7** Function $f(x) = x^\alpha$, with $\alpha \in \mathbb{R}$, can be written as $f(x) = e^{\alpha \log x}$. Thus, applying the chain rule,

$$(x^\alpha)' = e^{\alpha \log x}\frac{\alpha}{x} = x^\alpha \frac{\alpha}{x} = \alpha x^{\alpha-1}.$$

■

■ **Example 7.8** If $f(x) = \sin x$ in $[-\pi/2, \pi/2]$ then $f^{-1}(x) = \arcsin x$. Thus,

$$(\arcsin x)' = \frac{1}{\cos(\arcsin x)}.$$

But $\cos x = \sqrt{1-\sin^2 x}$ in $[-\pi/2, \pi/2]$, so

$$(\arcsin x)' = \frac{1}{\sqrt{1-\sin^2(\arcsin x)}} = \frac{1}{\sqrt{1-x^2}},$$

because $\sin(\arcsin x) = x$. ■

Exercise 7.2 Calculate the derivative of the functions $\tan x$ and $\arctan x$. ■

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
c	0	$\sin x$	$\cos x$	$\arctan x$	$\frac{1}{1+x^2}$
x^α	$\alpha x^{\alpha-1}$	$\cos x$	$-\sin x$	$\operatorname{arccot} x$	$\frac{-1}{1+x^2}$
e^x	e^x	$\tan x$	$\frac{1}{\cos^2 x} = 1+\tan^2 x$	$\sinh x$	$\cosh x$
a^x	$a^x \log a$	$\cot x$	$\frac{-1}{\sin^2 x} = -1-\cot^2 x$	$\cosh x$	$\sinh x$
$\log x$	$\frac{1}{x}$	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\tanh x$	$\frac{1}{\cosh^2 x} = 1-\tanh^2 x$
$\log_a x$	$\frac{1}{x \log a}$	$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$	$\coth x$	$\frac{-1}{\sinh^2 x} = 1-\coth^2 x$

Table 7.1: Derivatives of most elementary functions. Here $c, \alpha \in \mathbb{R}$, $a > 0$.

7.3 Derivatives and local behaviour

We will see here a set of results related to the local behaviour of a function (i.e., the behaviour within intervals). To begin with, we need to define local maxima and minima.

We say that a function f has a **local maximum** at a point a of its domain, if there is some interval $(a-\delta, a+\delta)$ such that $f(x) \leqslant f(a)$ for all $x \in (a-\delta, a+\delta)$.

We say that a function f has a **local minimum** at a point a of its domain, if there is some interval $(a-\delta, a+\delta)$ such that $f(x) \geqslant f(a)$ for all $x \in (a-\delta, a+\delta)$.

Local maxima and minima are collectively called **local extrema.** If local extrema remain extrema for all x in the domain of f, they are **absolute extrema.**

Theorem 7.3.1 — Derivatives at local extrema. If f has a local extremum at a point a where it is differentiable then $f'(a) = 0$.

Proof. Consider the case of a maximum (the proof for a minimum is analogous). By the definition of local maximum

$$\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} \geqslant 0$$

because $f(x) \leqslant f(a)$ near a, so $f(x) - f(a) \leqslant 0$, but on the left of a we have $x - a \leqslant 0$. On the other hand,

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \leqslant 0$$

because the numerator is again $f(x) - f(a) \leqslant 0$, but on the right of a we have $x - a \leqslant 0$. Since the derivative exists both limits must coincide, so the only possibility is that both are 0. Hence $f'(a) = 0$. ■

■ **Example 7.9** Consider the function $f(x) = |x(1-x)|$. We know that $x(1-x) \geqslant 0$ if $0 \leqslant x \leqslant 1$, and $x(1-x) < 0$ if $x < 0$ or $x > 1$. Then we can rewrite

$$f(x) = \begin{cases} x(1-x), & 0 \leqslant x \leqslant 1, \\ x(x-1), & x < 0 \text{ or } x > 1. \end{cases}$$

Let us compute the derivative,

$$f'(x) = \begin{cases} 1 - 2x, & 0 < x < 1, \\ 2x - 1, & x < 0 \text{ or } x > 1. \end{cases}$$

The derivative at $x = 0$ and $x = 1$ does not exists because, being $f(0) = 0$ and $f(x) = x(x-1)$ for $x < 0$,

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{x(x-1)}{x} = \lim_{x \to 0^-} (x-1) = -1.$$

However, since $f(x) = x(1-x)$ for $x > 0$,

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x(1-x)}{x} = \lim_{x \to 0^+} (1-x) = 1.$$

Since both one-sided limits are different the limit does not exist. For $x = 1$ the argument is similar.

Now to find the local extrema we need to look for the solutions of $f'(x) = 0$. This equation boils down to $2x = 1$, whose solution is $x = \frac{1}{2}$.

Figure 7.2 presents a plot of $f(x)$. One can clearly see that $x = \frac{1}{2}$ is indeed a local maximum —albeit not absolute, because there are points where $f(x) > f(1/2)$—; however, we can also see that $x = 0$ and $x = 1$ are local minima, but they are not contained in the equation $f'(x) = 0$. (Incidentally, these minima are both absolute.)

There is no contradiction with the theorem though, because, as we have just seen, the function is not differentiable at those points —a premise of the theorem.

This example brings about the point that, when looking for extrema, we need to check not only the solutions of $f'(x) = 0$, but also the points where $f'(x)$ does not exist. ■

R Notice also that $f'(c) = 0$ does not imply that c is an extremum. For instance take $f(x) = x^3$. Clearly $f'(0) = 0$, however there is no extremum at $x = 0$ because $f(x) > 0$ for $x > 0$ and $f(x) < 0$ for $x < 0$.

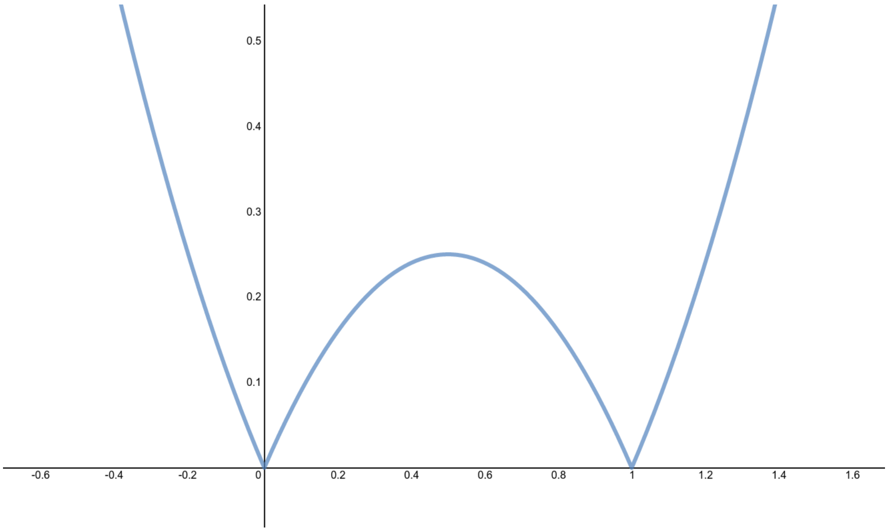


Figure 7.2: Plot of the function $f(x) = |x(1-x)|$.

Theorem 7.3.2 — Rolle's theorem. Let f be continuous in $[a,b]$, differentiable in (a,b), and such that $f(a) = f(b)$; then there exists $c \in (a,b)$ where $f'(c) = 0$.

Proof. The proof is very simple. Every continuous function in a closed interval reaches its absolute maximum and minimum within that interval. There are three possibilities:

(a) Both extrema are in (a,b). In that case f will be differentiable at both of them, c_{min} and c_{max}. According to Theorem 7.3.1 $f'(c_{\text{min}}) = f'(c_{\text{max}}) = 0$.

(b) One extremum is at $x = a$ or at $x = b$ and the other one is at $c \in (a,b)$. Then $f'(c) = 0$.

(c) One extremum is at $x = a$ and the other one at $x = b$. Then the function must be a constant because $f(a) = f(b)$, and its derivative will be $f'(x) = 0$ everywhere in (a,b).

In any of the three cases we see that $f'(c) = 0$ in at least one point of (a,b). ■

Theorem 7.3.3 — Mean value theorem. Let f be continuous in $[a,b]$ and differentiable in (a,b); then there exists $c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$.

Exercise 7.3 Prove the mean value theorem by applying Rolle's theorem to the function

$$g(x) = f(x) - \left(\frac{f(b)-f(a)}{b-a}\right)(x-a) - f(a).$$

(First check that g satifies the hypotheses of the theorem.) ■

There are practical consequences of the mean value theorem, which can be summarised in this corollary:

Corollary 7.3.4 With the hypothesis of the mean value theorem:

(i) If $f'(x) = 0$ for all $x \in (a,b)$ then f is constant in (a,b).

(ii) If $f'(x) = g'(x)$ for all $x \in (a,b)$ then $f(x) = g(x) + k$ in (a,b), with $k \in \mathbb{R}$ a constant.

(iii) If $f'(x) > 0$ for all $x \in (a,b)$ then f is strictly increasing in (a,b).

(iv) If $f'(x) < 0$ for all $x \in (a,b)$ then f is strictly decreasing in (a,b).

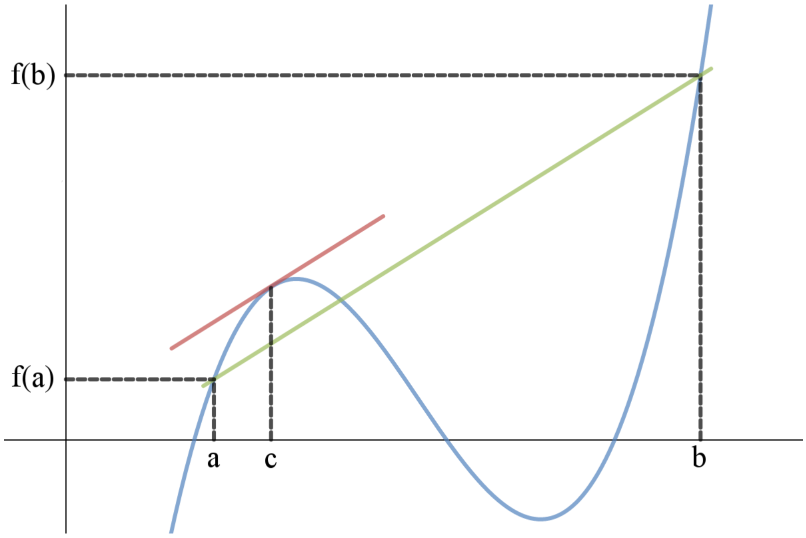


Figure 7.3: Geometric interpretation of the mean value theorem. The green straight line joining the points $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$ is defined by the equation $y = \left(\frac{f(b)-f(a)}{b-a}\right)(x-a)+f(a)$. The red parallel line proves that there is at least one point in the curve, $\big(c, f(c)\big)$, where the tangent has the same slope as (is parallel to) the straight line. This is the statement of the theorem.

Proof. Simply take $a < x < y < b$ arbitrary and apply the mean value theorem:

$$f(y) - f(x) = f'(c)(y-x),$$

where $x < c < y$.

If we assume $f'(c) = 0$ for any $c \in (a,b)$, then $f(y) = f(x)$. Since this is valid for any pair of points x, y in (a,b) this proves (i).

If we assume $f'(c) > 0$ for any $c \in (a,b)$, then $f(y) > f(x)$ whenever $y > x$. Since this is valid for any pair of points x, y in (a,b) this proves (iii).

If we assume $f'(c) < 0$ for any $c \in (a,b)$, then $f(y) < f(x)$ whenever $y > x$. Since this is valid for any pair of points x, y in (a,b) this proves (iv).

As for (ii), it is just a consequence of applying (i) to the function $f - g$. ■

These resuls are useful in identifying the nature of extrema, as this example illustrates:

■ **Example 7.10** Find the absolute extrema of the function $f(x) = 2x^{5/3} + 5x^{2/3}$ in the interval $[-8, 1]$.

There are four steps to solve a problem like this:

(1) Find the set where $f'(x)$ exists, and solve the equation $f'(x) = 0$ within that set.
(2) Take all solutions of $f'(x) = 0$ along with the points where $f'(x)$ does not exist.
(3) Check whether any of those point is a local extremum by checking the sign of f' on their left and on their right.
(4) Compare the value of $f(x)$ in all those points as well as the values at the extremes of the interval. Select the largest and the smallest and identify the absolute extrema.

In the case we are dealing with here

$$f'(x) = \frac{10}{3}(x^{2/3} + x^{-1/3}) = \frac{10}{3}(x+1)x^{-1/3}.$$

This function is well defined for all $x \neq 0$. At $x = 0$ the derivative does not exists because the limit

$$\lim_{x\to 0} \frac{2x^{5/3} + 5x^{2/3}}{x} = \lim_{x\to 0} \left(2x^{2/3} + 5x^{-1/3}\right)$$

diverges.

Now, the solution of $f(x) = 0$ is $x = -1$, and $f'(x) > 0$ for $x < -1$ (notice that $x^{-1/3} < 0$ whenever $x < 0$), but $f'(x) < 0$ for $-1 < x < 0$. The function thus increases on the left of $x = -1$ and decreases on the right, therefore there is a *local maximum* at $x = -1$.

As for $x = 0$, $f'(x) < 0$ for $-1 < x < 0$, but $f'(x) > 0$ for $x > 0$. Thus there is a *local minimum* at $x = 0$.

That is all for local extrema. Concerning absolute extrema we need to compute

$$f(-1) = 3, \qquad f(0) = 0, \qquad f(-8) = -44, \qquad f(1) = 7.$$

So the absolute maximum is at $x = 1$ (the rightmost extreme of the interval) and the absolute minimum is at $x = -8$ (the leftmost extreme of the interval).

Figure 7.4 illustrates what we have just found. ■

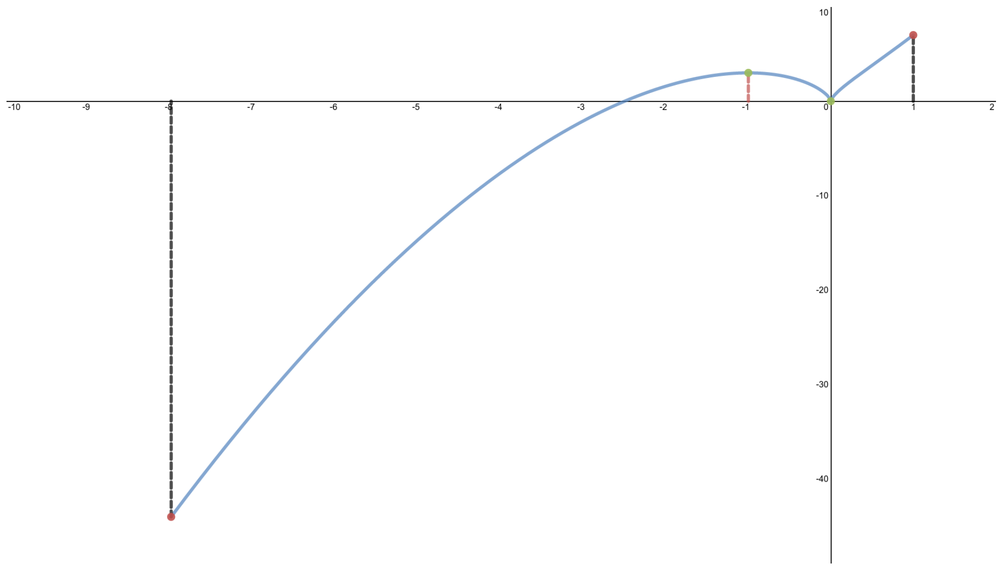


Figure 7.4: Plot of the function $f(x) = 2x^{5/3} + 5x^{2/3}$.

Theorem 7.3.5 — Cauchy's mean value theorem. Let f and g be both continuous in $[a,b]$ and differentiable in (a,b); then there exists $c \in (a,b)$ such that

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c). \tag{7.3}$$

Exercise 7.4 Prove Cauchy's mean value theorem by applying Rolle's theorem to the function

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x).$$

(First check that h satifies the hypotheses of the theorem.) ■

Cauchy's mean value theorem is the basis for the proof of an important result in the calculations of limits of indeterminacies of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. The theorem can also be applied to sequences—for which it is often an alternative to Stolz's theorem. The theorem (or rule, as it is customary referred to) is named after the 17th-century French mathematician Guillaume de l'Hôpital (1661–1704), although it was first introduced to him in 1694 by the Swiss mathematician Johann Bernoulli (1667–1748).

Theorem 7.3.6 — l'Hôpital's rule. Let f and g be two functions such that $g'(x) \neq 0$ in an environment of a (perhaps excluding a itself) and

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \ell.$$

If the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$, are both 0 or $\pm\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \ell.$$

R L'Hôpital rule remains valid even if $a = \pm\infty$ or if the limits are one-sided.

■ **Example 7.11** Let us see a limit that we already know, but obtained using l'Hôpital's rule:

$$\lim_{x\to 0} \frac{\sin x}{x}.$$

Since $(\sin x)' = \cos x$ and $(x)' = 1$, and

$$\lim_{x\to 0} \frac{\cos x}{1} = 1,$$

we can readily conclude that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

■

■ **Example 7.12** To calculate the limit

$$\lim_{x\to 0} \frac{e^x - x - \cos x}{\sin x^2},$$

which is a $\frac{0}{0}$ indeterminacy, we first compute

$$\frac{d}{dx}(e^x - x - \cos x) = e^x - 1 + \sin x, \qquad \frac{d}{dx}\sin x^2 = 2x\cos x^2,$$

and try to obtain

$$\lim_{x\to 0} \frac{e^x - 1 + \sin x}{2x\cos x^2}.$$

This remains a $\frac{0}{0}$ indeterminacy, so again we compute

$$\frac{d}{dx}(e^x - 1 + \sin x) = e^x + \cos x, \qquad \frac{d}{dx}(2x\cos x^2) = 2\cos x^2 - 4x^2\sin x^2.$$

Now

$$\lim_{x\to 0} \frac{e^x + \cos x}{2\cos x^2 - 4x^2\sin x^2} = \frac{1+1}{2-0} = 1,$$

therefore

$$\lim_{x\to 0} \frac{e^x - 1 + \sin x}{2x\cos x^2} = 1$$

and finally

$$\lim_{x\to 0} \frac{e^x - x - \cos x}{\sin x^2} = 1.$$

■

Exercise 7.5 Prove the equivalences, when $x \to 0$,

$$(1+x)^\alpha - 1 \sim \alpha x, \qquad e^x - 1 - x \sim \frac{x^2}{2}, \qquad x - \log(1+x) \sim \frac{x^2}{2}, \qquad x - \sin x \sim \frac{x^3}{6},$$

using l'Hôpital's rule. ■

R From the definition of limit it is clear that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \ell \quad \Rightarrow \quad \lim_{x\to\infty} \frac{f(n)}{g(n)} = \ell$$

because $x_n = n$ is a particular sequence $x_n \to \infty$. Therefore l'Hôpital's rule can be applied to sequences as well.

There is an important caveat to be made about l'Hôpital's rule: in general, it is not true that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

as it is typically read in (incorrect) applications of this result. L'Hôpital's theorem states that the existence of the second limit implies that the first limit is the same, but it may well happen that the second limit does not exist while the first one does, as the following example illustrates.

■ **Example 7.13** Consider the limit

$$\ell = \lim_{x\to 0} \frac{x^2 \sin(1/x)}{\sin x}.$$

Since $\sin x \sim x$ when $x \to 0$, then

$$\ell = \lim_{x\to 0} \frac{x^2 \sin(1/x)}{x} = \lim_{x\to 0} x \sin(1/x) = 0.$$

However, if we apply l'Hôpital's rule and try to calculate

$$\lim_{x\to 0} \frac{2x\sin(1/x) - \cos(1/x)}{\cos x}$$

this limit does not exist! ■

■ **Example 7.14** Before applying l'Hôpital's theorem one must check the hypothesis that the limits of the numerator and denominator are both 0 or $\pm\infty$. Otherwise the application of the theorem will lead to an incorrect result, as the following example illustrates:

$$\lim_{x\to 0} \frac{\cos x}{1 + \log(1+x)} = 1,$$

as it is easily verified. However, if one isists on applying l'Hôpital's theorem despite the fact that the limits of the numerator and denominator are neither 0 nor $\pm\infty$, this is what one would obtain:

$$\lim_{x\to 0} \frac{-\sin x}{\frac{1}{1+x}} = \lim_{x\to 0} \left[-(1+x)\sin x \right] = 0,$$

which is clearly wrong! ■

Problems

Problem 7.1 Let f and g be differentiable functions in $\mathbb{R}$. Write down the derivative of the following functions in their respective domains:

(i) $h(x) = \sqrt{f(x)^2 + g(x)^2}$;

(ii) $h(x) = \arctan\left(\dfrac{f(x)}{g(x)}\right)$;

(iii) $h(x) = f\big(g(x)\big)e^{f(x)}$;

(iv) $h(x) = \log\big(g(x)\sin f(x)\big)$;

(v) $h(x) = f(x)^{g(x)}$;

(vi) $h(x) = \dfrac{1}{\log\big(f(x)+g(x)^2\big)}$.

Problem 7.2

(a) Make up a continuous function in $\mathbb{R}$ which vanishes for $|x| \geqslant 2$ and equals 1 for $|x| \leqslant 1$.

(b) Do it again, but this time make sure that the function is differentiable in $\mathbb{R}$.

Problem 7.3 Check that the following functions satisfy the specified differential equations, where c, c_1, and c_2 are constants:

(i) $f(x) = \dfrac{c}{x}$ satisfies $xf' + f = 0$;

(ii) $f(x) = x\tan x$ satisfies $xf' - f - f^2 = x^2$;

(iii) $f(x) = c_1 \sin 3x + c_2 \cos 3x$ satisfies $f'' + 9f = 0$;

(iv) $f(x) = c_1 e^{3x} + c_2 e^{-3x}$ satisfies $f'' - 9f = 0$;

(v) $f(x) = c_1 e^{2x} + c_2 e^{5x}$ satisfies $f'' - 7f' + 10f = 0$;

(vi) $f(x) = \log(c_1 e^x + e^{-x}) + c_2$ satisfies $f'' + (f')^2 = 1$.

Problem 7.4 Prove the identities (valid only in the specified regions)

(i) $\arctan x + \arctan \dfrac{1}{x} = \dfrac{\pi}{2}$, for $x > 0$;

(ii) $\arctan \dfrac{1+x}{1-x} - \arctan x = \dfrac{\pi}{4}$, for $x < 1$;

(iii) $2\arctan x + \arcsin \dfrac{2x}{1+x^2} = \pi$, for $x \geqslant 1$.

HINT: Differentiate the equation and check one point of the specified region.

Problem 7.5 At which points does the graph of the function $f(x) = x + (\sin x)^{1/3}$ has a vertical tangent?

Problem 7.6 Given the function

$$f(x) = \begin{cases} \dfrac{x}{1+e^{1/x}}, & x \neq 0, \\ 0 & x = 0, \end{cases}$$

calculate the angle between the tangents on the left and on the right of its graph at $x = 0$.

Problem 7.7 Find the sets where the function $f(x) = \sqrt{x+2}\arccos(x+2)$ is continuous and differentiable.

Problem 7.8 Calculate the smallest α for which $f(x) = |\alpha x^2 - x + 3|$ is differentiable in $\mathbb{R}$.

Problem 7.9 Given the function

$$f(x) = \begin{cases} a + bx^2, & |x| \leqslant c, \\ \dfrac{1}{|x|}, & |x| > c, \end{cases} \qquad c > 0,$$

find a and b so that it is continuous and differentiable in $\mathbb{R}$.

Problem 7.10 Given the function

$$f(x) = \begin{cases} \dfrac{3-x^2}{2}, & x < 1, \\ \dfrac{1}{x}, & x \geqslant 1, \end{cases}$$

(a) determine the sets where it is continuous and where it is differentiable;
(b) check that the mean value theorem can be applied to this function in $[0,2]$ by determining the point(s) $c \in (0,2)$ where the theorem holds.

Problem 7.11 Function $f(x) = 1 - x^{2/3}$ vanishes in $x = \pm 1$; however $f'(x) \neq 0$ in $(-1,1)$. Find which hypothesis of Rolle's theorem is not satisfied.

Problem 7.12 Prove, using Rolle's theorem, the following statements about a function f that is continuous in $[a,b]$ and differentiable in (a,b):
(i) If f vanishes $k(\geqslant 2)$ times in $[a,b]$ then f' vanishes at least $k-1$ times in $[a,b]$.
(ii) If f is n-times differentiable in (a,b) and vanishes in $n+1$ different points of $[a,b]$, then $f^{(n)}$ vanishes at least once in $[a,b]$.

Problem 7.13 Using the mean value theorem, find an approximation to $26^{2/3}$ and $\log(3/2)$.

Problem 7.14 Calculate the limits

(i) $\displaystyle\lim_{x\to 0} \frac{e^x - \sin x - 1}{x^2}$;
(ii) $\displaystyle\lim_{x\to 0} \frac{\log|\sin 7x|}{\log|\sin x|}$;
(iii) $\displaystyle\lim_{x\to 1^+} \log x \log(x-1)$;
(iv) $\displaystyle\lim_{x\to\infty} x^{1/x}$;
(v) $\displaystyle\lim_{x\to 0} \frac{(1+x)^{1+x} - 1 - x - x^2}{x^3}$;
(vi) $\displaystyle\lim_{x\to\infty} x\left(\tan\frac{2}{x} - \tan\frac{1}{x}\right)$.

Problem 7.15 Suppose $h(x)$ is a twice-differentiable function and let

$$f(x) = \begin{cases} \dfrac{h(x)}{x^2}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Calculate $h(0)$, $h'(0)$, and $h''(0)$ so that f is continuous.

Problem 7.16 Calculate the limits

(i) $\displaystyle\lim_{x\to\infty} x\left[\left(1+\frac{1}{x}\right)^x - e\right]$;
(ii) $\displaystyle\lim_{x\to\infty} \frac{(1+\frac{1}{x})^{x^2}}{e^x}$;
(iii) $\displaystyle\lim_{x\to\infty} \left(\frac{2^{1/x}+18^{1/x}}{2}\right)^x$;
(iv) $\displaystyle\lim_{x\to\infty} \left(\frac{1}{p}\sum_{k=1}^{p} a_k^{1/x}\right)^x$, with $p \in \mathbb{N}$ and $a_k > 0$.

Problem 7.17 If f is a differentiable funtion such that

$$\lim_{x\to 0} \frac{f(2x^3)}{5x^3} = 1$$

and its derivative f' is continuous at $x = 0$,
(a) calculate $f(0)$;
(b) calculate $f'(0)$;

(c) calculate $\lim_{x\to 0} \frac{(f\circ f)(2x)}{f^{-1}(3x)}$.

Problem 7.18 The equation $e^{-f}f' = 2+\tan x$ together with the condition $f(0)=1$ define a one-to-one, differentiable function in the interval $[-\pi/4,\pi/4]$. If $g(x)=f^{-1}(x+1)$, calculate the limit

$$\lim_{x\to 0} \frac{e^x - e^{-\sin x}}{g(x)}.$$

Problem 7.19 Let $f(x)=|x^3(x-4)|-1$.

(a) Find where f is continuous and where it is differentiable.

(b) Determine its extrema.

(c) Prove that $f(x)=0$ has a unique solution in $[0,1]$.

Problem 7.20 Solve these optimisation problems:

(a) A factory that produces tomato sauce wants to can it in cylindrical cans of a fixed volume V. Determine their radius r and height h so that their fabrication consumes the least possible material.

(b) A recipient with square bottom and no cap must be covered by a thin layer of lead. If the volume of the recipient must be 32 litres, which dimensions should it have so that it requires the least possible amount of lead?

(c) Find two numbers $x,y>0$ such that $x+y=20$ and x^2y^3 is maximum.

(d) Find the rectangle inscribed in the ellipse $(x/a)^2+(y/b)^2=1$ with its sides parallel to the axes of the ellipse, such that its area is maximum.

(e) With a tangent to the parabola $y=6-x^2$ and the positive axes one can make a triangle. Determine which of those triangles has the smallest area and compute it.

(f) We need to construct a box with no cap with the shape of a parallelepiped whose base is an equilateral triangle, and whose volume is 128 cm^3. If the material for the base costs 0.20 euros/cm^2 and that for the lateral surfaces costs 0.10 euros/cm^2, what are the dimensions of the cheapest such box?

(g) A right triangle ABC has vertex A at the origin, vertex B on the circumference $(x-1)^2+y^2=1$ —side AB is the hypothenuse of the triangle— and side AC on the horizontal axis. Calculate the location of C that maximises the area of the triangle.

(h) Let $\mathrm{P}=(x_0,y_0)$ be a point of the first quadrant ($x_0,y_0>0$). A straight line through P cuts the axes at $\mathrm{A}=(x_0+\alpha,0)$ and $\mathrm{B}=(0,y_0+\beta)$. Calculate $\alpha>0$ and $\beta>0$ so as to minimise

(i) the length of segment AB;

(ii) the sum of the lengths of OA and OB;

(iii) the area of the triangle OAB.

HINT: Triangle similarity implies $\beta = x_0y_0/\alpha$.

Problem 7.21 Prove the following inequalities:

(a) $(1+x)^a \geqslant 1+ax$ for all $a\geqslant 1$, $x>-1$ (Bernoulli's inequality);

(b) $e^x \geqslant 1+x$ for all $x\in\mathbb{R}$;

(c) $\frac{x}{1+x} \leqslant \log(1+x) \leqslant x$ for all $x>-1$.

HINT: In all cases try to minimise the appropriate function.

Problem 7.22

(a) Prove that $\frac{\log x}{x} < \frac{1}{e}$ for all $x>0$, $x\neq e$.

(b) Prove that the previous inequality is equivalent to $e^x > x^e$ for all $x>0$, $x\neq e$.

Problem 7.23 Determine the number of solutions of the following equations in the specified domains:

(i) $x^7 + 4x = 3$ in $\mathbb{R}$;

(ii) $x^5 = 5x - 6$ in $\mathbb{R}$;

(iii) $x^4 - 4x^3 = 1$ in $\mathbb{R}$;

(iv) $\sin x = 2x - 1$ in $\mathbb{R}$;

(v) $x^x = 2$ in $[1, \infty)$;

(vi) $x^2 = \log \frac{1}{x}$ in $(1, \infty)$.

8. Taylor Expansions

8.1 Landau's 'o' notation

As we did for sequences, when it comes to functional limits comparing functions is worthwhile. Over time, a standard notation has been created that makes algebraic manipulations of functions easy. We are going to introduce and discuss that notation.

Definition 8.1.1 We say that function f is **negligible** with respect to g when $x \to a$ (where $-\infty \leqslant a \leqslant \infty$) if

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0.$$

We denote this as $f = o(g)$ $(x \to a)$, and read it "f is *small* o of g as x goes to a".

The intuitive meaning of $f = o(g)$ $(x \to a)$ is that the numerical value of $f(x)$ is much smaller than that of $g(x)$, the more so the closer is x to a.

This symbol bears a set of basic algebraic properties with which we can easily manipulate it:

Proposition 8.1.1 For given $-\infty \leqslant a \leqslant \infty$,

(a) if $f = o(g)$ $(x \to a)$ and $\lambda \in \mathbb{R}$, then $\lambda f = o(g)$ **scaling**
(b) if $f_1 = o(g)$ $(x \to a)$ and $f_2 = o(g)$ $(x \to a)$, then $f_1 + f_2 = o(g)$ $(x \to a)$ **additive**
(c) if $f_1 = o(g_1)$ $(x \to a)$ and $f_2 = o(g_2)$ $(x \to a)$, then $f_1 f_2 = o(g_1 g_2)$ $(x \to a)$ **multiplicative**
(d) if $f = o(g)$ $(x \to a)$ and $g = o(h)$ $(x \to a)$, then $f = o(h)$ $(x \to a)$. **transitive**
(e) if $f = o(g)$ $(x \to a)$ then $hf = o(hg)$ $(x \to a)$ **factorisation**

R The short-hand version of these properties is
(a) $\lambda\, o(g) = o(g)$,
(b) $o(g) + o(g) = o(g)$,
(c) $o(g_1)\, o(g_2) = o(g_1 g_2)$,
(d) $o\big(o(h)\big) = o(h)$,
(e) $h\, o(g) = o(hg)$,
where it is implicitly understood $x \to a$. These are the common manipulations of the o symbol.

Proof. (a) We are given that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=0;$$

therefore

$$\lim_{x\to a}\frac{\lambda f(x)}{g(x)}=\lambda\cdot\lim_{x\to a}\frac{f(x)}{g(x)}=0.$$

(b) We are given that

$$\lim_{x\to a}\frac{f_1(x)}{g(x)}=\lim_{x\to a}\frac{f_2(x)}{g(x)}=0;$$

therefore

$$\lim_{x\to a}\frac{f_1(x)+f_2(x)}{g(x)}=\lim_{x\to a}\frac{f_1(x)}{g(x)}+\lim_{x\to a}\frac{f_2(x)}{g(x)}=0.$$

(c) In this case

$$\lim_{x\to a}\frac{f_1(x)}{g_1(x)}=\lim_{x\to a}\frac{f_2(x)}{g_2(x)}=0;$$

therefore

$$\lim_{x\to a}\frac{f_1(x)f_2(x)}{g_1(x)g_2(x)}=\lim_{x\to a}\frac{f_1(x)}{g_1(x)}\cdot\lim_{x\to a}\frac{f_2(x)}{g_2(x)}=0.$$

(d) Now we know that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{g(x)}{h(x)}=0;$$

therefore

$$\lim_{x\to a}\frac{f(x)}{h(x)}=\lim_{x\to a}\frac{f(x)g(x)}{g(x)h(x)}=\lim_{x\to a}\frac{f(x)}{g(x)}\cdot\lim_{x\to a}\frac{g(x)}{h(x)}=0.$$

(e) Finally, if

$$\lim_{x\to a}\frac{f(x)}{g(x)}=0,$$

then

$$\lim_{x\to a}\frac{h(x)f(x)}{h(x)g(x)}=\lim_{x\to a}\frac{f(x)}{g(x)}=0.$$

■

■ **Example 8.1**

(a) If $a,\alpha,\gamma\in\mathbb{R}$ and $\alpha>0$, then $(x-a)^{\gamma+\alpha}=o\big((x-a)^{\gamma}\big)$ $(x\to a)$ because

$$\lim_{x\to a}\frac{(x-a)^{\gamma+\alpha}}{(x-a)^{\gamma}}=\lim_{x\to a}(x-a)^{\alpha}=0.$$

(b) If $\alpha, \gamma \in \mathbb{R}$, and $\alpha > 0$, then $\dfrac{1}{x^{\gamma+\alpha}} = o\left(\dfrac{1}{x^{\gamma}}\right)$ $(x \to \pm\infty)$, because

$$\lim_{x\to\pm\infty} \frac{x^{-\gamma-\alpha}}{x^{-\gamma}} = \lim_{x\to\pm\infty} \frac{1}{x^{\alpha}} = 0.$$

(c) $\sin x = o(\sqrt{x})$ $(x \to 0^+)$ because

$$\lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}} = \lim_{x\to 0^+} \sqrt{x}\,\frac{\sin x}{x} = \underbrace{\lim_{x\to 0^+} \sqrt{x}}_{=0} \cdot \underbrace{\lim_{x\to 0^+} \frac{\sin x}{x}}_{=1} = 0.$$

(d) $1 - \cos x = o(x)$ $(x \to 0)$ because

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} x\,\frac{1-\cos x}{x^2} = \underbrace{\lim_{x\to 0} x}_{=0} \cdot \underbrace{\lim_{x\to 0} \frac{1-\cos x}{x^2}}_{=\frac{1}{2}} = 0.$$

(e) Clearly, when $x \to \infty$,

$$(\log x)^r = o(x^q), \qquad x^q = o(a^x), \qquad a^x = o(x^x),$$

provided $r, q > 0$ and $a > 1$.

■

■ **Example 8.2** To illustrate how to manipulate expressions involving o terms let us calculate

$$[1 + x + x^2 + o(x^2)]^2.$$

By expanding this espression we obtain

$$\begin{aligned}
[1 + x + x^2 + o(x^2)]^2 =& [1 + x + x^2 + o(x^2)][1 + x + x^2 + o(x^2)] \\
=& 1 + x + x^2 + o(x^2) + x + x^2 + x^3 + x \cdot o(x^2) + x^2 + x^3 + x^4 + x^2 \cdot o(x^2) \\
& + o(x^2) + o(x^2) \cdot x + o(x^2) \cdot x^2 + o(x^2) \cdot o(x^2).
\end{aligned}$$

Our first simplification is to add equal powers and to transform products like $x \cdot o(x^2) = o(x^3)$ (rule (e)) or $o(x^2) \cdot o(x^2) = o(x^4)$ (rule (c)). Then we get

$$\begin{aligned}
[1 + x + x^2 + o(x^2)]^2 =& 1 + 2x + 3x^2 + 2x^3 + x^4 + 2 \cdot o(x^2) + 2 \cdot o(x^3) + 3 \cdot o(x^4) \\
=& 1 + 2x + 3x^2 + 2x^3 + x^4 + o(x^3) + o(x^2) + o(x^4).
\end{aligned}$$

Now we apply rule (d) and transform $o(x^3) = o(x^2)$ and $o(x^4) = o(x^2)$, so

$$[1 + x + x^2 + o(x^2)]^2 = 1 + 2x + 3x^2 + 2x^3 + x^4 + 3 \cdot o(x^2) = 1 + 2x + 3x^2 + 2x^3 + x^4 + o(x^2).$$

Finally, $2x^3 = o(x^2)$ and $x^4 = o(x^2)$, thus

$$[1 + x + x^2 + o(x^2)]^2 = 1 + 2x + 3x^2 + 3 \cdot o(x^2) = 1 + 2x + 3x^2 + o(x^2).$$

This is the simplest form of the result.

For a practical implementation of this calculation we do not really need to take all these intermediate steps. Once an $o(x^2)$ in present in the expression every other one, or every $o(x^n)$ with $n > 2$, or every power x^m with $m > 2$, can be simply neglected. ■

R Note that $f = o(1)$ $(x \to a)$ is another way to express that $\lim_{x\to a} f(x) = 0$.

Strongly related to the notion of negligible function is that of **equivalent** functions when $x \to a$. We will say that $f_1 = f_2 + o(g)$ $(x \to a)$ if $f_1 - f_2 = o(g)$ $(x \to a)$. The idea this represents is that functions f_1 and f_2 differ in a amount that is negligible compared to g when x approaches a.

8.2 Taylor's polynomial

Using the o notation we can re-express some features of a function f. For example, if we know that f is continuous at a we know that

$$\lim_{x\to a} f(x) = f(a).$$

But we can rewrite this expression as

$$\lim_{x\to a} [f(x) - f(a)] = 0,$$

which in terms of the o notation can be stated as

$$f(x) = f(a) + o(1) \quad (x \to a). \tag{8.1}$$

From this viewpoint, a continuous function at a can be approximated by its value at a if we are close enough to a.

We can go a step beyond. Let us assume that a function is differentiable at a. Then

$$\lim_{x\to a} \frac{f(x) - f(a)}{x-a} = f'(a).$$

But this can be rewritten as

$$0 = \lim_{x\to a} \left[\frac{f(x)-f(a)}{x-a} - f'(a)\right] = \lim_{x\to a} \frac{f(x)-f(a)-f'(a)(x-a)}{x-a},$$

in other words,

$$f(x) = f(a) + f'(a)(x-a) + o(x-a) \quad (x \to a). \tag{8.2}$$

Since $y = f(a) + f'(a)(x-a)$ is the equation of the tangent to $f(x)$ at $x = a$, the equation above expresses the fact that differentiable functions at a point a can be approximated by their tangent if x is sufficiently close to a. The $o(x-a)$, compared to the previous $o(1)$ of continuous functions, means that the approximation is better.

We can try to push the idea a bit further and see if this sort of approximations can be improved. Inspired by the two equations (8.1) (8.2) we may try to seek for an expression like

$$f(x) = f(a) + f'(a)(x-a) + c_2(x-a)^2 + o\big((x-a)^2\big) \quad (x \to a),$$

i.e., we can try to use a parabola to approximate better the function near a. For this expression to hold we must have

$$0 = \lim_{x\to a} \frac{f(x)-f(a)-f'(a)(x-a)-c_2(x-a)^2}{(x-a)^2} = \lim_{x\to a} \left[\frac{f(x)-f(a)-f'(a)(x-a)}{(x-a)^2} - c_2\right].$$

But this is equivalent to

$$\lim_{x\to a} \frac{f(x)-f(a)-f'(a)(x-a)}{(x-a)^2} = c_2.$$

Now, to calculate the limit we can apply l'Hôpital's rule (we know that f is differentiable at a). This yields

$$c_2 = \lim_{x\to a} \frac{f'(x)-f'(a)}{2(x-a)} = \frac{1}{2} \lim_{x\to a} \frac{f'(x)-f'(a)}{x-a}.$$

But the last limit is the definition of the derivative of f' at $x = a$, so if we assume that f is twice differentiable at a we conclude that

$$c_2 = \frac{f''(a)}{2}$$

and arrive to the formula

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + o\big((x-a)^2\big) \quad (x \to a). \tag{8.3}$$

Let us do it once more: let us look for an improved approximation of the form

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + c_3(x-a)^3 + o\big((x-a)^3\big) \quad (x \to a).$$

Thus,

$$\begin{aligned} 0 &= \lim_{x\to a} \frac{f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2}(x-a)^2 - c_3(x-a)^3}{(x-a)^3} \\ &= \lim_{x\to a} \left[\frac{f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2}(x-a)^2}{(x-a)^3} - c_3 \right], \end{aligned}$$

or equivalently

$$c_3 = \lim_{x\to a} \frac{f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2}(x-a)^2}{(x-a)^3}.$$

To compute the limit we apply l'Hôpital's rule twice (we know that f is twice differentiable at a) and get

$$c_3 = \lim_{x\to a} \frac{f''(x) - f''(a)}{3\cdot 2(x-a)} = \frac{1}{3!} \lim_{x\to a} \frac{f''(x) - f''(a)}{x-a}.$$

Again this limit is the expression of the derivative of f'' at $x = a$, so assuming f is three times differentiable at a

$$c_3 = \frac{f'''(a)}{3!}$$

(because $3 \cdot 2 = 3!$), which yields

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + o\big((x-a)^3\big) \quad (x \to a) \tag{8.4}$$

(note that $2 = 2!$).

By now we can easily put forth a conjecture: if we define the nth degree polynomial

$$P_{n,a}(x) \equiv f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n, \tag{8.5}$$

which we will refer to as the nth order **Taylor polynomial** of function f at the point a, the previous findings suggest the following theorem:

Theorem 8.2.1 — Taylor's theorem (I). If f is n times differentiable at a, then

$$f(x) = P_{n,a}(x) + o\big((x-a)^n\big) \quad (x \to a). \tag{8.6}$$

Proof. In order to prove that

$$f(x) = P_{n,a}(x) + o\big((x-a)^n\big) \quad (x \to a),$$

we need to calculate the limit

$$\lim_{x \to a} \frac{f(x) - P_{n,a}(x)}{(x-a)^n}$$

and show that it is zero. To do that we apply $n-1$ times l'Hôpital's rule, so that we need to calculate the limit

$$\lim_{x \to a} \frac{f^{(n-1)}(x) - P_{n,a}^{(n-1)}(x)}{n!(x-a)}.$$

But differentiating $n-1$ times $P_{n,a}(x)$ all powers smaller than $n-1$ disappear, and there remain the last two terms. Now,

$$\frac{d^{n-1}}{dx^{n-1}}(x-a)^{n-1} = (n-1)!, \qquad \frac{d^{n-1}}{dx^{n-1}}(x-a)^n = n!(x-a),$$

therefore

$$P_{n,a}^{(n-1)}(x) = f^{(n-1)}(a) + f^{(n)}(a)(x-a).$$

Then the limit becomes

$$\lim_{x \to a} \frac{f^{(n-1)}(x) - f^{(n-1)}(a) - f^{(n)}(a)(x-a)}{n!(x-a)} = \frac{1}{n!} \lim_{x \to a} \left[\frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{x-a} - f^{(n)}(a) \right] = 0$$

because

$$\lim_{x \to a} \frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{x-a}$$

is the definition of the derivative of $f^{(n-1)}$ at a. This completes the proof. ■

R We will write $P_{n,a}(x)$ in a more compact way as

$$P_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k, \tag{8.7}$$

where we define $f^{(0)}(a) = f(a)$ and $0! = 1$.

■ **Example 8.3** Consider the function $f(x) = (1+x)^\alpha$, where $\alpha \in \mathbb{R}$. Then $f(0) = 1$ and

$$\begin{aligned}
f'(x) &= \alpha(1+x)^{\alpha-1}, & f'(0) &= \alpha, \\
f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2}, & f''(0) &= \alpha(\alpha-1), \\
f'''(x) &= \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}, & f'''(0) &= \alpha(\alpha-1)(\alpha-2), \\
&\vdots & &\vdots \\
f^{(n)}(x) &= \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}, & f^{(n)}(0) &= \alpha(\alpha-1)\cdots(\alpha-n+1).
\end{aligned}$$

Therefore

$$(1+x)^\alpha = 1+\alpha x+\frac{\alpha(\alpha-1)}{2}x^2+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n+o(x^n) \quad (x\to 0).$$

There is an interesting notation for this expression derived from the formula for the binomial coefficients. From equation (B.4), if $\alpha \in \mathbb{N}$,

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

Since this formula is meaningful even if $\alpha \in \mathbb{R}$, we use it as a definition and thus write

$$(1+x)^\alpha = \sum_{k=0}^{n}\binom{\alpha}{k}x^k+o(x^n) \quad (x\to 0).$$

This is the famous binomial formula as it was first obtained by Newton in 1665. ■

8.3 Calculating limits

Taylor's theorem can be applied to calculating complicated limits. As a matter of fact, it is more powerful than l'Ĥopital's rule in dealing with indeterminacies. A few examples will illustrate the procedure.

■ **Example 8.4** Suppose we want to calculate the limit

$$\lim_{x\to 0}\frac{\cos x-e^x+x}{x^2}.$$

All we need to do is to use the Taylor expansions

$$\cos x = 1-\frac{x^2}{2}+o(x^2), \qquad e^x = 1+x+\frac{x^2}{2}+o(x^2).$$

Then

$$\cos x-e^x+x = 1-\frac{x^2}{2}+o(x^2)-1-x-\frac{x^2}{2}+o(x^2)+x = -x^2+o(x^2).$$

Now

$$\lim_{x\to 0}\frac{\cos x-e^x+x}{x^2} = \lim_{x\to 0}\frac{-x^2+o(x^2)}{x^2} = -1+\lim_{x\to 0}\frac{o(x^2)}{x^2}.$$

But by definition the last limit is 0, so

$$\lim_{x\to 0}\frac{\cos x-e^x+x}{x^2} = -1.$$

■

■ **Example 8.5** In Example 7.12 we calculated the limit

$$\lim_{x\to 0}\frac{e^x-x-\cos x}{\sin x^2},$$

by applying twice l'Hôpital's rule. Let us do the same using Taylor expansion. As for the denominator, since $\sin y = y+o(y)$,

$$\sin x^2 = x^2+o(x^2).$$

$f(x)$	$P_{k,0}(x)$	$o(x^k)$
$(1+x)^\alpha$	$1+\alpha x+\frac{\alpha(\alpha-1)}{2}x^2+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n$	$o(x^n)$
$\log(1+x)$	$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+(-1)^{n+1}\frac{x^n}{n}$	$o(x^n)$
e^x	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\cdots+\frac{x^n}{n!}$	$o(x^n)$
$\sin x$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}$	$o(x^{2n+2})$
$\sinh x$	$x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\cdots+\frac{x^{2n+1}}{(2n+1)!}$	$o(x^{2n+2})$
$\cos x$	$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}$	$o(x^{2n+1})$
$\cosh x$	$1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots+\frac{x^{2n}}{(2n)!}$	$o(x^{2n+1})$
$\arcsin x$	$x+\frac{x^3}{2\cdot 3}+\frac{1\cdot 3}{2\cdot 4\cdot 5}x^5+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7+\cdots+\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)(2n+1)}x^{2n+1}$	$o(x^{2n+2})$
$\operatorname{arcsinh} x$	$x-\frac{x^3}{2\cdot 3}+\frac{1\cdot 3}{2\cdot 4\cdot 5}x^5-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7+\cdots+(-1)^n\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)(2n+1)}x^{2n+1}$	$o(x^{2n+2})$
$\arctan x$	$x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots+(-1)^n\frac{x^{2n+1}}{2n+1}$	$o(x^{2n+2})$
$\operatorname{arctanh} x$	$x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\cdots+\frac{x^{2n+1}}{2n+1}$	$o(x^{2n+2})$
$\tan x$	$x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7$	$o(x^8)$
$\tanh x$	$x-\frac{1}{3}x^3+\frac{2}{15}x^5-\frac{17}{315}x^7$	$o(x^8)$

Table 8.1: Taylor polynomials of some elementary functions as $x \to 0$. (Here $\alpha \in \mathbb{R}$.)

This suggests expanding the numerator up to x^2. Thus,

$$e^x - x - \cos x = 1 + x + \frac{x^2}{2} + o(x^2) - x - \left[1 - \frac{x^2}{2} + o(x^2)\right] = x^2 + o(x^2)$$

Therefore

$$\lim_{x\to 0} \frac{e^x - x - \cos x}{\sin x^2} = \lim_{x\to 0} \frac{x^2 + o(x^2)}{x^2 + o(x^2)} = \lim_{x\to 0} \frac{1+o(1)}{1+o(1)} = 1.$$

■

■ **Example 8.6** Here is a complicated limit:

$$\lim_{x\to\infty} x^2 \left(1 - \sec\frac{1}{x}\right).$$

To calculate this limit it is convenient to change to the variable $t = 1/x$, so that

$$\lim_{x\to\infty} x^2\left(1-\sec\frac{1}{x}\right) = \lim_{t\to 0^+}\frac{1}{t^2}\left(1-\frac{1}{\cos t}\right).$$

Now, $\cos t = 1-\frac{t^2}{2}+o(t^2)$ $(t\to 0)$, therefore the limit becomes

$$\lim_{t\to 0^+}\frac{1}{t^2}\left(1-\frac{1}{1-\frac{t^2}{2}+o(t^2)}\right) = \lim_{t\to 0^+}\frac{1}{t^2}\frac{1-\frac{t^2}{2}+o(t^2)-1}{1-\frac{t^2}{2}+o(t^2)} = \lim_{t\to 0^+}\frac{-\frac{1}{2}+o(1)}{1+o(1)} = -\frac{1}{2}.$$

Here we have used that $o(t^2)/t^2 = o(1)$, $-\frac{t^2}{2} = o(1)$, and $o(t^2) = o(1)$, as $t\to 0$. ■

8.4 Remainder and Taylor's theorem

The difference between the function $f(x)$ and its Taylor polynomial $P_{n,a}(x)$ is called the remainder, and denoted $R_{n,a}(x)$. It is the error we make when approximating $f(x)$ by its Taylor polynomial. So far we only know that $R_{n,a}(x) = o\left((x-a)^n\right)$ as $x\to a$, but it would be interesting to have a quantitative estimate of that error. As a matter of fact, Cauchy's mean value theorem can help us in deriving such an expression. The result is a second version of Taylor's theorem that yields an explicit form for the remainder.

Theorem 8.4.1 — Taylor's theorem (II). Let f be a function $n+1$ times differentiable in an interval I, and let $a\in I$. Then, for every $x\in I$ there exists some c between a and x such that

$$f(x) = P_{n,a}(x)+\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}. \tag{8.8}$$

Proof. For each $b\in I$ let us define two functions:

$$F(x) = f(b)-P_{n,x}(b), \qquad G(x) = (b-x)^{n+1}, \qquad x\in I.$$

Note that

$$P_{n,x}(b) = \sum_{k=0}^{n}\frac{f^{(k)}(x)}{k!}(b-x)^k$$

is no longer a polynomial, but a complicated combination of the function $f(x)$ and its n first derivatives. Also note that both functions, F and G, are differentiable in I.

Under this conditions Cauchy's mean value theorem states that there exists c between a and b such that

$$\frac{F(b)-F(a)}{G(b)-G(a)} = \frac{F'(c)}{G'(c)}.$$

Let us now compute all these terms and see what this expression amounts to.

First of all $P_{n,b}(b) = f(b)$, hence $F(b) = 0$. Then, $F(a) = f(b)-P_{n,a}(b)$. Now, $G(b) = 0$ and $G(a) = (b-a)^{n+1}$. Thus,

$$\frac{F(b)-F(a)}{G(b)-G(a)} = \frac{F(a)}{G(a)} = \frac{f(b)-P_{n,a}(b)}{(b-a)^{n+1}}.$$

On the other hand $G'(x) = -(n+1)(b-x)^n$, and

$$\frac{d}{dx}P_{n,x}(b) = f'(x) + \sum_{k=1}^{n}\left[\frac{f^{(k+1)}(x)}{k!}(b-x)^k + \frac{f^{(k)}(x)}{k!}k(b-x)^{k-1}(-1)\right]$$
$$= f'(x) + \sum_{k=1}^{n}\left[\frac{f^{(k+1)}(x)}{k!}(b-x)^k - \frac{f^{(k)}(x)}{(k-1)!}(b-x)^{k-1}\right].$$

The sum in the last expression is telescoping. Thus,

$$\sum_{k=1}^{n}\left[\frac{f^{(k+1)}(x)}{k!}(b-x)^k - \frac{f^{(k)}(x)}{(k-1)!}(b-x)^{k-1}\right] = \frac{f^{(n+1)}(x)}{n!}(b-x)^n - f'(x),$$

which implies

$$F'(x) = -\frac{d}{dx}P_{n,x}(b) = -\frac{f^{(n+1)}(x)}{n!}(b-x)^n.$$

Accordingly,

$$\frac{F'(c)}{G'(c)} = \frac{-\frac{f^{(n+1)}(c)}{n!}(b-c)^n}{-(n+1)(b-c)^n} = \frac{f^{(n+1)}(c)}{(n+1)!}.$$

The theorem thus reads

$$\frac{f(b) - P_{n,a}(b)}{(b-a)^{n+1}} = \frac{f^{(n+1)}(c)}{(n+1)!},$$

or equivalently

$$f(b) = P_{n,a}(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Changing the name of b to x this is the stament of the theorem. ■

The theorem provides an expression for the remainder that bears the name **Lagrange's remainder.** Given that $c = (1-\theta)a + \theta x$ for some $0 < \theta < 1$, it is customary to write it in the form

$$R_{n,a}(x) = \frac{f^{(n+1)}\big((1-\theta)a + \theta x\big)}{(n+1)!}(x-a)^{n+1}, \qquad 0 < \theta < 1. \tag{8.9}$$

This is not the only possible form in which we can write the remainder. For instance, in the proof of the theorem Cauchy chose to apply Cauchy's mean value theorem to the same function $F(x)$ but to $G(x) = b - x$. The expression this yields for the remainder is

$$R_{n,a}(x) = \frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-a).$$

So we have **Cauchy's remainder** that can be expressed as

$$R_{n,a}(x) = \frac{f^{(n+1)}\big((1-\theta)a+\theta x\big)}{n!}(1-\theta)^n(x-a), \qquad 0<\theta<1. \tag{8.10}$$

In further chapters we will see yet another form for the remainder involving integrals.

Corollary 8.4.2 $P_n(x) = P_{n,a}(x)$ for any polynomial $P_n(x)$ of degree n and every $a \in \mathbb{R}$.

Proof. If $f(x) = P_n(x)$ then $R_{n,a}(x) = 0$ no matter what a is because $f^{n+1}(a) = 0$ for every $a \in \mathbb{R}$. Hence Taylor formula is exact. ■

■ **Example 8.7** To write down the polynomial $P(x) = 1-2x^2+x^3$ in powers of $x-1$ all we need to do is to obtain its Taylor's polynomial $P_{3,1}(x)$. Since

$$\begin{aligned}
P(x) &= 1-2x^2+x^3, & P(1) &= 0,\\
P'(x) &= -4x+3x^2, & P'(1) &= -1,\\
P''(x) &= -4+6x, & P''(1) &= 2,\\
P'''(x) &= 6, & P'''(1) &= 6,
\end{aligned}$$

we have $P(x) = P_{3,1}(x) = -(x-1)+(x-1)^2+(x-1)^3$. (Check that it is indeed the same polynomial by expanding the two binomials and simplifying.) ■

8.5 Taylor series

Suppose we have a function f that can be differentiated infinitely often in an interval containing a. For this function we have a formula

$$f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{k,a}(x)$$

for each $k \in \mathbb{N}$ and every x in the interval. We can then take the limit when $k \to \infty$ in this expression. Since the left-hand side does not depend on k we will obtain

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n + \lim_{k\to\infty} R_{k,a}(x). \tag{8.11}$$

Those functions for which

$$\lim_{k\to\infty} R_{k,a}(x) = 0 \tag{8.12}$$

for every x in the interval are given by their **Taylor series**

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \tag{8.13}$$

as long as the series converges.

The expression we have just obtained is a particular case of a class of series referred to as **power series.** These are series of the form

$$\sum_{n=0}^{\infty} a_n(x-a)^n, \qquad a_n \in \mathbb{R}. \tag{8.14}$$

We can assess the absolute convergence of these series using the root test. Accordingly, one such series will converge absolutely if

$$\lim_{n\to\infty}\sqrt[n]{|a_n||x-a|^n}<1 \qquad\Leftrightarrow\qquad \left(\lim_{n\to\infty}\sqrt[n]{|a_n|}\right)|x-a|<1. \tag{8.15}$$

We can define the number $\rho>0$ by the formula

$$\frac{1}{\rho}\equiv\lim_{n\to\infty}\sqrt[n]{|a_n|}. \tag{8.16}$$

We refer to ρ as the **convergence radius** of the series because condition (8.15) holds for every x such that

$$|x-a|<\rho. \tag{8.17}$$

In other words, the series (8.14) converges absolutely in the interval $(a-\rho,a+\rho)$.

What happens outside this interval? If $|x-a|>\rho$ then

$$\lim_{n\to\infty}\sqrt[n]{|a_n||x-a|^n}=\frac{|x-a|}{\rho}\equiv\ell>1.$$

Hence, for any $\varepsilon>0$,

$$\ell-\varepsilon<\sqrt[n]{|a_n||x-a|^n}<\ell+\varepsilon \qquad\Rightarrow\qquad (\ell-\varepsilon)^n<|a_n||x-a|^n$$

if n is large enough. As we can take ε so that $\ell-\varepsilon>1$ (e.g., $\varepsilon=(\ell-1)/2$), then

$$\lim_{n\to\infty}(\ell-\varepsilon)^n=\infty \qquad\Rightarrow\qquad \lim_{n\to\infty}|a_n||x-a|^n=\infty \qquad\Rightarrow\qquad \lim_{n\to\infty}a_n(x-a)^n\neq 0.$$

Therefore the power series does not converge.

In summary, the power series converges if $x\in(a-\rho,a+\rho)$ and diverges otherwise, except maybe at $x=a\pm\rho$ (where the root test yields a limit 1 and does not decide). At these two points the analysis has to be done on a case-by-case basis.

R As a consequence of Corollary 3.4.4, the convergence radius can also be obtained as

$$\frac{1}{\rho}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

provided this limit exists.

■ **Example 8.8** Consider the Taylor expansion of $f(x)=e^x$ with remainder. Given that $f^{(n)}(x)=e^x$ we will have

$$e^x=\sum_{k=0}^{n}\frac{x^n}{n!}+R_{n,0}(x), \qquad R_{n,0}(x)=e^{\theta x}\frac{x^{n+1}}{(n+1)!}, \qquad 0<\theta<1.$$

Since the exponential is an increasing function, $e^{\theta x}<\max\{1,e^x\}$ —that includes the cases $x>0$ and $x<0$. Therefore

$$0<R_{n,0}(x)<\max\{1,e^x\}\frac{x^n}{n!} \xrightarrow[n\to\infty]{} 0$$

for any given $x\in\mathbb{R}$. Hence

$$e^x=\sum_{k=0}^{\infty}\frac{x^n}{n!}.$$

■

Table 8.2 is a version of Table 8.1 containing a list with the Taylor series of some elementary fractions, along with their convergence radii. For some of these series is easy to prove the equality with the function —as the case of the exponential—, for other it is more difficult, but for all of them it can be proven that $R_{n,0}(x) \to 0$ as $n \to \infty$.

A nice —and very useful— property of power series is that they can be differentiated within their interval of convergence. The precise statement of this property is:

Theorem 8.5.1 Let $f(x)$ be an infinitely often differentiable function such that

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n, \qquad |x-a| < \rho.$$

Then

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-a)^n,$$

and the convergence radius of this series is also ρ.

Aside from the practical applications of this theorem, there is a very important consequence that we can extract from it:

Corollary 8.5.2 The power series of $f(x)$ in Theorem 8.5.1 is unique.

Proof. According to Theorem 8.5.1, the series for f can be differentiated term by term infinitely often, and

$$f^{(k)}(x) = \sum_{n=0}^{\infty} (n+k)(n+k-1)\cdots(n+1) a_{n+k} (x-a)^n,$$

so $f^{(k)}(a) = k! a_k$, which implies $a_k = f^{(k)}(a)/k!$—hence the coefficients are uniquely determined by the function and its derivatives at $x = a$. ■

■ **Example 8.9** The differentiability of power series helps finding some series. For instance, suppose we want to obtain a power series, in powers of x, of $f(x) = \arctan x$. We know of this function that is odd, so its powers series can only contain odd powers. In other words,

$$\arctan x = \sum_{n=0}^{\infty} a_n x^{2n+1}.$$

But we also know that

$$(\arctan x)' = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)},$$

so

$$\sum_{n=0}^{\infty} (2n+1) a_n x^{2n} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \qquad |x| < 1.$$

As power series are unique, the coefficients of the two series that represent the same function must be the same, therefore $a_n = (-1)^n/(2n+1)$ and

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}, \qquad |x| < 1.$$

■

$f(x)$	Taylor series	ρ
$(1+x)^\alpha$	$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$	1
$\log(1+x)$	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$	1
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	∞
$\sin x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	∞
$\sinh x$	$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$	∞
$\cos x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	∞
$\cosh x$	$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$	∞
$\arcsin x$	$\sum_{n=0}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)(2n+1)} x^{2n+1}$	1
$\operatorname{arcsinh} x$	$\sum_{n=0}^{\infty} (-1)^n \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)(2n+1)} x^{2n+1}$	1
$\arctan x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	1
$\operatorname{arctanh} x$	$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$	1

Table 8.2: Taylor series of some elementary functions as powers of x, along with their convergence radii ρ. (Here $\alpha \in \mathbb{R}$.)

8.6 Numerical approximations

With the expression of the remainder we can find bounds to the error that we incur when approximating a function by its Taylor polynomial of a certain degree. This allows us to obtain numerical values of transcendental functions —which would otherwise be difficult to obtain. Some examples illustrate the method.

■ **Example 8.10** We know that

$$\sin x = x - \frac{x^3}{6} + R_{4,0}(x), \qquad R_{4,0}(x) = \frac{\cos(\theta x)}{120} x^5, \quad 0 < \theta < 1.$$

We of course ignore the value of θ (otherwise $\sin x$ could be exactly computed), but we know that irrespective of θ and x, $|\cos(\theta x)| \leqslant 1$. Thus,

$$|R_{4,0}(x)| \leqslant \frac{|x|^5}{120}.$$

Suppose we want to compute $\sin(0.1)$. From the previous inequality $|R_{4,0}(x)| \leqslant 8.3333 \times 10^{-8}$. Now compare:

$$\sin(0.1) = \mathbf{0.09983}341664\ldots, \qquad P_{4,0}(0.1) = \mathbf{0.09983}33333\ldots$$

The error incurred using this simple approximation is 8.3313×10^{-8}, very close to our estimate.

Suppose we do not want our error to be larger than 10^{-5}. What is the largest x for which we can use this approximation? To answer this question we simply set the estimate to the error tolerance and find $|x|$:

$$\frac{|x|^5}{120} = 10^{-5} \qquad \Rightarrow \qquad |x| = \sqrt[5]{120} \times 0.1 \approx 0.26.$$

■

■ **Example 8.11** Imagine that we want to compute $\sqrt{3.8}$. We can do it by expanding the function $\sqrt{4-x}$ around $x = 0$. Thus,

$$f(x) = \sqrt{4-x}, \qquad f(0) = \sqrt{4} = 2,$$

$$f'(x) = \frac{-1}{2\sqrt{4-x}}, \qquad f'(0) = \frac{-1}{2\sqrt{4}} = -\frac{1}{4},$$

$$f''(x) = \frac{-1}{4(4-x)^{3/2}}, \qquad f''(0) = \frac{-1}{4 \cdot 4^{3/2}} = -\frac{1}{32},$$

$$f'''(x) = \frac{-3}{8(4-x)^{5/2}}.$$

Then

$$\sqrt{4-x} = 2 - \frac{x}{4} - \frac{x^2}{64} + R_{2,0}(x), \qquad R_{2,0}(x) = \frac{-1}{16(4-\theta x)^{5/2}} x^3, \quad 0 < \theta < 1.$$

If $x > 0$,

$$|R_{2,0}(x)| < \frac{x^3}{16(4-x)^{5/2}} = \frac{x^3}{16\left(\sqrt{4-x}\right)^5}.$$

Now we can estimate

$$\sqrt{3.8} = P_{2,0}(0.2) = 2 - \frac{0.2}{4} - \frac{(0.2)^2}{64} = 1.949375\ldots$$

and use this estimation in the error bound

$$|R_{2,0}(x)| < \frac{(0.2)^3}{16(1.949375\ldots)^5} \approx 1.78 \times 10^{-5}.$$

As a matter of fact,

$$\sqrt{3.8} = \mathbf{1.949}\mathbf{3}58869\ldots, \qquad P_{2,0}(0.2) = \mathbf{1.9493}75,$$

the difference being 1.61×10^{-5}. ■

8.7 Local behaviour of functions

We saw in Corollary 7.3.4 that the sign of $f'(x)$ determines wether the function is increasing (positive) or decreasing (negative) at x, and Theorem 7.3.1 showed that at local extrema the function satisfies $f'(x) = 0$ (provided it is differentiable). In its second formulation —with the remainder— Taylor's theorem provides a more detailed information about the local behaviour of a function which has higher order derivatives.

Before getting into it, we need to characterise another qualitative feature of functions: whether their slope increases or decreases. This feature is called *convexity.*

We say that f is **convex** at $x=a$ if it is locally *above* its tangent at that point, i.e., $f(x) > f(a)+f'(a)(x-a)$ for all $0<|x-a|<\varepsilon$, for some $\varepsilon>0$.

Likewise, we say that it is **concave** at $x=a$ if it is locally *below* its tangent a that point, i.e., $f(x)<f(a)+f'(a)(x-a)$ for all $0<|x-a|<\varepsilon$, for some $\varepsilon>0$.

Finally, we say that f has an **inflection point** at $x=a$ if the sign of $f(x)-f(a)-f'(a)(x-a)$ is different for $x<a$ and for $x>a$.

Figure 8.1 illustrates these three behaviours.

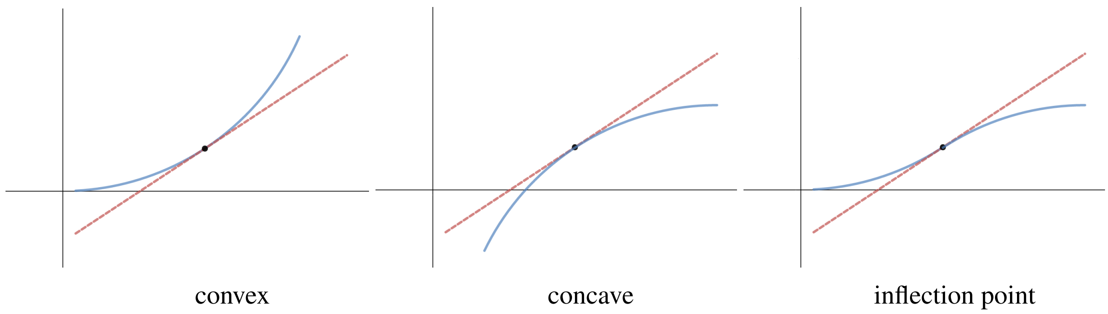


Figure 8.1: Local behaviour of a function with respect to its tangent at a point (convexity).

Suppose that a function f can be differentiated several times (posibly infinitely many) in a certain interval and that the first nonzero derivative beyond the first at $x=a$ is $f^{(n)}(a)$. We can use Taylor's theorem —with Lagrange's remainder— to write

$$f(x)=f(a)+f'(a)(x-a)+\frac{f^{(n)}(c)}{n!}(x-a)^n,$$

where $c=(1-\theta)a+\theta x$ with $0<\theta<1$. One important point to stress here is that, since $f^{(n)}(a)\neq 0$ —so it is either positive or negative—, when x is sufficiently close to a —and so is c— $f^{(n)}(c)$ will have the same sign as $f^{(n)}(a)$. This is key for the argument to come.

Since we can write the Taylor expansion as

$$f(x)-f(a)-f'(a)(x-a)=\frac{f^{(n)}(c)}{n!}(x-a)^n,$$

the sign of the left-hand side —which decides the convexity— will be determined by sign of the product $f^{(n)}(c)(x-a)^n$ or, given what we have just argued, by the sign of the product $f^{(n)}(a)(x-a)^n$.

Now, if n is odd, the sign of $f^{(n)}(a)$ is irrelevant because $(x-a)^n$ has a different sign for $x<a$ and for $x>a$. Therefore a will be an *inflection point.*

If n is even then $(x-a)^n>0$ for all $x\neq a$. Then the sign is determined by that of $f^{(n)}(a)$. We will then have two possibilities:

(a) $f^{(n)}(a)>0$, and then the function is *convex,* or

(b) $f^{(n)}(a)<0$, and then the function is *concave.*

If added to that we have that $f'(a)=0$, then for n odd nothing changes —hence $x=a$ still is an inflection point—, but for n even the point $x=a$ is a local extremum. A convex extremum ($f^{(n)}(a)>0$) is a *local minimum* and a concave extremum ($f^{(n)}(a)<0$) is a *local maximum.*

All these results are summarised in Table 8.3.

n	sign of $f^{(n)}(a)$	$f'(a) \neq 0$	$f'(a) = 0$
odd	$+/-$	inflection point	inflection point
even	$+$	convex	local minimum
even	$-$	convex	local maximum

Table 8.3: Classification of the local behaviour of a function according to the sign of the first nonzero derivative $f^{(n)}(a)$ with $n > 1$.

8.8 Function graphing

All the local information provided by the derivatives can be gathered to sketch a qualitative graph of any function $f(x)$. The steps to follow in graphing a function are these (some of them might not be necessary):

1. **Domain:** Determine precisely the set of points where the function $f(x)$ is defined.
2. **Symmetries:** It is helpful to know whether the function has one of these symmetries:
   (a) *Even:* $f(-x) = f(x)$.
   (b) *Odd:* $f(-x) = -f(x)$.
   (c) *Periodic:* $f(x+c) = f(x)$ for some $c > 0$.
   In the first two cases it is enough to represent the function for $x \geqslant 0$ (for $x < 0$ it is represented using the symmetry). In the last case it is enough to represent the function in the interval $[0,c]$ (or any other interval of the same lenght) and then reproduce its graph periodically.
   Other symmetries might be possible (e.g., $f(a+x) = \pm f(a-x)$, i.e., f is even/odd around the vertical axis $x = a$).
3. **Continuity and differentiability:** Discontinuities ("jumps") and points where $f'(x)$ does not exists ("cusps") are relevant features of the function, and might be useful in detecting local extrema.
4. **Zeroes:** Finding the solutions of $f(x) = 0$ determines where f crosses the X axis. These points separate regions where the sign of f remains constant.
5. **Growth:** Finding the solutions of $f'(x) = 0$ determines the regions where f increases ($f'(x) > 0$) or decreases ($f'(x) > 0$). Usually this is enough to locate the extrema of f.
6. **Convexity:** The convex/concave regions are usually determined by the sign of $f''(x)$. Inflections points can be inferred from that information (as points where the concavity changes).

7. **Asymptotes:** These are known curves (usually straight lines) which $f(x)$ approaches when it gets close to some points or to $\pm\infty$. The main ones are:
   (a) *Vertical asymptotes:* These are the vertical straight lines through the points $x = a$ where $\lim_{x\to a^\pm} f(x) = \pm\infty$.
   (b) *Horizontal asymptotes:* These are the horizontal straight lines $y = \ell$ where ℓ is such that $\lim_{x\to\pm\infty} f(x) = \ell$.
   (c) *Inclined asymptotes:* We say that $y = mx + b$ is an asymptote of $f(x)$ when $x \to \pm\infty$ if
   $$m = \lim_{x\to\pm\infty} \frac{f(x)}{x}, \qquad b = \lim_{x\to\pm\infty} [f(x) - mx].$$
   (In other words, $f(x) = mx + b + o(1)$ $(x \to \pm\infty)$.)
   Other types of asymptote are possible. In general, the curve $y = g(x)$ is an asymptote of f when $x \to \pm\infty$ if $f(x) = g(x) + o(1)$ $(x \to \pm\infty)$.

■ **Example 8.12** Sketch the graph of

$$f(x) = \frac{3x^2 + x + 1}{x + 2}.$$

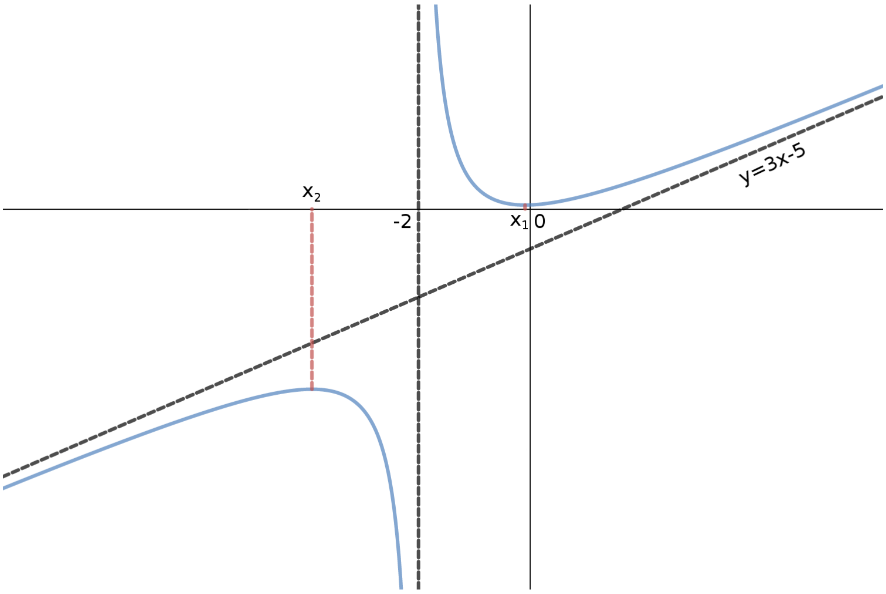


Figure 8.2: Sketch of $f(x) = \dfrac{3x^2+x+1}{x+2}$.

The domain of this function is $\mathbb{R} - \{-2\}$ (because the denominator vanishes at that point.) It has no obvious symmetries and, being a rational function, it is continuous and differentiable (an infinite number of times) in all its domain.

We can obtain the derivative as

$$f'(x) = \frac{(6x+1)(x+2)-(3x^2+x+1)}{(x+2)^2} = \frac{6x^2+13x+2-3x^2-x-1}{(x+2)^2} = \frac{3x^2+12x+1}{(x+2)^2}.$$

This derivative vanishes when $3x^2+12x+1=0$. The roots of this parabola are $x = -2 \pm \sqrt{11/3}$, i.e., $x_1 \approx -0.085$, $x_2 \approx -3.91$. For $x < x_2$ and $x > x_1$ function f increases ($f' > 0$) and for $x_2 < x < x_1$ it decreases ($f' < 0$).

f has no zeros because $3x^2+x+1 > 0$ for all $x \in \mathbb{R}$ (the parabola has no roots). So $f(x) < 0$ for $x < -2$ and $f(x) > 0$ for $x > -2$.

It is not necessary to analyse the concavity, as it can be inferred from all the other information, including that of the asymptotes. We know there is a vertical asymptote at $x = -2$ because

$$\lim_{x\to -2^-} f(x) = -\infty, \qquad \lim_{x\to -2^+} f(x) = +\infty.$$

There are no horizontal asymptotes because f diverges when $x \to \pm\infty$. However, we can express the polynomial $P(x) = 3x^2+x+1$ in powers of $x+2$ using Taylor's polynomial, because $P_{2,-2}(x) = P(x)$. As

$$\begin{aligned} P(x) &= 3x^2+x+1, & \qquad P(-2) &= 11, \\ P'(x) &= 6x+1, & P'(-2) &= -11, \\ P''(x) &= 6, & P''(-2) &= 6, \end{aligned}$$

we have $P(x) = 11 - 11(x+2) + 3(x+2)^2$. Therefore

$$\begin{aligned} f(x) &= \frac{3x^2+x+1}{x+2} = \frac{11-11(x+2)+3(x+2)^2}{x+2} = \frac{11}{x+2} - 11 + 3(x+2) = \frac{11}{x+2} - 5 + 3x \\ &= 3x - 5 + o(1) \quad (x \to \pm\infty), \end{aligned}$$

i.e., $y = 3x - 5$ is an inclined asymptote both when $x \to \pm\infty$.

$f(x)$ is represented in Figure 8.2. ■

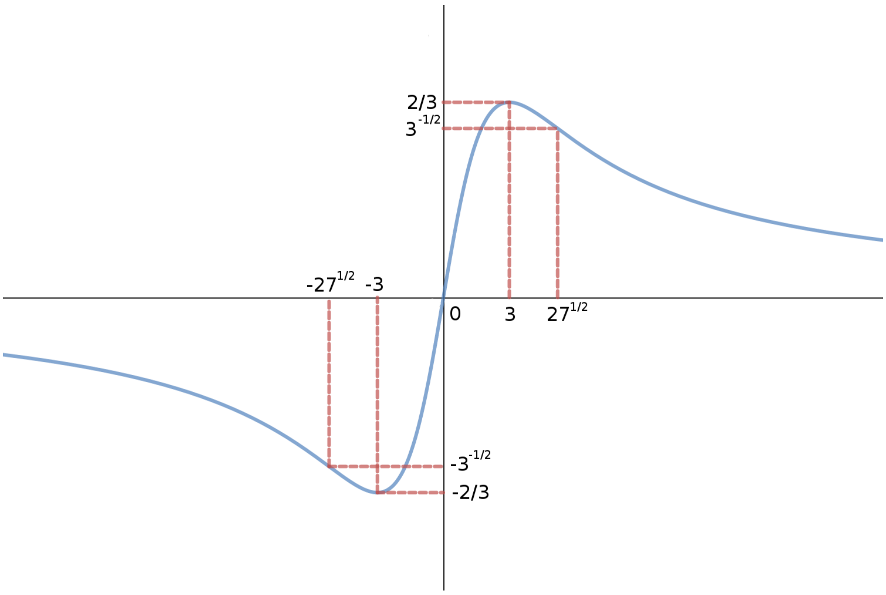


Figure 8.3: Sketch of $f(x) = \dfrac{4x}{x^2+9}$.

■ **Example 8.13** Sketch the graph of

$$f(x) = \frac{4x}{x^2+9}.$$

The domain of this function is $\mathbb{R}$, and it is continuous and differentiable everywhere. It is an odd function because

$$f(-x) = \frac{4(-x)}{(-x)^2+9} = -\frac{4x}{x^2+9} = -f(x),$$

so we only need to care about the region $x \geqslant 0$. As every odd continuous function $f(0) = 0$, and this is the only point where f croses the X axis. Besides $f(x) > 0$ for $x > 0$.

Its derivative is

$$f'(x) = \frac{4(x^2+9) - 4x \cdot 2x}{(x^2+9)^2} = \frac{4x^2+36-8x^2}{(x^2+9)^2} = \frac{4(9-x^2)}{(x^2+9)^2}.$$

Thus, in $x \geqslant 0$ we have $f'(x) > 0$ for $x < 3$ and $f'(x) < 0$ for $x > 3$. The function grows up to $x = 3$, where it has a local maximum, and then decreases beyond that point.

As for the second derivative,

$$\begin{aligned} f''(x) &= \frac{-8x(x^2+9)^2 - (36-4x^2)2(x^2+9)2x}{(x^2+9)^4} = \frac{-8x(x^2+9) - (36-4x^2)4x}{(x^2+9)^3} \\ &= \frac{8x^3 - 216x}{(x^2+9)^3} = \frac{8x(x^2-27)}{(x^2+9)^3}, \end{aligned}$$

so f is concave ($f'' < 0$) for $x < \sqrt{27} = 3\sqrt{3}$ and convex ($f'' > 0$) for $x > 3\sqrt{3}$. At $x = 3\sqrt{3}$ there is an inflection point.

Finally, there are no vertical asymptotes (f is defined in the whole $\mathbb{R}$), but since $\lim_{x\to\infty} f(x) = 0$, the X axis is a horizontal asymptote.

$f(x)$ is represented in Figure 8.3. ■

Problems

Problem 8.1 Write the Taylor polynomial $P_{5,0}(x)$ for these functions:

(i) $e^x \sin x$;
(ii) $e^{-x^2}\cos 2x$;
(iii) $\sin x \cos 2x$;
(iv) $e^x \log(1-x)$;
(v) $\sin^2 x$;
(vi) $\dfrac{1}{1-x^3}$.

Problem 8.2 Write the polynomial $x^4 - 5x^3 + x^2 - 3x + 4$ in powers of $x-4$.

Problem 8.3 Write the Taylor polynomial $P_{n,a}(x)$ for these functions around the specified a:

(i) $f(x) = 1/x$ around $a = -1$;
(ii) $f(x) = xe^{-2x}$ around $a = 0$;
(iii) $f(x) = (1+e^x)^2$ around $a = 0$;
(iv) $f(x) = \sin x$ around $a = \pi$.

Problem 8.4 Consider the function

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0 & x = 0. \end{cases}$$

(i) Prove by induction that $f^{(n)}(x) = Q_n(1/x)e^{-1/x^2}$ for $x \neq 0$, where $Q_n(t)$ is some polynomial.
(ii) Prove by induction that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.
(iii) Write the Taylor polynomial $P_{n,0}(x)$ of $f(x)$. What can you conclude from that?

Problem 8.5 Prove that

(i) $\sin x = o(x^\alpha)$ $(x \to 0)$ for all $\alpha < 1$;
(ii) $\log(1+x^2) = o(x)$ $(x \to 0)$;
(iii) $\log x = o(x)$ $(x \to \infty)$;
(iv) $\tan x - \sin x = o(x^2)$ $(x \to 0)$.

Problem 8.6 Calculate the following limits using Taylor's theorem:

(i) $\displaystyle\lim_{x\to 0} \frac{e^x - \sin x - 1}{x^2}$;
(ii) $\displaystyle\lim_{x\to 0} \frac{\sin x - x + x^3/6}{x^5}$;
(iii) $\displaystyle\lim_{x\to 0} \frac{\cos x - \sqrt{1-x}}{\sin x}$;
(iv) $\displaystyle\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$;
(v) $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x(1-\cos 3x)}$;
(vi) $\displaystyle\lim_{x\to 0} \frac{\cos x + e^x - x - 2}{x^3}$;
(vii) $\displaystyle\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$;
(viii) $\displaystyle\lim_{x\to 0} \frac{1}{x}\left(\frac{1}{x} - \cot x\right)$;
(ix) $\displaystyle\lim_{x\to\infty} x^{3/2}\left(\sqrt{x+1} + \sqrt{x-1} - 2\sqrt{x}\right)$;
(x) $\displaystyle\lim_{x\to\infty} \left[x - x^2 \log\left(1 + \frac{1}{x}\right)\right]$.

Problem 8.7 If $f(x) = -\dfrac{x}{2} - \dfrac{x^2}{4} + o(x^2)$ $(x \to 0)$, calculate

$$\lim_{x\to 0} \frac{\log[1+f(x)] + x/2}{x^2}.$$

Problem 8.8 Prove that the function

$$f(x) = \begin{cases} \dfrac{1}{x} - \dfrac{1}{e^x - 1}, & \text{if } x \neq 0, \\ \dfrac{1}{2}, & \text{if } x = 0, \end{cases}$$

is differentiable at $x = 0$ by calculating $f'(0)$ from the definition.

Problem 8.9 Determine the first nonzero order in the Taylor expansion of the following functions:

(i) $f(x) = \tan(\sin x) - \sin(\tan x)$;

(ii) $f(x) = \dfrac{1}{R^2} - \dfrac{1}{(R+x)^2}$;

(iii) $f(x) = \sqrt[3]{\dfrac{1+x}{1-x}} - \sqrt[3]{\dfrac{1-x}{1+x}}$.

Problem 8.10 Consider the function

$$f(x) = \frac{1-\cos x}{1+\cos x}.$$

This function is even and $f(0) = 0$, so its Taylor expansion up to 7th order will be

$$f(x) = Ax^2 + Bx^4 + Cx^6 + o(x^7), \quad (x \to 0).$$

Then

$$1 - \cos x = \left[Ax^2 + Bx^4 + Cx^6 + o(x^7)\right](1+\cos x).$$

Using the Taylor expansion of $\cos x$ up to 7th order find the coefficients A, B, and C from this equation.

Problem 8.11 Find coefficients a and b so that

(i) $x - (a + b\cos x)\sin x = o(x^4)$ $(x \to 0)$;

(ii) $\cot x - \dfrac{1+ax^2}{x+bx^3} = o(x^4)$ $(x \to 0)$.

Problem 8.12 Find constants $a, b, c, d \in \mathbb{R}$ such that

$$e^x = \frac{1+ax+bx^2}{1+cx+dx^2} + o(x^4) \quad (x \to 0).$$

Problem 8.13 Given that $\sqrt{1+x} = 1 + \frac{x}{2} + o(x)$ $(x \to 0)$, prove:

(i) $\lim\limits_{n\to\infty} \sin\left(\pi\sqrt{1+n^2}\right) = 0$;

(ii) $\sum\limits_{n=0}^{\infty} \sin^2\left(\pi\sqrt{1+n^2}\right) < \infty$.

Problem 8.14 Calculate the Taylor polynomial $P_{4,0}(x)$ for $f(x) = 1 + x^3 \sin x$. Given the result, does f have a local maximum, minimum or inflection point at $x = 0$?

Problem 8.15 Use a Taylor polynomial of the specified degree to provide an approximation to these numbers, and give an upper bound for the error incurred:

(i) $\dfrac{1}{\sqrt{1.1}}$, degree 3;

(ii) $\sqrt[3]{28}$, degree 2.

Problem 8.16 Given the function $f(x) = \cos x + e^x$,

(i) find its Taylor polynomial $P_{3,0}(x)$;

(ii) estimate an upper bound for the error incurred if $-1/4 \leqslant x \leqslant 1/4$.

Problem 8.17 What is the smallest degree Taylor polynomial necessary to approximate the function $f(x) = e^x$ in $[-1, 1]$ with at least three exact decimal places?

Problem 8.18 Determine the convergence radius of the following power series, and specify the interval where they converge absolutely:

(i) $\sum_{n=1}^{\infty} \frac{x^n}{2^n n^2}$; (iii) $\sum_{n=1}^{\infty} \frac{x^n}{n10^{n-1}}$; (v) $\sum_{n=0}^{\infty} (3-2x)^n$;

(ii) $\sum_{n=1}^{\infty} \frac{n!x^n}{n^n}$; (iv) $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$; (vi) $\sum_{n=1}^{\infty} \frac{(x-2)^n}{\sqrt{2n}}$.

Problem 8.19 Expand the function $f(x) = \dfrac{1}{(1-x)^k}$ for $k = 1$, 2, and 3.

Problem 8.20 Consider the power series

$$\frac{1}{x^2+x+1} = \sum_{n=0}^{\infty} a_n x^n.$$

What are the values of the coefficients a_{300}, a_{301}, and a_{302}?
HINT: Recall that $1-x^3 = (1-x)(x^2+x+1)$.

Problem 8.21 Calculate the derivatives $f^{(100)}(0)$ and $f^{(231)}(0)$ of the function $f(x) = \log\left(1+x^2\right)$.

Problem 8.22 Determine the convergence radius of the following power series, and calculate their sums:

(i) $\sum_{n=1}^{\infty} \frac{x^n}{n}$; (ii) $\sum_{n=0}^{\infty} (n+1)2^{-n}x^n$.

Problem 8.23 Expand in power series the following functions, specifying the domain of validity of those expansions:

(i) $f(x) = \sin^2 x$; (iii) $f(x) = \dfrac{x}{a+bx}$; (v) $f(x) = \dfrac{1+x-(1-x)e^{2x}}{e^x}$.

(ii) $f(x) = \log\sqrt{\dfrac{1+x}{1-x}}$; (iv) $f(x) = \dfrac{1}{2-x^2}$;

Problem 8.24 Sum the following series:

(i) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$; (iii) $\sum_{n=1}^{\infty} \frac{1}{n2^n}$;

(ii) $\sum_{n=1}^{\infty} \frac{n}{2^n}$; (iv) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

Problem 8.25 Given the function $f(x) = \sum_{n=1}^{\infty} \dfrac{n^x}{n!}$, compute the values $f(0)$, $f(1)$, and $f(2)$.

Problem 8.26 Find a function $f(x)$ that can be expaded in power series of x and such that it satisfies the equation $f'(x) = f(x) + x$ with the condition $f(0) = 2$.

Problem 8.27 Prove that if f and g are twice differentiable, convex functions, and f is increasing, then $h = f \circ g$ is convex.

Problem 8.28 Discuss the convexity of the following functions:

(i) $f(x) = (x-2)x^{2/3}$; (ii) $f(x) = |x|e^{|x|}$; (iii) $f(x) = \log(x^2-6x+8)$.

Problem 8.29

(i) Sketch the graph of the function $f(x) = x + \log|x^2-1|$.

(ii) Based on the previous graph, plot function $g(x) = |x| + \log|x^2-1|$ and $h(x) = \big|x + \log|x^2-1|\big|$.

Problem 8.30 Sketch a plot of the following functions:

(i) $f(x) = e^x \sin x$;

(ii) $f(x) = \sqrt{x^2-1} - 1$;

(iii) $f(x) = xe^{1/x}$;

(iv) $f(x) = x^2 e^x$;

(v) $f(x) = (x-2)x^{2/3}$;

(vi) $f(x) = (x^2-1)\log\left(\dfrac{1+x}{1-x}\right)$;

(vii) $f(x) = \dfrac{x}{\log x}$;

(viii) $f(x) = \dfrac{x^2-1}{x^2+1}$;

(ix) $f(x) = \dfrac{e^{1/x}}{1-x}$;

(x) $f(x) = \log[(x-1)(x-2)]$;

(xi) $f(x) = \dfrac{e^x}{x(x-1)}$;

(xii) $f(x) = 2\sin x + \cos 2x$;

(xiii) $f(x) = \dfrac{x-2}{\sqrt{4x^2+1}}$;

(xiv) $f(x) = \sqrt{|x-4|}$;

(xv) $f(x) = \dfrac{1}{1+e^x}$;

(xvi) $f(x) = \dfrac{e^{2x}}{e^x-1}$;

(xvii) $f(x) = e^{-x}\sin x$;

(xviii) $f(x) = x^2 \sin\dfrac{1}{x}$.

Problem 8.31 Draw the graph of the following functions:

(i) $f(x) = \min\{\log|x^3-3|, \log|x+3|\}$;

(ii) $f(x) = \dfrac{1}{|x|-1} - \dfrac{1}{|x-1|}$;

(iii) $f(x) = \dfrac{1}{1+|x|} - \dfrac{1}{1+|x-a|}$, $(a>0)$;

(iv) $f(x) = x\sqrt{x^2-1}$;

(v) $f(x) = \arctan\log|x^2-1|$;

(vi) $f(x) = 2\arctan x + \arcsin\left(\dfrac{2x}{1+x^2}\right)$.

Problem 8.32 Plot the function

$$f(x) = \begin{cases} \dfrac{e^{1/x}}{1+x}, & x \neq 0, \\ 0 & x = 0, \end{cases}$$

and discuss how many real solutions has the equation $\dfrac{e^{1/x}}{1+x} = x^3$.

Problem 8.33 Given the function $f(x) = \dfrac{1+x}{3+x^2}$ plot the functions $g(x) = \sup_{y>x} f(y)$ and $h(x) = \inf_{y>x} f(y)$.

Problem 8.34 Determine the equations of the tangents to $f(x) = \log(1+x^2)$ at its inflection points and plot them along with the graph of $f(x)$.

IV Integral Calculus

9. Primitives

Differentiating is an operation that transforms an appropriate function f into another function f', which we refer to as its derivative. It makes sense to wonder about the inverse operation, i.e., given the function f' to determine f.

Definition 9.0.1 — Primitive. A function F is called a **primitive** of f if $F' = f$. We denote this operation

$$\int f(x)\,dx = F(x). \tag{9.1}$$

(Function f is called **integrand.**)

The first question we can ask is whether the primitive, if it exists, is unique. According to Corollary 7.3.4 the answer is no —but almost so. The reason is that if F and G are such that $F' = G' = f$ (i.e., F and G are two primitives of f), then $F(x) = G(x) + c$ for some constant c. Thus, primitives are unique up to an additive constant.

Some properties of primitives are inherited from those of derivatives. For instance, primitives are *linear,* i.e., given functions f and g and constants $a, b \in \mathbb{R}$,

$$\int [af(x) + bg(x)]\,dx = a\int f(x)\,dx + b\int g(x)\,dx. \tag{9.2}$$

We can obtain a few elementary primitives by reversing the derivatives Table 7.1. The list is shown in Table 9.1.

Some primitives have the pattern

$$\int f'\big(g(x)\big)g'(x)\,dx = f\big(g(x)\big) + c. \tag{9.3}$$

We call this primitives *immediate.*

$f(x)$	$F(x)$	$f(x)$	$F(x)$	$f(x)$	$F(x)$
x^α $(\alpha \neq -1)$	$\dfrac{x^{\alpha+1}}{\alpha+1}$	$\sin x$	$-\cos x$	$\dfrac{1}{1+x^2}$	$\arctan x$
x^{-1}	$\log\lvert x\rvert$	$\cos x$	$\sin x$	$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x$
e^x	e^x	$\sinh x$	$\cosh x$	$\dfrac{1}{\cos^2 x}$	$\tan x$
a^x	$\dfrac{a^x}{\log a}$	$\cosh x$	$\sinh x$	$\dfrac{1}{\cosh^2 x}$	$\tanh x$

Table 9.1: Primitives $F(x)$ of some elementary functions $f(x)$ (up to the additive constant) as obtained by reversing Table 7.1. Here $\alpha \in \mathbb{R}$, $a > 0$.

Here are some important special cases:

$$\int \frac{g'(x)}{g(x)}\,dx = \log|g(x)| + c, \qquad \int g'(x)[g(x)]^\alpha\,dx = \frac{g(x)^{\alpha+1}}{\alpha+1}, \quad \alpha \neq -1, \tag{9.4}$$

$$\int \frac{g'(x)}{1+g(x)^2}\,dx = \arctan g(x) + c, \qquad \int \frac{g'(x)}{\sqrt{1-g(x)^2}}\,dx = \arcsin g(x) + c. \tag{9.5}$$

■ **Example 9.1** The primitive

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

has nearly the form

$$\int \frac{g'(x)}{g(x)}\,dx$$

because $(\cos x)' = -\sin x$. Then

$$\int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\,dx = -\int \frac{(\cos x)'}{\cos x}\,dx.$$

Therefore

$$\int \tan x\,dx = -\log|\cos x| + c. \tag{9.6}$$

By a similar argument

$$\int \cot x\,dx = \log|\sin x| + c. \tag{9.7}$$

■

■ **Example 9.2** Here is a more involved example:

$$\int \sec x\,dx = \int \frac{dx}{\cos x}.$$

In order to find this primitive, let us first compute

$$(\sec x)' = \frac{\sin x}{\cos^2 x} = \sec x \tan x, \qquad (\tan x)' = \sec^2 x.$$

Adding up these two equations we get

$$(\sec x + \tan x)' = \sec x \tan x + \sec^2 x = \sec x(\tan x + \sec x).$$

Therefore

$$\frac{(\sec x + \tan x)'}{\sec x + \tan x} = \sec x$$

and from this we conclude

$$\int \sec x \, dx = \log|\sec x + \tan x| + c = \log\left|\frac{1+\sin x}{\cos x}\right| + c. \tag{9.8}$$

Similarly we obtain

$$\int \csc x \, dx = -\log|\csc x + \cot x| + c = \log\left|\frac{\sin x}{1+\cos x}\right| + c. \tag{9.9}$$

Notice that

$$\begin{aligned}
\left(\frac{1+\sin x}{\cos x}\right)^2 &= \frac{(1+\sin x)^2}{1-\sin^2 x} = \frac{1+\sin x}{1-\sin x} = \frac{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}} \\
&= \left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right)^2 = \left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)^2 = \tan^2\left(\frac{x}{2}+\frac{\pi}{4}\right), \\
\left(\frac{\sin x}{1+\cos x}\right)^2 &= \frac{1-\cos^2 x}{(1+\cos x)^2} = \frac{1-\cos x}{1+\cos x} = \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}} = \tan^2\frac{x}{2},
\end{aligned}$$

therefore, we have the alternative expressions

$$\begin{aligned}
\log\left|\frac{1+\sin x}{\cos x}\right| &= \frac{1}{2}\log\left(\frac{1+\sin x}{1-\sin x}\right) = \log\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right) = \log\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|, \\
\log\left|\frac{\sin x}{1+\cos x}\right| &= \frac{1}{2}\log\left(\frac{1-\cos x}{1+\cos x}\right) = \log\left|\tan\frac{x}{2}\right|
\end{aligned}$$

■

9.1 Integration by parts

Theorem 9.1.1 If f and g are two differentiable functions, then

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx. \tag{9.10}$$

Proof. Since $f(x)g(x)$ is the primitive of $\left[f(x)g(x)\right]'$, we have

$$f(x)g(x) = \int \left[f(x)g(x)\right]' dx = \int \left[f'(x)g(x) + f(x)g'(x)\right] dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

From here equation (9.10) follows straight away. ■

This is one of the most useful integration techniques, as a few examples will reveal.

■ **Example 9.3** A classic example is the integral

$$\int xe^x\,dx.$$

Since $e^x = (e^x)'$ it is easy to recognise the left-hand side of (9.10). Therefore

$$\int xe^x\,dx = xe^x - \int 1\cdot e^x\,dx = xe^x - e^x + c = (x-1)e^x + c.$$

■

This example can be generalised whenever we have a function $g'(x)$ easy to integrate several times (e.g., an exponential, a power, a trigonometric function...) multiplied by a polynomial. The polynomial plays the role of function $f(x)$, and we have to apply integration by parts as many times as the degree of the polynomial. (The example above is one of those cases, in which the polynomial has degree 1.)

Exercise 9.1 Calculate

$$\int (x^2+1)\sin(2x-1)\,dx.$$

■

■ **Example 9.4** Often we cannot see $g'(x)$ explicitly because $g'(x) = 1$. For example, in the integral

$$\int \log x\,dx.$$

If $g'(x) = 1$ then $g(x) = x$, therefore

$$\int \log x\,dx = x\log x - \int x\cdot\frac{1}{x}\,dx = x\log x - \int dx = x\log x - x + c.$$

Thus, we can add the primitive of yet another elementary function to our list:

$$\int \log x\,dx = x\log x - x + c. \tag{9.11}$$

■

We can generalise this example to obtain the primitive of an inverse f^{-1} if we know that $F(x)$ is a primitive of $f(x)$:

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int x\left(f^{-1}\right)'(x)\,dx.$$

But $x = f\left(f^{-1}(x)\right)$, therefore

$$\int x(f^{-1})'(x)\,dx = \int f\left(f^{-1}(x)\right)\left(f^{-1}\right)'(x)\,dx = F\left(f^{-1}(x)\right)$$

because the last integral matches the pattern of an immediate integral. Thus, we can conclude:

Theorem 9.1.2 If function f has an inverse f^{-1} and $F'(x) = f(x)$, then

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - F\left(f^{-1}(x)\right). \tag{9.12}$$

■ **Example 9.5** We know that

$$\int \tan x\,dx = -\log|\cos x| \equiv F(x)$$

is a primitive of $f(x) = \tan x$. Therefore

$$\int \arctan x\,dx = x\arctan x + \log|\cos(\arctan x)| + c.$$

We can simplify this expression if we rewrite the cosine in terms of the tangent. Since

$$\cos^2 x = \frac{1}{1+\tan^2 x} \qquad \Rightarrow \qquad \cos x = (1+\tan^2 x)^{-1/2},$$

then

$$\log|\cos(\arctan x)| = \log|(1+x^2)^{-1/2}| = -\frac{1}{2}\log(1+x^2).$$

Thus

$$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\log(1+x^2) + c. \tag{9.13}$$

■

Exercise 9.2 Prove that

$$\int \arcsin x\,dx = x\arcsin x + \sqrt{1-x^2} + c. \tag{9.14}$$

■

■ **Example 9.6** Another typical use of the integration by parts is to recover the same integral after applying the formula. Such is the case of

$$\int \frac{\log x}{x}\,dx.$$

since $1/x = (\log x)'$,

$$\int \frac{\log x}{x}\,dx = (\log x)^2 - \int \log x \frac{dx}{x}.$$

From this we conclude that

$$2\int \frac{\log x}{x}\,dx = (\log x)^2$$

thus

$$\int \frac{\log x}{x}\,dx = \frac{1}{2}(\log x)^2 + c.$$

■

■ **Example 9.7** The integrals

$$\int e^x \sin x\,dx, \qquad \int e^x \cos x\,dx,$$

are another example of the same technique, where we have to integrate by parts more than once. In the first integration we identify $g'(x) = \sin x$ and get

$$\int e^x \sin x\,dx = e^x(-\cos x) - \int (-\cos x)e^x\,dx = -e^x \cos x + \int \cos x e^x\,dx.$$

In the second integration we identify $g'(x) = \cos x$, so

$$\int \cos x e^x\,dx = e^x \sin x - \int \sin x e^x\,dx.$$

Therefore, if we denote

$$S \equiv \int e^x \sin x\,dx, \qquad C \equiv \int e^x \cos x\,dx,$$

what we have obtained are the equations

$$S = -e^x \cos x + C, \qquad C = e^x \sin x - S.$$

Solving this system we obtain

$$S = \frac{e^x}{2}(\sin x - \cos x) + c, \qquad C = \frac{e^x}{2}(\sin x + \cos x) + c.$$

■

■ **Example 9.8** Another technique associated to the integration by parts is the construction of recurrence formulas. This is illustrated by the example

$$I_n(x) = \int \frac{dx}{(1+x^2)^n},$$

whose case $n = 1$ is straightforward: $I_1(x) = \arctan x$. In order to find the recurrence we proceed as follows:

$$I_{n+1}(x) = \int \frac{dx}{(1+x^2)^{n+1}} = \int \frac{1+x^2}{(1+x^2)^{n+1}}\,dx - \int \frac{x^2}{(1+x^2)^{n+1}}\,dx = I_n(x) - \frac{1}{2}\int x \frac{2x}{(1+x^2)^{n+1}}\,dx.$$

We now integrate by parts

$$\int x \frac{2x}{(1+x^2)^{n+1}}\,dx = -\frac{x}{n}\frac{1}{(1+x^2)^n} + \frac{1}{n}\int \frac{dx}{(1+x^2)^n} = \frac{I_n(x)}{n} - \frac{x}{n(1+x^2)^n}.$$

Thus

$$I_{n+1}(x) = I_n(x)\left(1 - \frac{1}{2n}\right) + \frac{x}{2n(1+x^2)^n} = \frac{2n-1}{2n} I_n(x) + \frac{1}{2n}\frac{x}{(1+x^2)^n}.$$

For instance,

$$I_2(x) = \frac{1}{2}\arctan x + \frac{1}{2}\frac{x}{(1+x^2)},$$

$$I_3(x) = \frac{3}{4} I_2(x) + \frac{1}{4}\frac{x}{(1+x^2)^2} = \frac{3}{8}\arctan x + \frac{3}{8}\frac{x}{(1+x^2)} + \frac{1}{4}\frac{x}{(1+x^2)^2},$$

etc. ■

9.2 Primitives of rational functions

Rational functions can be integrated thanks to a partial fractions decomposition. First of all, we can focus on rational functions

$$R(x) = \frac{P(x)}{Q(x)}, \tag{9.15}$$

where the degree of $P(x)$ is smaller than the degree of $Q(x)$, and $Q(x)$ is a monic polynomial. The reason is that if this is not true, then we can divide $P(x)$ by $Q(x)$, obtain a quotient polynomial $C(x)$ and a remainder $M(x)$, so that we can write

$$R(x) = \frac{P(x)}{Q(x)} = C(x) + \frac{M(x)}{Q(x)}.$$

In the last rational fraction the degree of $M(x)$ is smaller than the degree of $Q(x)$, and the polynomial $C(x)$ can be readily integrated.

Any monic polynomial $Q(x)$ can be factored out into a series of elementary factors, i.e.,

$$Q(x) = (x-a_1)^{n_1} \cdots (x-a_r)^{n_r} \left(x^2+p_1x+q_1\right)^{m_1} \cdots \left(x^2+p_sx+q_s\right)^{m_s}. \tag{9.16}$$

Numbers $a_1, a_2, \ldots, a_r$ are real roots of the polynomial and $n_1, n_2, \ldots, n_r$ their respective multiplicities. The quadratic factors $\left(x^2+p_jx+q_j\right)^{m_j}$ are irreducible (i.e., $p_j^2 < 4q_j$) and correspond to complex roots of the polynomial. Numbers m_j are their respective multiplicities.

It turns out that the rational function (9.15) with denominator (9.16) can be expanded as

$$R(x) = \sum_{i=1}^{r} \left[\frac{A_{i1}}{x-a_i} + \cdots + \frac{A_{in_i}}{(x-a_i)^{n_i}}\right] + \sum_{j=1}^{s} \left[\frac{B_{j1}x+C_{j1}}{x^2+p_jx+q_j} + \cdots + \frac{B_{jm_j}x+C_{jm_j}}{(x^2+p_jx+q_j)^{m_j}}\right] \tag{9.17}$$

These partial fractions are easier to integrate. A few examples will illustrate the method.

■ **Example 9.9** Calculate

$$\int \frac{2x^2-4x+6}{(x-1)^3}\, dx.$$

According to the partial fractions decomposition (9.17),

$$\frac{2x^2-4x+6}{(x-1)^3} = \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{x-1}.$$

There are several ways to find A, B, and C. For instance, we can multiply the equation above by $(x-1)^3$ and get

$$2x^2-4x+6 = A + B(x-1) + C(x-1)^2.$$

Then setting $x=1$ we obtain $A=4$. Substituting this value of A in the previous equation and simplifying yields

$$2x^2-4x+2 = B(x-1) + C(x-1)^2 \qquad \Rightarrow \qquad 2(x-1)^2 = B(x-1) + C(x-1)^2,$$

hence $B=0$ and $C=2$.

An alternative is to obtain the Taylor polynomial for $2x^2-4x+6$ in powers of $x-1$. It is $4+2(x-1)^2$. Then

$$\frac{2x^2-4x+6}{(x-1)^3} = \frac{4+(x-1)^2}{(x-1)^3} = \frac{4}{(x-1)^3} + \frac{2}{x-1}.$$

Finally

$$\int \frac{2x^2-4x+6}{(x-1)^3}\,dx = \int \frac{4}{(x-1)^3}\,dx + \int \frac{2}{x-1}\,dx = -\frac{2}{(x-1)^2} + 2\log|x-1| + c.$$

■

■ **Example 9.10** Calculate

$$\int \frac{x+2}{x(x-1)(x-2)}\,dx.$$

The partial fraction decomposition yields

$$\frac{x+2}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}.$$

Multiplying by $x(x-1)(x-2)$ leads to

$$x+2 = A(x-1)(x-2) + Bx(x-2) + Cx(x-1).$$

Setting $x=0$ we get $2=2A$, i.e., $A=1$. Setting $x=1$ we get $3=-B$, i.e., $B=-3$. Finally, setting $x=2$ we get $4=2C$, i.e., $C=2$. Thus

$$\begin{aligned}\int \frac{x+2}{x(x-1)(x-2)}\,dx &= \int \frac{dx}{x} - 3\int \frac{dx}{x-1} + 2\int \frac{dx}{x-2} \\ &= \log|x| - 3\log|x-1| + 2\log|x-2| + c.\end{aligned}$$

■

■ **Example 9.11** Calculate

$$\int \frac{x^2+1}{x^2(x-1)(x+1)}\,dx.$$

The partial fraction decomposition yields

$$\frac{x^2+1}{x^2(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+1}.$$

Multiplying by $x^2(x-1)(x+1)$ leads to

$$x^2+1 = Ax(x-1)(x+1) + B(x-1)(x+1) + Cx^2(x+1) + Dx^2(x-1).$$

Setting $x=0$ leads to $1=-B$, i.e., $B=-1$. Setting $x=1$ leads to $2=2C$, i.e., $C=1$. Setting $x=-1$ leads to $2=-2D$, i.e., $D=-1$. Now, substituting these constants

$$\begin{aligned}x^2+1 &= Ax(x-1)(x+1) - (x-1)(x+1) + x^2(x+1) - x^2(x-1) \\ &= Ax(x-1)(x+1) - x^2 + 1 + x^3 + x^2 - x^3 + x^2 = Ax(x-1)(x+1) + x^2 + 1,\end{aligned}$$

so $A=0$.

Now,

$$\int \frac{x^2+1}{x^2(x-1)(x+1)}\,dx = -\int \frac{dx}{x^2} + \int \frac{dx}{x-1} - \int \frac{dx}{x+1} = \frac{1}{x} + \log|x-1| - \log|x+1| + c.$$

■

■ **Example 9.12** Calculate

$$\int \frac{5x^2-x+3}{x(x^2+1)}\,dx.$$

The partial fraction decomposition yields

$$\frac{5x^2-x+3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

Multiplying by $x(x^2+1)$leads to

$$5x^2-x+3 = A(x^2+1)+(Bx+C)x.$$

Setting $x=0$ leads to $A=3$. Substituting this value

$$5x^2-x+3 = 3(x^2+1)+(Bx+C)x \qquad \Rightarrow \qquad 2x^2-x=(Bx+C)x \qquad \Rightarrow$$
$$2x-1 = Bx+C,$$

so $B=2$ and $C=-1$.

Then

$$\int \frac{5x^2-x+3}{x(x^2+1)}\,dx = 3\int \frac{dx}{x} + \int \frac{2x}{x^2+1}\,dx - \int \frac{dx}{x^2+1} = 3\log|x| + \log(x^2+1) - \arctan x + c.$$

■

■ **Example 9.13** Calculate

$$\int \frac{2x+4}{x^2+2x+2}\,dx.$$

In order to perform a partial fraction decomposition we need to find the roots of the denominator. However these roots are $-1\pm i$, so x^2+2x+2 is an irreducible square factor. The way to proceed in these cases is to take the first two terms and complete the square. In other words, we write $x^2+2x=(x+1)^2-1$. Thus,

$$\begin{aligned}\int \frac{2x+4}{x^2+2x+2}\,dx &= \int \frac{2(x+2)}{(x+1)^2+1}\,dx = \int \frac{2(x+1)}{(x+1)^2+1}\,dx + 2\int \frac{dx}{(x+1)^2+1}\\ &= \log\left[(x+1)^2+1\right] + 2\arctan(x+1)+c\\ &= \log(x^2+2x+2) + 2\arctan(x+1)+c.\end{aligned}$$

■

■ **Example 9.14** Calculate

$$\int \frac{2x+4}{(x^2+2x+2)^2}\,dx.$$

Using the same transformation as in the previous example

$$\int \frac{2x+4}{(x^2+2x+2)^2}\,dx = \int \frac{2(x+1)}{[(x+1)^2+1]^2}\,dx + 2\int \frac{dx}{[(x+1)^2+1]^2}.$$

The first integral is immediate,

$$\int \frac{2(x+1)}{[(x+1)^2+1]^2}\,dx = -\frac{1}{(x+1)^2+1} = -\frac{1}{x^2+2x+2}.$$

The second integral can be done using the recurrence derived in Example 9.8,

$$2\int \frac{dx}{[(x+1)^2+1]^2} = \arctan(x+1) + \frac{x+1}{x^2+2x+2}.$$

Thus,

$$\int \frac{2x+4}{(x^2+2x+2)^2}\,dx = \arctan(x+1) + \frac{x}{x^2+2x+2} + c.$$

■

9.3 Change of variable

Let $F(x)$ be one primitive of $f(x)$, and let $x = g(t)$ be a change from variable x to the new variable t. By the chain rule

$$\frac{d}{dt}F\big(g(t)\big) = f\big(g(t)\big)g'(t),$$

thus, integrating this equation,

$$F\big(g(t)\big) = \int f\big(g(t)\big)g'(t)\,dt.$$

But using the change $x = g(t)$ and the fact that $F(x) = \int f(x)\,dx$, we can rewrite this identity as

$$\int f(x)\,dx = \int f\big(g(t)\big)g'(t)\,dt. \tag{9.18}$$

This is the equation ruling a change of variable in the calculation of a primitive.

R A simple way to remember this rule is to rewrite dx according to

$$dx = \frac{dx}{dt}\,dt = g'(t)\,dt.$$

■ **Example 9.15** Calculate

$$\int \frac{e^x}{e^{2x}+1}\,dx.$$

Here the obvious change of variable is $e^x = t$ or $x = \log t$. Then $dx = dt/t$ and

$$\int \frac{e^x}{e^{2x}+1}\,dx = \int \frac{t}{t^2+1}\frac{dt}{t} = \int \frac{dt}{t^2+1} = \arctan t + c = \arctan(e^x) + c.$$

■

■ **Example 9.16** Calculate

$$\int \frac{dx}{\sqrt[3]{(1-2x)^2} - \sqrt{1-2x}}.$$

Here the change of variable is $t^m = 1-2x$, choosing m so that all roots disappear. The simplest choice is the least common multiple of 2 and 3 in this case, i.e., $m=6$. So $x=(1-t^6)/2$ and therefore $dx = -3t^5\,dt$. Then,

$$\begin{aligned}\int \frac{dx}{\sqrt[3]{(1-2x)^2}-\sqrt{1-2x}} &= \int \frac{-3t^5}{t^4-t^3}\,dt = -3\int \frac{t^2}{t-1}\,dt = -3\int \left(t+1+\frac{1}{t-1}\right)dt \\ &= -\frac{3}{2}(t+1)^2 - 3\log|t-1| + c \\ &= -\frac{3}{2}\left(1+\sqrt[6]{1-2x}\right)^2 - 3\log\left|1-\sqrt[6]{1-2x}\right| + c\end{aligned}$$

■

■ **Example 9.17** Calculate

$$\int \frac{dx}{x\sqrt{1-x^2}}.$$

Whenever we have an expression like $\sqrt{1-x^2}$ one possible change of variable is $x=\sin t$, for then $\sqrt{1-x^2}=\cos t$ and $dx=\cos t\,dt$. In this case this leads to

$$\int \frac{dx}{x\sqrt{1-x^2}} = \int \frac{\cos t}{\sin t\cos t}\,dt = \int \frac{dt}{\sin t} = \log\left|\frac{\sin t}{1+\cos t}\right| + c = \log\left(\frac{|x|}{1+\sqrt{1-x^2}}\right) + c.$$

■

Suggested changes of variables:

(I) If there appear $\sqrt{1+x^2}$ then $x=\tan t$ transforms

$$\sqrt{1+x^2} = \frac{1}{\cos t}, \qquad dx = \frac{dt}{\cos^2 t},$$

or $x=\sinh t$ transforms

$$\sqrt{1+x^2} = \cosh t, \qquad dx = \cosh t\,dt.$$

(II) If there appear $\sqrt{x^2-1}$ then $x=\sec t$ transforms

$$\sqrt{x^2-1} = \tan t, \qquad dx = \sec t\tan t\,dt,$$

or $x=\cosh t$ transforms

$$\sqrt{x^2-1} = \sinh t, \qquad dx = \sinh t\,dt.$$

(III) As a last resource, in rational functions of sines and cosines we can use $t=\tan(x/2)$, which transforms

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}, \qquad dx = \frac{2\,dt}{1+t^2}.$$

Problems

Problem 9.1 Obtain the following immediate (or nearly so) primitives:

(i) $\displaystyle\int \frac{dx}{\cos^2 x}$; (ii) $\displaystyle\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx$; (iii) $\displaystyle\int \frac{x}{(x^2+1)^{5/2}}\,dx$;

(iv) $\displaystyle\int \frac{1+\sin x}{1+\cos x}\,dx$; (v) $\displaystyle\int \frac{dx}{1-\sin x}\,dx$; (vi) $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\,dx$;

(vii) $\displaystyle\int \frac{1+\sqrt{1-\sqrt{x}}}{\sqrt{x}}\,dx$; (viii) $\displaystyle\int \frac{\cos^3 x}{\sin^4 x}\,dx$; (ix) $\displaystyle\int x^3\sqrt{1-x^2}\,dx$.

HINTS: (iv) multiply and divide by $1-\cos x$ and expand; (v) idem with $1+\sin x$; (vii) alternatively $t=\sqrt{1-\sqrt{x}}$; (viii) $\cos^3 x = (1-\sin^2 x)\cos x$ and expand; (ix) write $x^3 = x(x^2-1)+x$ and expand.

Problem 9.2 Obtain the primitives of the following rational functions:

(i) $\displaystyle\int \frac{x^2}{(x-1)^3}\,dx$; (ii) $\displaystyle\int \frac{dx}{(x-1)^2(x^2+x+1)}$;

(iii) $\displaystyle\int \frac{2x^2+3}{x^2(x-1)}\,dx$; (iv) $\displaystyle\int \frac{2}{x^2-2x+2}\,dx$;

(v) $\displaystyle\int \frac{4x^4-x^3-46x^2-20x+153}{x^3-2x^2-9x+18}\,dx$; (vi) $\displaystyle\int \frac{x^5-2x^3}{x^4-2x^2+1}\,dx$.

HINTS: (ii) $x^2+x+1=(x+1/2)^2+3/4$; (v) $x^3-2x^2-9x+18=(x-2)(x-3)(x+3)$; (vi) $x^4-2x^2+1=(x-1)^2(x+1)^2$.

Problem 9.3 Obtain the following primitives doing an appropriate change of variable:

(i) $\displaystyle\int x^2\sqrt{x-1}\,dx$; (ii) $\displaystyle\int x^2\sin\sqrt{x^3}\,dx$; (iii) $\displaystyle\int \cos(\log x)\,dx$; (iv) $\displaystyle\int \sin(\log x)\,dx$; (v) $\displaystyle\int \cos^2(\log x)\,dx$; (vi) $\displaystyle\int \frac{\sqrt{x}+1}{x+3}\,dx$; (vii) $\displaystyle\int \frac{(x+1)^3}{\sqrt{1-(x+1)^2}}\,dx$; (viii) $\displaystyle\int \frac{x^3}{(1+x^2)^3}\,dx$;

(ix) $\displaystyle\int \frac{dx}{(2+x)\sqrt{1+x}}$; (x) $\displaystyle\int \frac{dx}{1+\sqrt[3]{1-x}}$; (xi) $\displaystyle\int \frac{e^{4x}}{e^{2x}+2e^x+2}\,dx$; (xii) $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}\,dx$; (xiii) $\displaystyle\int \sqrt{e^x-1}\,dx$; (xiv) $\displaystyle\int \frac{\sin^2 x\cos^5 x}{\tan^3 x}\,dx$; (xv) $\displaystyle\int \frac{dx}{3+\sqrt{2x+5}}$; (xvi) $\displaystyle\int \sqrt{\frac{x-1}{x+1}}\,dx$;

(xvii) $\displaystyle\int \sqrt{\sqrt{x}+1}\,dx$; (xviii) $\displaystyle\int \frac{\sqrt{x+2}}{1+\sqrt{x+2}}\,dx$; (xix) $\displaystyle\int \sqrt{2+e^x}\,dx$; (xx) $\displaystyle\int \frac{\sin x+3\cos x}{\sin x+2\cos x}\,dx$; (xxi) $\displaystyle\int \frac{\sin x+3\cos x}{\sin x\cos x+2\sin x}\,dx$; (xxii) $\displaystyle\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x}}\,dx$; (xxiii) $\displaystyle\int \frac{dx}{(x+1)\sqrt[3]{x+2}}$; (xxiv) $\displaystyle\int \frac{dx}{e^x-4e^{-x}}\,dx$.

HINTS: (i) $t=\sqrt{x-1}$ (or int. by parts twice); (ii) $t^2=x^3$; (iii)–(v) $t=\log x$; (vi) $t=\sqrt{x}$; (vii) $t=\sqrt{1-(x+1)^2}$; (viii) $t=1+x^2$; (ix) $t^2=1+x$; (x) $t^3=1-x$; (xi) $t=e^x$; (xii) $t^2=e^{2x}-1$; (xiii) $t^2=e^x-1$; (xiv) $t=\cos x$; (xv) $t=3+\sqrt{2x+5}$; (xvi) $t=\sqrt{(x-1)/(x+1)}$; (xvii) $t=\sqrt{\sqrt{x}+1}$; (xviii) $t=\sqrt{x+2}$; (xix) $t=\sqrt{2+e^x}$; (xx) $t=\tan x$; (xxi) $t=\tan(x/2)$; (xxii) $t=\sqrt{1+x^{1/3}}$; (xxiii) $t^3=x+2$; (xxiv) $t=e^x$.

Problem 9.4 Obtain the following primitives with the help of some trigonometric identity:

(i) $\int \sin^2 x\,dx$; (ii) $\int \cos^2 x\,dx$; (iii) $\int \sin^4 x\,dx$; (iv) $\int \cos^4 x\,dx$; (v) $\int \cos^6 x\,dx$; (vi) $\int \sin^2 x\cos^2 x\,dx$; (vii) $\int \tan^2 x\,dx$; (viii) $\int \tan^4 x\,dx$; (ix) $\int \frac{dx}{\cos^4 x}$; (x) $\int \sin^5 x\,dx$; (xi) $\int \cos^3 x\sin^2 x\,dx$; (xii) $\int \sec^6 x\,dx$; (xiii) $\int \sin^3 x\cos^2 x\,dx$; (xiv) $\int \tan^3 x\,dx$; (xv) $\int \tan^3 x\sec^4 x\,dx$.

HINTS: Identities to use: $2\cos^2 x = 1+\cos 2x$; $2\sin^2 x = 1-\cos 2x$; $\cos^2 x+\sin^2 x = 1$; $\sec^2 x = 1+\tan^2 x$.

Problem 9.5 Integrate by parts to obtain the following primitives:

(i) $\int x\tan^2(2x)\,dx$; (ii) $\int e^x \sin \pi x\,dx$; (iii) $\int e^x \cos 2x\,dx$; (iv) $\int \sec^3 x\,dx$; (v) $\int \tan^2(3x)\sec^3(3x)\,dx$; (vi) $\int e^{\sin x}\cos^3 x\,dx$; (vii) $\int x^2 \log x\,dx$; (viii) $\int x^m \log x\,dx$; (ix) $\int (\log x)^3\,dx$; (x) $\int x(\log x)^2\,dx$; (xi) $\int \frac{x\log x}{(1+x^2)^2}\,dx$; (xii) $\int \arctan \sqrt[3]{x}\,dx$.

Problem 9.6 Obtain the following primitives by performing a trigonometric substitution:

(i) $\int \frac{x^2+1}{\sqrt{x^2-1}}\,dx$; (ii) $\int \frac{x^2}{(x^2+1)^{5/2}}\,dx$; (iii) $\int \frac{x^2}{(1-x^2)^{3/2}}\,dx$; (iv) $\int \frac{dx}{x^2\sqrt{1-x^2}}$; (v) $\int \frac{dx}{x^2\sqrt{9-x^2}}$.

Problem 9.7 Find recurrence formulas for the following integrals:

(i) $I_m = \int \sin^m x\,dx \longrightarrow I_m = -\frac{1}{m}\sin^{m-1} x\cos x + \frac{m-1}{m} I_{m-2}$;

(ii) $I_m = \int (\log x)^m\,dx \longrightarrow I_m = x(\log x)^m - mI_{m-1}$;

(iii) $I_m = \int x^m e^{-x}\,dx \longrightarrow I_m = -x^m e^{-x} + mI_{m-1}$;

(iv) $I_m = \int \tan^m x\,dx \longrightarrow I_m = \frac{1}{m-1}\tan^{m-1} x - I_{m-2}$;

(v) $I_m = \int \sec^m x\,dx \longrightarrow I_m = \frac{1}{m-1}\tan x\sec^{m-2} x + \frac{m-2}{m-1} I_{m-2}$;

(vi) $I_m = \int x^m e^{x^2}\,dx \longrightarrow I_m = \frac{1}{2}x^{m-1}e^{x^2} - \frac{m-1}{2} I_{m-2}$;

(vii) $I_{m,n} = \int \sin^m x\cos^n x\,dx \longrightarrow I_{m,n} = -\frac{1}{m+n}\sin^{m-1} x\cos^{n+1} x + \frac{m-1}{m+n} I_{m-2,n}$.

Problem 9.8 Without calculating the integral, prove that

$$\int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}\,dx = Ax + B\log|c\cos x + d\sin x| + \text{const.}$$

by determining the constants A and B as functions of a, b, c, and d.

10. Fundamental Theorem of Calculus

Integration is a device that was invented to calculate areas of figures limited by curved sides. The idea can be traced back at least to Archimedes. He is well known —among many other things— by calculating the area of a circle of unit diameter, A, in terms of its perimeter, π (π is the initial of περίμετρος = *perimeter*), obtaining the celebrated formula $A = \pi/4$. He did that by using two sequence of polygons, both circumscribed to and inscribed in the circumference, and then taking the limit of the number of sides going to infinity (see Figure 10.1).

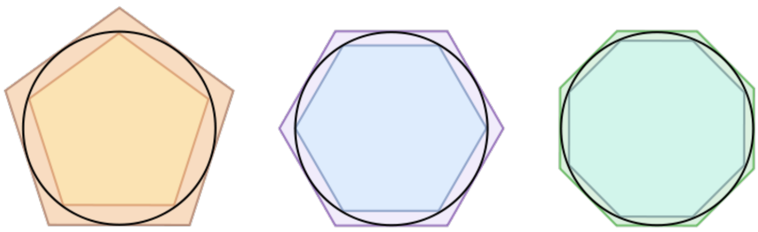


Figure 10.1: Archimedes's construction to obtain the relation between the area and the perimeter of a circle.

A similar idea was employed to obtain the area under more complicated curves. If we define a signed area as in Figure 10.2(a) (i.e., it adds if $f(x) > 0$ and substracts if $f(x) < 0$), the problem is how to calculate the total area enclosed by a curved within a given interval. Following Archimedes, one way to estimate that area is to approximate it as a sum of rectangles, as in Figure 10.2(b). In the limit when the width of these rectangles goes to zero we obtain the value of the seeked area.

■ **Example 10.1** As an example of this procedure, let us calculate, using this method, the area below the curve $f(x) = x^2$ within the interval $[0, a]$. To do that, we divide the interval in n rectangles of width a/n and heights $(ak/n)^2$, with $k = 1, 2, \ldots, n$. The areas of these rectangles will then be

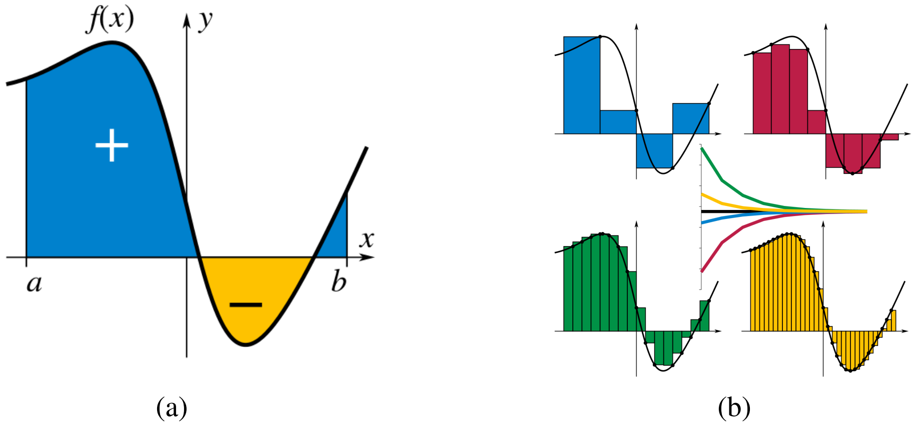


Figure 10.2: (a) Area "under" a curve: above the X axis area has a positive sign and below the X axis has a negative sign. (b) Approximations to that area as sums of thiner and thiner rectangles.

a^3k^2/n^3. This yields the following approximation to the area:

$$A_n = \sum_{k=1}^{n} \frac{k^2}{n^3} = \frac{a^3}{n^3} \sum_{k=1}^{n} k^2.$$

It is a know result that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6},$$

thus

$$A_n = a^3 \frac{n(n+1)(2n+1)}{6n^3}.$$

Therefore

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} a^3 \frac{n(n+1)(2n+1)}{6n^3} = \frac{a^3}{3}$$

is the area we are seeking. ■

10.1 Riemann's integral

The problem with the heuristic idea exposed above is that, for that procedure to make sense, the result should not depend on the division in rectangles that we propose. In other words, irrespective of whether we choose all rectangles to have the same or different widths, the limit process should yield the same area. Thus we need a more rigorous construction and limit process.

To this purpose, given an interval $[a,b]$ we will define a **partition** of the interval as as the set $P = \{x_0, x_1, \dots, x_n\}$, where $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.

Now, for any function f bounded in $[a,b]$, if we define

$$m_i \equiv \inf_{x_{i-1} \leqslant x \leqslant x_i} f(x), \qquad M_i \equiv \sup_{x_{i-1} \leqslant x \leqslant x_i} f(x), \tag{10.1}$$

then the (signed) area between the X axis and $f(x)$ within the interval $[x_{i-1}, x_i]$ —provided it can be defined— will be bounded from below by $m_i(x_i - x_{i-1})$ and from above by $M_i(x_i - x_{i-1})$ —the areas of two rectangles (see Figure 10.3). Thus, the two numbers

$$L(f,P) = \sum_{i=1} m_i(x_i - x_{i-1}), \qquad U(f,P) = \sum_{i=1} M_i(x_i - x_{i-1}), \tag{10.2}$$

respectively called **lower sum** and **upper sum** of f with respect to the partition P, will be an upper and a lower bound to the (signed) area between $f(x)$ and the X axis within the interval $[a,b]$. By construction $L(f,P) \leqslant U(f,P)$.

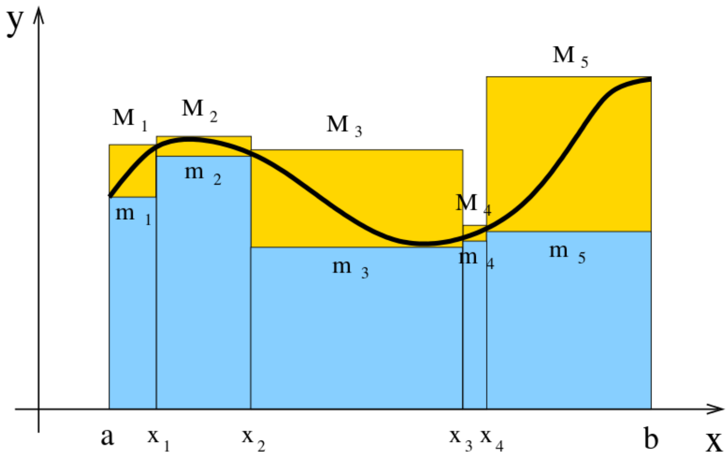


Figure 10.3: Definition of the upper sum and lower sum for a function $f(x)$ with respecto to a partition of the interval $[a,b]$. The (signed) area between the X axis and $f(x)$ is bounded between them two.

Partitions can be defined by adding more points to it. Thus, Q is a **refinement** of P if $P \subset Q$. Upon refining partitions we increase the lower sum and decrease the upper sum, i.e.,

$$L(f,P) \leqslant L(f,Q), \qquad U(f,Q) \leqslant U(f,P).$$

Accordingly, if P_1 and P_2 are two partitions of $[a,b]$, then $Q = P_1 \cup P_2$ will be a refinement of both of them and therefore

$$L(f,P_1) \leqslant L(f,Q) \leqslant U(f,Q) \leqslant U(f,P_2).$$

In other words, $L(f,P_1) \leqslant U(f,P_2)$ irrespective of the partitions P_1 and P_2.

This is summarised in the statement

$$\sup_P L(f,P) \leqslant \inf_P U(f,P). \tag{10.3}$$

This led Riemann to invent the following definition:

Definition 10.1.1 — Integral. A function f bounded in $[a,b]$ is **integrable** in $[a,b]$ if

$$\sup_P L(f,P) = \inf_P U(f,P) = \int_a^b f. \tag{10.4}$$

The number $\int_a^b f$ is known as the (Riemann's) **integral** of f in $[a,b]$.

R It is customary to use Leibniz's notation for the integral and write

$$\int_a^b f = \int_a^b f(x)\,dx.$$

This notation reminds the definition of the integral as a sum (hence the sign $\int$) of the areas of rectagles of with dx and height $f(x)$, for all $a \leqslant x \leqslant b$.

■ **Example 10.2** Not all bounded functions can be integrated. For instance the function

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}, \end{cases}$$

does not satisfy the definition because for every partition P of the interval $[a,b]$ we have $L(f,P) = 0$ and $U(f,P) = b - a$ —because every subterval $[x_{k-1}, x_k]$ contains both rational and irrational numbers. ■

■ **Example 10.3** The function

$$f(x) = \begin{cases} 1, & x = \frac{1}{2}, \\ 0, & x \neq \frac{1}{2}, \end{cases}$$

can be integrated in e.g. $[0,1]$. Let P be any partition of that interval. Then $L(f,P) = 0$ because in every interval of P f takes the value 0. On the other hand, $U(f,P) = \Delta x$, where Δx is the length of the interval containing the point $x = 1/2$. Since by refining the partition Δx can be made arbitrarily small,

$$L(f,P) = \inf_P U(f,P) = 0 \qquad \Rightarrow \qquad \int_a^b f = 0.$$

■

An important result that justifies the heuristic construction is this:

Theorem 10.1.1 The bounded function f is integrable in $[a,b]$ if and only if there exists a sequence of partitions $\{P_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} U(f,P_n).$$

In other words, to prove the existence of an integral we simply have to take a partition P_n of the interval $[a,b]$ into n equal segments, compute $L(f,P_n)$ and $U(f,P_n)$ and take the limits.

Exercise 10.1 Transform Example 10.1 into a rigorous proof that $\displaystyle\int_0^a x^2\,dx = \frac{a^3}{3}$. ■

The full characterisation of the set of functions that can be integrated according to Riemann's definition is out of the scope of this course. However, this set includes important classes of functions worth mentioning:

Theorem 10.1.2 If f is continuous in $[a,b]$ then it is integrable in $[a,b]$.

The idea of the proof of this result is that continuous functions have the property that the difference between their maximum and minimum values in a closed interval is smaller the smaller the interval. This means that we can make the difference between $L(f,P)$ and $U(f,P)$ arbitrarily small by simply refining the partition sufficiently.

Theorem 10.1.3 If f is monotonic in $[a,b]$ then it is integrable in $[a,b]$.

Proof. Let us assume that f is increasing (the proof is analogous for decreasing functions). The idea of the proof is that, within the interval $[x_{i-1}, x_i]$, the maximum of f is $f(x_i)$ and the minimum is $f(x_{i-1})$. Thus, if P_n is the partition of $[a,b]$ into n equal size intervals,

$$L(f,P_n) = \sum_{i=1}^{n} f(x_{i-1})\frac{b-a}{n}, \qquad U(f,P_n) = \sum_{i=1}^{n} f(x_i)\frac{b-a}{n},$$

and therefore

$$0 \leqslant U(f,P_n) - L(f,P_n) = \frac{b-a}{n}\sum_{i=1}^{n}[f(x_i) - f(x_{i-1})] = \frac{b-a}{n}[f(b) - f(a)] \quad \xrightarrow[n\to\infty]{} \quad 0.$$

■

Notice that monotonic functions need not be continuous, so this result is not contained in the previous one.

10.2 Properties of the integral

Theorem 10.2.1 Let f and g be two integrable functions in $[a,b]$. Then the following properties hold:

(i) $\displaystyle\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$ for all $\alpha, \beta \in \mathbb{R}$ **linearity**

(ii) $\displaystyle\int_a^b f \leqslant \int_a^b g$ whenever $f \leqslant g$ in $[a,b]$ **boundedness**

(iii) $|f|$ is integrable in $[a,b]$ and $\displaystyle\left|\int_a^b f\right| \leqslant \int_a^b |f|$ **absolute integrability**

A consequence of (ii) is that if $f \geqslant 0$ then $\displaystyle\int_a^b f \geqslant 0$.

Another consequence is that if $M = \sup\limits_{x\in[a,b]} f(x)$ and $m = \inf\limits_{x\in[a,b]} f(x)$, then

$$m(b-a) \leqslant \int_a^b f \leqslant M(b-a). \tag{10.5}$$

Theorem 10.2.2 — Interval additivity. Given $a < b < c$, function f is integrable in $[a,c]$ if and only if it is integrable in $[a,b]$ and $[b,c]$. Besides

$$\int_a^c f = \int_a^b f + \int_b^c f. \tag{10.6}$$

Notice that this formula implies

$$\int_a^b f = \int_a^c f - \int_b^c f,$$

so interval additivity will be preserved beyond the constraint $a < b < c$ if we *define*

$$\int_c^b f = -\int_b^c f. \tag{10.7}$$

10.3 Riemann's sums

Let f be a bounded function in $[a,b]$. For any partition P of this interval the expression

$$S(f,P) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}), \tag{10.8}$$

for any choice of points $x_{i-1} \leqslant c_i \leqslant x_i$ is referred to as a **Riemann's sum.**

It is clear from the definition that Riemann's sums satisfy $L(f,P) \leqslant S(f,P) \leqslant U(f,P)$. Therefore, if f is integrable in $[a,b]$ and $\{P_n\}_{n=1}^{\infty}$ is a sequence of partitions such that

$$\lim_{n\to\infty}[U(f,P_n) - L(f,P_n)] = 0,$$

then

$$\lim_{n\to\infty} S(f,P_n) = \int_a^b f. \tag{10.9}$$

This result is very useful in calculating some limits, as the examples illustrate.

■ **Example 10.4** Suppose we need to calculate the limit

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}.$$

This limit does not define a series, because the terms in the sum change not only with k but also with n.

In order to calculate this limit we need to rewrite the sum as

$$\sum_{k=1}^{n}\frac{1}{n+k} = \sum_{k=1}^{n}\frac{1}{1+(k/n)}\cdot\frac{1}{n}.$$

The right-hand side is the expression of $S(f,P_n)$, where $f(x) = 1/(1+x)$, $c_k = k/n$ and P_n is a partition of $[0,1]$ in n equal-size intervals. Since f is continuous —hence integrable—, then

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k} = \lim_{n\to\infty} S(f,P_n) = \int_0^1 \frac{dx}{1+x}.$$

■

■ **Example 10.5** Let us calculate

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)^{1/n}.$$

If we denote the limit ℓ, then

$$\log\ell = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\log\left(1+\frac{k}{n}\right) = \lim_{n\to\infty} S(f,P_n),$$

where P_n is a partition of $[0,1]$ in n equal-size intervals and $f(x) = \log(1+x)$. Thus,

$$\log\ell = \int_0^1 \log(1+x)\,dx.$$

■

10.4 Fundamental theorem of calculus

The basic idea of the connection between integrals and derivatives —the essence of the fundamental theorem of calculus— is this. Let us denote $A(x)$ the (signed) area between the X axis and the function f within the interval $[a,x]$. Suppose that we increase the inverval by a very small amount

h. In practical terms, we are enlarging the area by adding almost a rectangle of width h and height $\approx f(x)$. In other words,

$$A(x+h) \approx A(x) + f(x)h \qquad \Rightarrow \qquad f(x) \approx \frac{A(x+h)-A(x)}{h}.$$

If we now take the limit $h \to 0$ we obtain the connection $A'(x) = f(x)$. This is the basic result that both Newton and Leibniz were aware of and which renders calculus such a powerful tool.

We are going to obtain this result in a more rigorous way by using our definition of (Riemann's) integral.

To begin with, let us first prove that integrals always define continuous functions:

Theorem 10.4.1 If f is integrable in $[a,b]$, then $F(x) = \displaystyle\int_a^x f(t)\,dt$ defines a continuous function in $[a,b]$.

Proof. Take any point $c \in [a,b]$. Since f is integrable in $[a,b]$ it is also bounded, so let $M = \sup\limits_{x\in[a,b]} |f(x)|$. Then

$$|F(x)-F(c)| = \left|\int_a^x f(t)\,dt - \int_a^c f(t)\,dt\right| = \left|\int_c^x f(t)\,dt\right| \leqslant \left|\int_c^x |f(t)|\,dt\right| \leqslant \left|\int_c^x M\,dt\right| = M|x-c|.$$

By the sandwich rule,

$$\lim_{x\to c} |x-c| = 0 \qquad \Rightarrow \qquad \lim_{x\to c} |F(x)-F(c)| = 0 \qquad \Rightarrow \qquad \lim_{x\to c} F(x) = F(c).$$

This proves that F is continuous at any $c \in [a,b]$. ■

Notice that this result requieres nothing from f apart from its integrability. In particular, f needs not be a continuous function.

■ **Example 10.6** Let

$$f(x) = \begin{cases} 0, & x \leqslant \frac{1}{2}, \\ 1, & x > \frac{1}{2} \end{cases}$$

be a function with a jump discointinuity at $x = 1/2$. Now, for any $x \leqslant 1/2$,

$$F(x) = \int_0^x f(t)\,dt = \int_0^x 0\,dt = 0,$$

whereas for any $x > 1/2$,

$$F(x) = \int_0^x f(t)\,dt = \int_0^{1/2} f(t)\,dt + \int_{1/2}^x f(t)\,dt = \int_0^{1/2} 0\,dt + \int_{1/2}^x dt = x - \frac{1}{2}.$$

Thus,

$$F(x) = \begin{cases} 0, & x \leqslant \frac{1}{2}, \\ x-\frac{1}{2}, & x > \frac{1}{2}, \end{cases}$$

which is continuous everywhere. ■

Theorem 10.4.2 — First fundamental theorem of calculus. If f is continuous in $[a,b]$ then $F(x)=\int_a^x f(t)\,dt$ is differentiable in (a,b) and $F'(x)=f(x)$.

Proof. First of all

$$\frac{F(x+h)-F(x)}{h}=\frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

Now, in the interval $[x,x+h]$ (or $[x+h,x]$ if $h<0$) f reaches its maximum M_h and minimum m_h values —as every continuous function in a closed interval. Then, if $h>0$

$$m_h h \leqslant \int_x^{x+h} f(t)\,dt \leqslant M_h h \qquad \Rightarrow \qquad m_h \leqslant \frac{1}{h}\int_x^{x+h} f(t)\,dt \leqslant M_h,$$

and if $h<0$

$$m_h(-h) \leqslant \int_{x+h}^{x} f(t)\,dt \leqslant M_h(-h) \qquad \Rightarrow \qquad m_h \leqslant \frac{1}{(-h)}\int_{x+h}^{x} f(t)\,dt \leqslant M_h$$

$$\Rightarrow \qquad m_h \leqslant \frac{1}{h}\int_x^{x+h} f(t)\,dt \leqslant M_h.$$

In any case, the number $\frac{1}{h}\int_x^{x+h} f(t)\,dt$ is an intermediate value between m_h and M_h. Any continuous function in a closed interval reaches all intermediate values between its maximum and its minimum, so there must be a point $c_h \in [x,x+h]$ (or in $[x+h,x]$ if $h<0$) such that

$$f(c_h)=\frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

Clearly $c_h \to x$ when $h\to 0$. Therefore

$$F'(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\lim_{h\to 0} f(c_h)=f(x).$$

■

The take-home message of this theorem is that integrals of functions are primitives of those functions. Here is the connection between differentiation and integration. From now on, calculating the area between the X axis and a given curve $f(x)$ is as simple as finding the right primitive of f. Actually, the problem is even easier: any primitive will do, because of this second version of the fundamental theorem of calculus:

Theorem 10.4.3 — Second fundamental theorem of calculus (Barrow's rule). If f is continuous in $[a,b]$ and G is any primitive of f in (a,b), then

$$\int_a^b f(x)\,dx = G(b)-G(a).$$

Proof. According to the first version of this theorem $F(x)=\int_a^x f(t)\,dt$ is a primitive of f in (a,b). Therefore $G(x)=F(x)+c$. Now $F(a)=\int_a^a f(t)\,dt=0$, hence $G(a)=F(a)+c=c$. In other words, $F(x)=G(x)-G(a)$. Then

$$\int_a^b f(x)\,dx = F(b)=G(b)-G(a).$$

■

R Often primitives are referred to as "indefinite integrals" and denoted $\int f(x)\,dx$, whereas integrals of the form $\int_a^b f(x)\,dx$ are called "definite integrals".

Corollary 10.4.4 If f is continuous in $[a,b]$ and g_1, g_2 are differentiable in (a,b) then

$$H(x) = \int_{g_1(x)}^{g_2(x)} f(t)\,dt \tag{10.10}$$

is also differentiable in (a,b) and

$$H'(x) = f\big(g_2(x)\big)g_2'(x) - f\big(g_1(x)\big)g_1'(x). \tag{10.11}$$

Proof. Let $F(x)$ be a primitive of $f(x)$ in (a,b). Then $H(x) = F\big(g_2(x)\big) - F\big(g_1(x)\big)$. Since F, g_1, g_2 are all differentiable, so is H. Finally, the derivative of H will be, by the chain rule,

$$H'(x) = F'\big(g_2(x)\big)g_2'(x) - F'\big(g_1(x)\big)g_1'(x) = f\big(g_2(x)\big)g_2'(x) - f\big(g_1(x)\big)g_1'(x)$$

because $F'(x) = f(x)$. ■

■ **Example 10.7** If

$$F(x) = \int_0^{x^3} \cos t\,dt,$$

then $F'(x) = 3x^2\cos(x^3)$. ■

Applying Barrow's rule we can obtain particular versions of the integration by parts and change of variable theorems:

Theorem 10.4.5 — Integration by parts. If f and g are two differentiable functions in (a,b), then

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx. \tag{10.12}$$

The symbol in the right-hand side is a short-hand for

$$f(x)g(x)\Big|_a^b = f(b)g(b) - f(a)g(a). \tag{10.13}$$

Theorem 10.4.6 — Change of variable. If g is continuous in $[a,b]$ and differentiable in (a,b), and f is continuous in $g\big([a,b]\big)$, then

$$\int_{g(a)}^{g(b)} f(u)\,du = \int_a^b f\big(g(x)\big)g'(x)\,dx. \tag{10.14}$$

Proof. On the one hand, if F is a primitive of f then

$$\int_{g(a)}^{g(b)} f(u)\,du = F\big(g(b)\big) - F\big(g(a)\big).$$

On the other hand, by the chain rule,

$$\frac{d}{dx}F\big(g(x)\big) = F'\big(g(x)\big)g'(x) = f\big(g(x)\big)g'(x),$$

therefore $F\big(g(x)\big)$ is a primitive of $f\big(g(x)\big)g'(x)$ and, according to Barrow's rule,

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = F\big(g(b)\big) - F\big(g(a)\big).$$

The result follows from the fact that the right-hand side is the same for both integrals. ■

■ **Example 10.8** Let us calculate the area of a circle of radius a. The equation of its circumference is $x^2+y^2=a^2$, from which we obtain $y=\pm\sqrt{a^2-x^2}$. Clearly the area between the X axis and the function $f(x)=\sqrt{a^2-x^2}$ within the interval $[-a,a]$ is half the area we want to calculate, therefore

$$A = 2\int_{-a}^{a}\sqrt{a^2-x^2}\,dx.$$

We can introduce the variable $t=x/a$, or $x=at$, so that $\dfrac{dx}{dt}=a$, and the limits $x=-a \;\to\; t=-1$ and $x=a \;\to\; t=1$. Thus

$$A = 2\int_{-1}^{1}\sqrt{a^2-a^2t^2}\,a\,dt = 2a^2\int_{-1}^{1}\sqrt{1-t^2}\,dt.$$

Let us now introduce a second change of variable: $t=\sin\theta$. Then $\dfrac{dt}{d\theta}=\cos\theta$, and the limits $t=-1 \;\to\; \theta=-\pi/2$ and $t=1 \;\to\; \theta=\pi/2$. The integral then becomes

$$A = 2a^2\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta = a^2\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta = a^2\left(\pi+\underbrace{\frac{1}{2}\sin 2\theta\Big|_{-\pi/2}^{\pi/2}}_{=0}\right) = \pi a^2.$$

■

■ **Example 10.9 — One last integration trick.** Suppose one has to compute the integral

$$I = \int_a^b f(x)\,dx.$$

A simple change of variable is given by $x=a+b-t$, which transforms the integral into

$$I = \int_a^b f(a+b-t)\,dt$$

(because $dx=-dt$ and $t=b$ for $x=a$ and $t=a$ for $x=b$). Then, an alternative way of writing the original integral is as an average of these two expressions, namely

$$\int_a^b f(x)\,dx = \frac{1}{2}\int_a^b \big[f(x)+f(a+b-x)\big]\,dx.$$

As an illustrative example, let us calculate the integral

$$I = \int_0^{\pi/2}\frac{dx}{1+\sqrt{\tan x}}.$$

A first remark about this integral is that the integrand is a bounded, continuous function in $[0,\pi/2]$, because $\tan x \geqslant 0$ in this interval and, although it diverges when $x \to \left(\frac{\pi}{2}\right)^-$,

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{1}{1+\sqrt{\tan x}} = 0.$$

A second remark is that performing this integral by any other standard method poses a real challenge (give it a try!) With this last trick though, it is a piece of cake.

According to the formula we have just derived,

$$I = \frac{1}{2}\int_0^{\pi/2}\left(\frac{1}{1+\sqrt{\tan x}}+\frac{1}{1+\sqrt{\cot x}}\right)dx$$

because $\tan(\pi/2-x) = \cot x$. But

$$\begin{aligned}\frac{1}{1+\sqrt{\tan x}}+\frac{1}{1+\sqrt{\cot x}} &= \frac{1+\sqrt{\cot x}+1+\sqrt{\tan x}}{(1+\sqrt{\tan x})(1+\sqrt{\cot x})} = \frac{2+\sqrt{\cot x}+\sqrt{\tan x}}{1+\sqrt{\tan x}+\sqrt{\cot x}+\sqrt{\tan x\cot x}} \\ &= \frac{2+\sqrt{\cot x}+\sqrt{\tan x}}{2+\sqrt{\tan x}+\sqrt{\cot x}} = 1,\end{aligned}$$

where we have just used the fact that $\tan x\cot x = 1$. Then

$$I = \frac{1}{2}\int_0^{\pi/2} dx = \frac{\pi}{4}.$$

■

Exercise 10.2 Use the method above to prove that

$$\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi^2}{4}.$$

■

Problems

Problem 10.1 Find a continuous function f such that $f(0) = 0$ and

$$f'(x) = \begin{cases} \dfrac{4-x^2}{(4+x^2)^2}, & x < 0, \\ e^{\sqrt{x}}, & x > 0. \end{cases}$$

Problem 10.2

(a) Prove that if f is odd then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

(b) Prove that if f is even then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(c) Calculate the integral $\displaystyle\int_{6}^{10} \sin\left(\sin\left((x-8)^3\right)\right) dx$.

Problem 10.3 Calculate the following limits:

(i) $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2}$; (ii) $\displaystyle\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sqrt[n]{e^{2k}}$; (iii) $\displaystyle\lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{1}{\sqrt{n^2-k^2}}$.

Problem 10.4 Calculate $F(x) = \displaystyle\int_{-1}^{x} f(t)\,dt$, with $-1 \leqslant x \leqslant 1$, for the following functions:

(i) $f(x) = |x|e^{-|x|}$;

(ii) $f(x) = |x - 1/2|$;

(iii) $f(x) = \begin{cases} -1, & -1 \leqslant x < 0, \\ 1, & 0 \leqslant x \leqslant 1; \end{cases}$

(iv) $f(x) = \begin{cases} x^2, & -1 \leqslant x < 0, \\ x^2 - 1, & 0 \leqslant x \leqslant 1; \end{cases}$

(v) $f(x) = \begin{cases} 1, & -1 \leqslant x \leqslant 0, \\ x+1, & 0 < x \leqslant 1; \end{cases}$

(vi) $f(x) = \begin{cases} 1+x, & -1 \leqslant x \leqslant -\frac{1}{2}, \\ \frac{1}{2}, & -\frac{1}{2} < x < \frac{1}{2}, \\ 1-x, & \frac{1}{2} \leqslant x \leqslant 1; \end{cases}$

(vii) $f(x) = \max\left\{\sin(\pi x/2), \cos(\pi x/2)\right\}$.

Problem 10.5 Calculate the following integrals:

(i) $\displaystyle\int_{0}^{\log 2}\sqrt{e^x - 1}\,dx$; (ii) $\displaystyle\int_{1}^{2}\frac{\sqrt{x^2-1}}{x}\,dx$.

Problem 10.6 Calculate the derivative of the following functions:

(i) $F(x) = \displaystyle\int_{x^2}^{x^3}\frac{e^t}{t}\,dt$;

(ii) $F(x) = \displaystyle\int_{-x^3}^{x^3}\frac{dt}{1+\sin^2 t}$;

(iii) $F(x) = \displaystyle\int_{3}^{\int_1^x \sin^3 t\,dt}\frac{dt}{1+\sin^6 t + t^2}$;

(iv) $F(x) = \displaystyle\int_{2}^{\exp\left\{\int_1^{x^2}\tan\sqrt{t}\,dt\right\}}\frac{ds}{\log s}$;

(v) $F(x) = \displaystyle\int_{0}^{x} x^2 f(t)\,dt$, with f continuous in $\mathbb{R}$;

(vi) $F(x) = \sin\left(\displaystyle\int_{0}^{x}\sin\left(\int_{0}^{y}\sin^3 t\,dt\right)dy\right)$.

Problem 10.7 Find the absolute maximum and minimum in the interval $[1,\infty)$ of the function

$$f(x) = \int_{0}^{x-1}\left(e^{-t^2} - e^{-2t}\right) dt.$$

HINT: $\lim_{x\to\infty}\int_0^x e^{-t^2}\,dt = \sqrt{\pi}/2$.

Problem 10.8 Prove that the equation

$$\int_0^x e^{t^2}\,dt = 1$$

has a unique solution in $\mathbb{R}$ and that it can be found in the interval $(0,1)$.

Problem 10.9 Let $f(x)$ be a continuous function such that $f(x) > 0$ for all $0 \leqslant x \leqslant 1$, and consider the function

$$F(x) = 2\int_0^x f(t)\,dt - \int_x^1 f(t)\,dt.$$

Determine how many solutions the equation $F(x) = 0$ has in $[0,1]$.

Problem 10.10 Find and classify the local extrema within $(0,\infty)$ of the function

$$G(x) = \int_0^{x^2} \sin t\, e^{\sin t}\,dt.$$

Problem 10.11 Write the equation of the straight tangent to the curve

$$y = \int_{x^2}^{\sqrt{\pi}/2} \tan(t^2)\,dt$$

at the point $x = \sqrt[4]{\pi/4}$.

Problem 10.12 Given the function

$$f(x) = \begin{cases} \dfrac{e^x - 1 - x}{x^2}, & x < 0, \\[2ex] a + b\displaystyle\int_0^x e^{-t^4}\,dt, & x \geqslant 0, \end{cases}$$

calculate a and b so that it is continuous and differentiable.

Problem 10.13 Calculate the following limits:

(i) $\displaystyle\lim_{x\to 0} \frac{1}{x^3}\left(\int_0^x e^{t^2}\,dt - x\right)$; (ii) $\displaystyle\lim_{x\to 0} \frac{\cos x}{x^4}\int_0^x \sin(t^3)\,dt$.

Problem 10.14 Calculate the two one-sided limits at $x = 0$ of the function

$$f(x) = \frac{1}{2x^3}\int_0^{x^2} \tan\sqrt{t}\,dt.$$

Problem 10.15 Consider the function $f(x) = \displaystyle\int_0^{x^2} \frac{\sin t}{t}\,dt$.

(a) Using the Taylor series of $\sin t$ in powers of t, find that of f in powers of x.

(b) Calculate $\displaystyle\lim_{x\to 0} \frac{f(x)}{1-\cos x}$.

(c) Discuss the convergence of the series $\displaystyle\sum_{n=1}^{\infty} f(1/n)$.

Problem 10.16 Let $f(x) = \displaystyle\int_{-1/x}^{x} \frac{dt}{a^2 + t^2}$. Determine, without computing the integral, for which values of a the function f is constant.

Problem 10.17 Consider the functions $f(x) = e^{x^2} - x^2 - 1$ and $g(x) = \displaystyle\int_0^x f(t)\,dt$.

(a) Write the Taylor series of g in powers of x.
(b) Determine if g has a maximum, a minimum, or an inflection point at $x = 0$.

Problem 10.18
(a) Use the change of variable $t = \sin^2\theta$ to calculate the integral

$$\int_0^1 \arcsin\sqrt{t}\,dt.$$

(b) Consider the function

$$f(x) = \int_0^{\sin^2 x} \arcsin\sqrt{t}\,dt + \int_0^{\cos^2 x} \arccos\sqrt{t}\,dt.$$

Prove that $f(x) = c$, a constant, in the interval $[0, \pi/2]$.
(c) Determine the value of the constant c.

Problem 10.19 The equation

$$\int_0^{g(x)} \left(e^{t^2} + e^{-t^2}\right) dt = x^3 + 3\arctan x$$

defines an injective, differentiable function g in $\mathbb{R}$. Calculate:
(a) $g(0)$, $g'(0)$, and $\left(g^{-1}\right)'(0)$.
(b) $\lim\limits_{x\to 0} \dfrac{g^{-1}(x)}{g(x)}$.

Problem 10.20 Let $f : [-1,1] \mapsto \mathbb{R}$ be any integrable function.
(a) Prove that

$$\int_0^\pi x f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx.$$

HINT: Do the change of variables $y = \pi - x$.
(b) Calculate the integral

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx.$$

Problem 10.21 Let f be a differentiable function such that

$$\int_0^x f(t)\,dt = \int_x^1 t^2 f(t)\,dt + \frac{x^{16}}{8} + \frac{x^{18}}{9} + c.$$

Find $f(x)$ and the constant c.

Problem 10.22 Prove that

$$\int_0^x e^{t^2}\,dt \sim \frac{e^{x^2}}{2x} \quad (x\to\infty).$$

Problem 10.23 Let f be a function $n+1$ times differentiable in an interval I, and let $a, x \in I$. Assume that the integral defining the function

$$R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt, \qquad n = 0, 1, \ldots$$

exists.
(a) Calculate $R_0(x)$.
(b) Integrating by parts, find a recurrence formula for $R_n(x)$.
(c) Solve the recurrence and interpret the result.

11. Geometric Applications of Integrals

11.1 Area of flat figures

Given two functions f and g such that $f(x) \leqslant g(x)$ for all $x \in [a,b]$, we can obtain the area of the flat figure S delimited by them and the vertical lines at $x = a$ and $x = b$ (see Figure 11.1(a)) as

$$\mathscr{A}(S) = \int_a^b \left[g(x) - f(x)\right] dx. \tag{11.1}$$

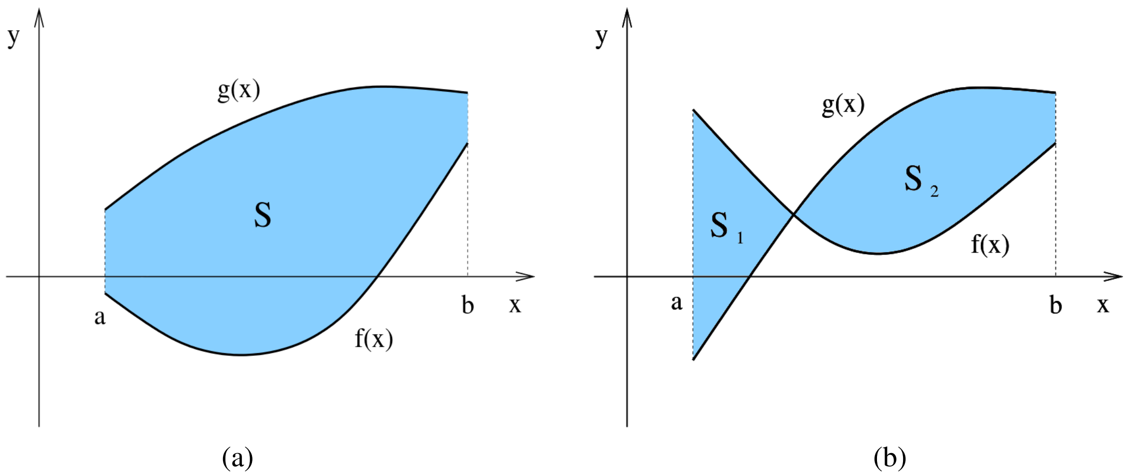


Figure 11.1: Flat figure delimited by the functions $f(x)$ and $g(x)$ and the vertical lines at $x = a$ and $x = b$. (a) Simple case where $f \leqslant g$. (b) Case in which f and g cross each other, so the figure is actually the union of several figures.

In a general case, where f and g can cross one or several times within the interval $[a,b]$, the figure S is made of the union of several figures —joined at the crossing points (see Figure 11.1(b)). Strictly speaking we should then decompose the calculation between consecutive crossing points and apply formula (11.1) taking into account which function is the largest in each subinterval. This can be done automatically by extending formula (11.1) as

$$\mathscr{A}(S) = \int_a^b \left|g(x) - f(x)\right| dx. \tag{11.2}$$

■ **Example 11.1** Let us calculate the area between $f(x) = x(x-2)$ and $g(x) = x/2$ within the interval $[0,2]$. Since in that interval $f(x) \leqslant 0$ and $g(x) \geqslant 0$,

$$\begin{aligned}\mathscr{A}(S) &= \int_0^2 \left(\frac{x}{2} - x(x-2)\right) dx = \int_0^2 \left(\frac{x}{2} - x^2 + 2x\right) dx = \frac{x^2}{4} - \frac{x^3}{3} + x^2\bigg|_0^2 \\ &= 1 - \frac{8}{3} + 4 = \frac{7}{3}.\end{aligned}$$

■

■ **Example 11.2** Let us now calculate the area between the curves $f(x) = x$ and $g(x) = x^3/4$ within the interval $[-1,2]$. First we need to find the crossing points:

$$x = \frac{x^3}{4} \qquad \Rightarrow \qquad x = 0,\ x = \pm 2.$$

Between -2 and 0 we have $g \geqslant f$, but between 0 and 2 the opposite holds. Thus,

$$\begin{aligned}\mathscr{A}(S) &= \int_{-1}^0 \left(\frac{x^3}{4} - x\right) dx + \int_0^2 \left(x - \frac{x^3}{4}\right) dx = \left(\frac{x^4}{16} - \frac{x^2}{2}\right)\bigg|_{-1}^0 + \left(\frac{x^2}{2} - \frac{x^4}{16}\right)\bigg|_0^2 \\ &= 2 - 1 - \frac{1}{16} + \frac{1}{2} = \frac{23}{16}.\end{aligned}$$

■

11.2 Area of flat figures in polar coordinates

Suppose we have a curve given as $r = f(\theta)$, where r is the distance to the origin and θ the angle with the positive X axis. These two variables are known as polar coordinates, and its relation to the cartesian coordinates is given by the transformation (see Figure 11.2(a))

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{11.3}$$

Figure 11.2(b) illustrates a curve $r = f(\theta)$ expressed in polar coordinates.

The problem we face now is that of calculating the area of the figure formed by the curve $r = f(\theta)$ and the radii at angles $\theta = a$ and $\theta = b$, i.e., the figure $S = \big\{(r,\theta) : a \leqslant \theta \leqslant b,\ 0 \leqslant r \leqslant f(\theta)\big\}$. In order to achieve that we can introduce the analogue of upper and lowe sums. If we introduce the partition $P = \{\theta_0, \theta_1, \dots, \theta_n\}$, where $a = \theta_0 < \theta_1 < \cdots < \theta_{n-1} < \theta_n = b$ and define

$$m_i = \inf_{\theta_{i-1} \leqslant \theta \leqslant \theta_i} f(\theta), \qquad M_i = \sup_{\theta_{i-1} \leqslant \theta \leqslant \theta_i} f(\theta), \tag{11.4}$$

then $\mathscr{A}(S)$, the area of S, should be bounded as

$$\sum_{i=1}^n \frac{1}{2} m_i^2 (\theta_i - \theta_{i-1}) \leqslant \mathscr{A}(S) \leqslant \sum_{i=1}^n \frac{1}{2} M_i^2 (\theta_i - \theta_{i-1}). \tag{11.5}$$

If f^2 is an integrable function, this is equivalent to the Riemann definition of the integral. Thus, in the limit we will find

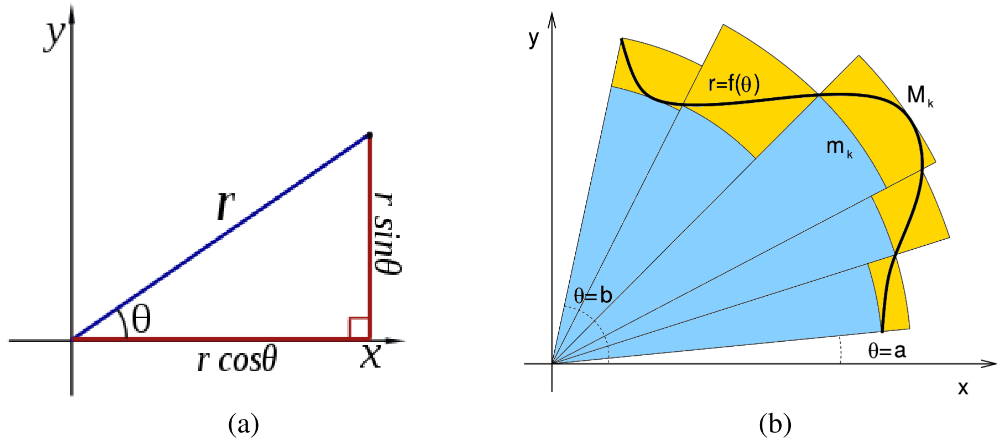


Figure 11.2: (a) Polar coordinates. (b) A curve expressed in polar coordinates $r = f(\theta)$ and its associated upper and lower sums construction. Here rectangles are replaced by circular sectors.

$$\mathscr{A}(S) = \frac{1}{2}\int_a^b f(\theta)^2\, d\theta. \tag{11.6}$$

■ **Example 11.3** Let us calculate the area enclosed by the curve $(x^2+y^2)^3 = y^2$ —represented in Figure 11.3.

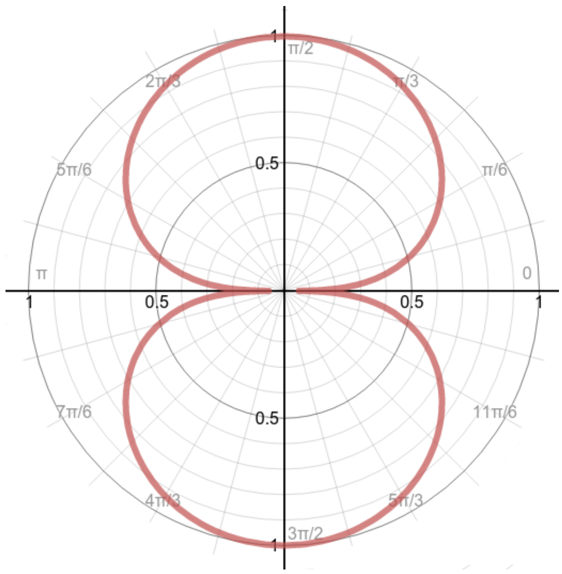


Figure 11.3: Curve $(x^2+y^2)^3 = y^2$.

To begin with, we will rewrite the curve in polar coordinates as

$$(r^2)^3 = r^2\sin^2\theta \qquad \Rightarrow \qquad r^4 = \sin^2\theta \qquad \Rightarrow \qquad r = \sqrt{|\sin\theta|} = f(\theta), \quad 0 \leqslant \theta \leqslant 2\pi.$$

Accordingly

$$\mathscr{A}(S) = \frac{1}{2}\int_0^{2\pi} |\sin\theta|\, d\theta = \int_0^{\pi} \sin\theta\, d\theta = (-\cos\theta)\Big|_0^{\pi} = 2.$$

■

11.3 Volumes

11.3.1 Solids of a given section

Suppose we have a solid S whose sections parallel to certain plane at a distance u from that plane are given by the function $a_S(u)$. Suppose further that the solid spans all distances $a \leqslant u \leqslant b$. Then the volume of S will be given by

$$\mathscr{V}(S) = \int_a^b a_S(u)\, du. \tag{11.7}$$

The proof of this result —first used by Pappus of Alexandria in the 4th century— follows the same spirit as Riemann's upper and lower sums construction. Suppose we make a partition $P = \{a = u_0, u_1, \ldots, u_{n-1}, u_n = b\}$ of the segment $[a,b]$. Any sum

$$\sum_{i=1}^{n} a_S(c_i)(u_i - u_{i-1}), \qquad u_{i-1} \leqslant c_i \leqslant u_i,$$

is a Riemann sum of the function $a_S(u)$ which represent the sum of the volumes of a stack of parallelepipeds that provides an estimate of the volume of the solid S. Thus we obtain formula (11.7) as long as $a_S(u)$ is an integrable function.

■ **Example 11.4** As an example let us calculate the volume of a square pyramid of base side l and height h. If we take the basal plane as the reference plane, sections parallel to the base at height u are squares of side x (see Figure 11.4). The value of x can be determined by triangle similarity as

$$\frac{h}{l/2} = \frac{u}{(l-x)/2} \qquad \Rightarrow \qquad x = l\left(1 - \frac{u}{h}\right).$$

Accordingly

$$a_S(u) = l^2\left(1 - \frac{u}{h}\right)^2$$

and therefore

$$\mathscr{V}(s) = \int_0^h l^2\left(1 - \frac{u}{h}\right)^2 du = l^2\left[-\frac{h}{3}\left(1 - \frac{u}{h}\right)^3\right]\Bigg|_0^h = \frac{l^2 h}{3}.$$

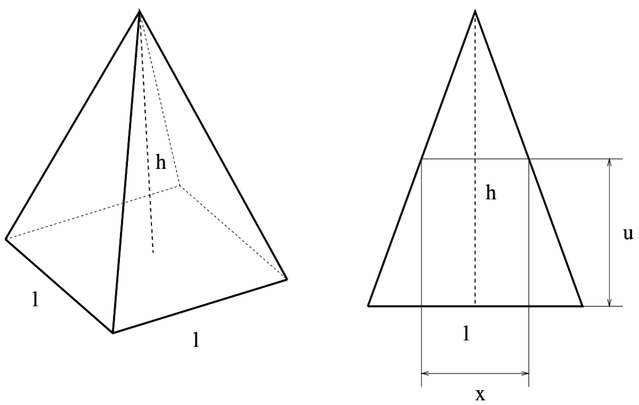


Figure 11.4: Volume of a square pyramid of base side l and height h calculated through its sections.

■

11.3.2 Solids of revolution

As a special application of formula (11.7) we can obtain formulas for solids of revolution —i.e., solids generated by the turn of a plane figure around the X or the Y axis.

Around the X axis

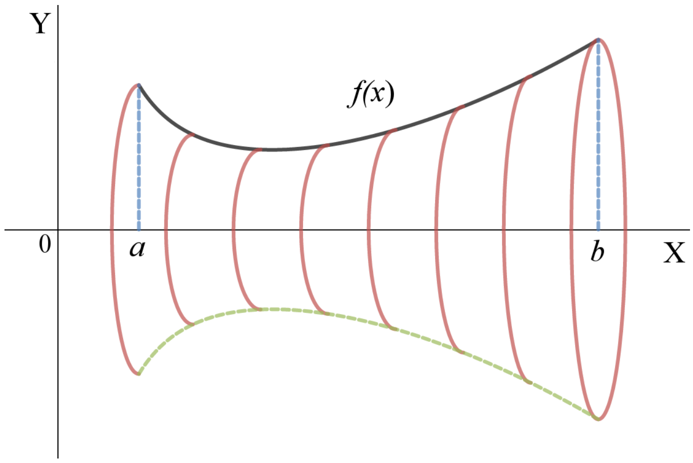


Figure 11.5: Solid of revolution generated by the curve $y = f(x)$ with $a \leqslant x \leqslant b$ as it revolves around the X axis.

As illustrated in Figure 11.5, the portion of the curve $y = f(x)$ within the interval $[a,b]$ generates, as it revolves around the X axis, the surface of a solid of revolution S. Sections of this solid perpendicular to the X axis are disks. If the section is taken at $a \leqslant x \leqslant b$, then the radius of the disk is $f(x)$. Therefore $a_S(x) = \pi f(x)^2$ and consequently

$$\mathscr{V}(S) = \pi \int_a^b f(x)^2 \, dx. \tag{11.8}$$

■ **Example 11.5** To determine the volume of a sphere of radius a we can take as a solid of revolution of the curve $f(x) = \sqrt{a^2 - x^2}$ within the interval $[-a,a]$. Thus,

$$\mathscr{V}(S) = \pi \int_{-a}^{a} (a^2 - x^2) \, dx = \pi \left(a^2 x - \frac{x^3}{3} \right) \Big|_{-a}^{a} = 2\pi \left(a^3 - \frac{a^3}{3} \right) = \frac{4\pi}{3} a^3.$$

■

Around the Y axis

In the case that the flat figure between the curve $y = f(x)$ and the X axis delimited by the interval $[a,b]$ revolves around the Y axis, we obtain a solid of revolution S as that of Figure 11.6. To calculate its volume we need to adapt Pappus's construction a little bit, because each interval of a partition $P = \{a = x_0, x_1, \ldots, x_{n-1}, x_n = b\}$, along with the corresponding heights $f(c_i)$ (where $x_{i-1} \leqslant c_i \leqslant x_i$), generates a hollow cylinder (a tube). The volume of that cylinder is the difference between that of the outer cylinder $\pi f(c_i) x_i^2$ and that of the inner cylinder $\pi f(c_i) x_{i-1}^2$. Thus, the Riemann sum

$$\sum_{i=1}^{n} \pi (x_i^2 - x_{i-1}^2) f(c_i) \tag{11.9}$$

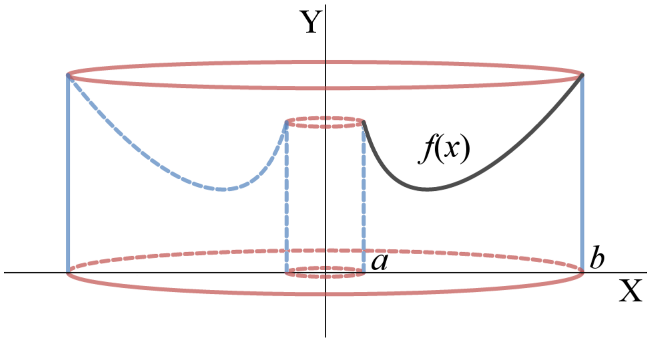


Figure 11.6: Solid of revolution generated by the curve $y = f(x)$ with $a \leqslant x \leqslant b$ as it revolves around the Y axis.

provides an estimate of the volume of the solid S. This is valid for any c_i, but if we choose $c_i = (x_i + x_{i-1})/2$ we can rewrite in a much better way as

$$\sum_{i=1}^{n} 2\pi c_i f(c_i)(x_i - x_{i-1}). \tag{11.10}$$

This is a Riemann sum of the function $g(x) = 2\pi x f(x)$, so if g is integrable in $[a,b]$ then

$$\mathscr{V}(S) = 2\pi \int_a^b x f(x)\,dx. \tag{11.11}$$

■ **Example 11.6** Let us calculate the volume of a doughnut (a *torus* in mathematical parlance), the solid represented in Figure 11.7. This volume will be twice the volume generated by the half-disk delimited by the function $f(x) = \sqrt{a^2 - (x-R)^2}$ and the X axis within the interval $[R-a, R+a]$, as it revolves around the Y axis. Hence

$$\mathscr{V}(S) = 4\pi \int_{R-a}^{R+a} x\sqrt{a^2 - (x-R)^2}\,dx$$

With the change of variable $x = R + a\sin\theta$ (thus $x' = a\cos\theta$) we transform the integral into

$$\mathscr{V}(S) = 4\pi \int_{-\pi/2}^{\pi/2} (R + a\sin\theta)a^2\cos^2\theta\,d\theta$$

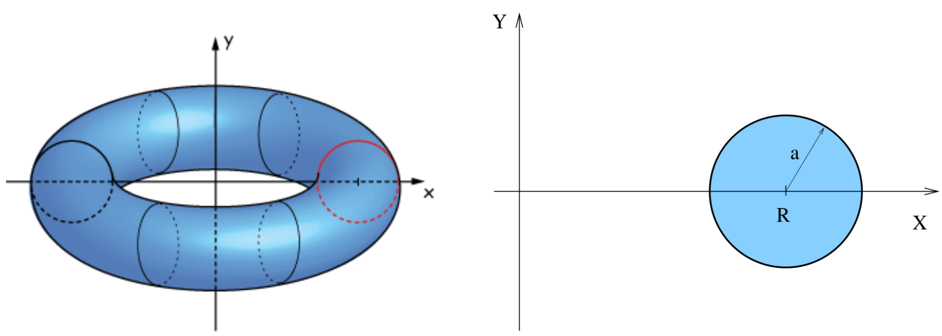


Figure 11.7: Solid of revolution generated by the curve $(x-R)^2 + y^2 = a^2$ as it revolves around the Y axis. This solid —actually a doughnut— is called in mathematics *torus*.

Now, $\sin\theta\cos^2\theta$ is an odd function, so

$$\int_{-\pi/2}^{\pi/2} \sin\theta\cos^2\theta\, d\theta = 0$$

and therefore

$$\mathscr{V}(S) = 4\pi R a^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta = 2\pi R a^2 \int_{-\pi/2}^{\pi/2} (1+\cos 2\theta)\, d\theta = 2\pi R a^2 \left(\pi + \underbrace{\frac{1}{2}\sin 2\theta\Big|_{-\pi/2}^{\pi/2}}_{=0}\right)$$

$$= (2\pi R)(\pi a^2).$$

■

11.4 Length of curves

Consider a parametric curve $C = \{\mathbf{r}(t) \in \mathbb{R}^n : a \leqslant t \leqslant b\}$, and the partition of the interval $[a,b]$ defined by $P = \{a = t_0, t_1, \dots, t_{n-1}, t_n = b\}$. If we join with straight segments the point $\mathbf{r}(t_0)$ with $\mathbf{r}(t_1)$, the point $\mathbf{r}(t_1)$ with $\mathbf{r}(t_2)$, and so on and so forth up to $\mathbf{r}(t_{n-1})$ with $\mathbf{r}(t_n)$, we obtain a polygonal curve $\Pi(P)$ that approximates the curve C (see Figure 11.8). The length of $\Pi(P)$ is easy to calculate:

$$\mathscr{L}\big(\Pi(P)\big) = \sum_{i=1}^{n} \|\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})\|. \tag{11.12}$$

If we multiply and divide each term of this sum by the length of the interval of the parameter we obtain

$$\mathscr{L}\big(\Pi(P)\big) = \sum_{i=1}^{n} \left\| \frac{\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})}{t_i - t_{i-1}} \right\| (t_i - t_{i-1}). \tag{11.13}$$

As we refine more and more the partition

$$\frac{\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})}{t_i - t_{i-1}} \xrightarrow[t_i - t_{i-1} \to 0]{} \mathbf{r}'(t_i)$$

and $\mathscr{L}\big(\Pi(P)\big)$ gets closer and closer to a Riemann sum of the function $\|\mathbf{r}'(t)\|$. Thus, if this function is integrable,

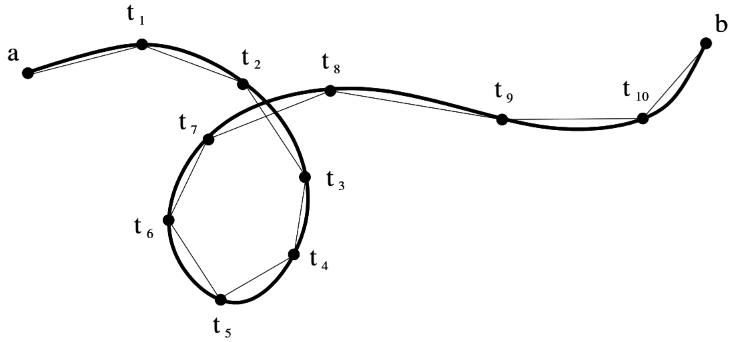


Figure 11.8: A curve along with its polygonal curve associated to a partition.

$$\mathscr{L}(C) = \int_a^b \|r'(t)\|\,dt. \tag{11.14}$$

■ **Example 11.7** A circumference of radius a is a plane curve C given by the parametrisation $\mathbf{r}(t) = (a\cos t, a\sin t)$, where $0 \leqslant t \leqslant 2\pi$. Thus, its length can be obtained as

$$\mathscr{L}(C) = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt = \int_0^{2\pi} a\,dt = 2\pi a.$$

■

Problems

Problem 11.1 Calculate the area delimited by the following curves:

(i) $y = x^2$, $y = (x-2)^2$, $y = (2-x)/6$;
(ii) $x^2 + y^2 = 1$, $x^2 + y^2 = 2x$;
(iii) $y = \dfrac{1-x}{1+x}$, $y = \dfrac{2-x}{1+x}$, $y = 0$, $y = 1$;
(iv) one loop of the curve $y^2 = (x-a)(x-b)^2$, with $a < b$.

Problem 11.2 Determine the area between the curve $f(x) = \dfrac{x(x^2-1)}{(x^2+1)^{3/2}}$ and the X axis.

Problem 11.3 Calculate the area delimited by the following curves:

(i) $r = a\theta$ (Archimedes's spiral), $0 \leqslant \theta \leqslant 2\pi$, and the segment $\{(x,0) : 0 \leqslant x \leqslant 2\pi a\}$;
(ii) a petal of the three-petal rose $r = a\cos 3\theta$, $-\pi/6 \leqslant \theta \leqslant \pi/6$;
(iii) half a *lemniscata* $r = a\sqrt{\cos 2\theta}$, $-\pi/4 \leqslant \theta \leqslant \pi/4$.

Problem 11.4 Let A the plane figure limited by the curves $y = x^2$ and $y = \sqrt{x}$. Determine:

(a) the area of A;
(b) the volume of the solid generated when A revolves around the X axis.

Problem 11.5 Compute the volume of the solids generated when the following sets revolve around the X asis:

(i) $0 \leqslant y \leqslant 1 + \sin x$, $0 \leqslant x \leqslant 2\pi$;
(ii) $R^2 \leqslant x^2 + y^2 \leqslant 4R^2$;
(iii) plane figure delimited by the curves $y = \sin x$ and $y = x$ with $0 \leqslant x \leqslant \pi$.

Problem 11.6 Compute the volume of the following solids:

(i) the solid generated when the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ revolves around the X axis;
(ii) same thing around the Y axis;
(iii) the solid whose base is the ellipse above and whose sections perpendicular to the X axis are isosceles triangles of height 2.

Problem 11.7

(a) Calculate the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.
(b) Calculate the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.
(c) Check the result of Problem 11.6 (i) and (ii) as particular cases of the previous result.

HINT: Notice that intersecting the ellipsoid by planes parallel to the coordinate planes ($x = 0$, $y = 0$, or $z = 0$) we obtain ellipses.

Problem 11.8 Calculate the length of the following curves:

(i) catenary: $y = e^{x/2} + e^{-x/2}$, $0 \leqslant x \leqslant 2$;
(ii) cycloid: $x(t) = a(t - \sin t)$, $y(t) = a(1 - \cos t)$, $0 \leqslant t \leqslant 2\pi$;
(iii) hypocycloid or astroid: $x^{2/3} + y^{2/3} = 4$;
(iv) tractrix: $y = a\log\left(\dfrac{a + \sqrt{a^2 - x^2}}{x}\right) - \sqrt{a^2 - x^2}$, $a/2 \leqslant x \leqslant a$;
(v) cardioid: $r = 1 + \cos\theta$, $0 \leqslant \theta \leqslant 2\pi$.

V Appendices

A. Sums and products

Sums and products are commonly denoted by the symbols

$$\sum_{k=1}^{n} a_k \equiv a_1 + a_2 + \cdots + a_n, \qquad \prod_{k=1}^{n} a_k \equiv a_1 a_2 \cdots a_n. \tag{A.1}$$

This is a handy symbolic notation that helps doing calculations thanks to a set of properties that allow simple manipulations. Some of these properties are common to sums and products:

Common properties of symbolic sums and products:

(i) *Trivial sum/product:*

$$\sum_{k=1}^{1} a_k = \prod_{k=1}^{1} a_k = a_1.$$

(ii) *Separating out some elements:*

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_r + \sum_{k=r+1}^{n} a_k = \left(\sum_{k=1}^{n-r} a_k \right) + a_{n-r+1} + a_{n-r+2} + \cdots + a_n,$$

$$\prod_{k=1}^{n} a_k = a_1 a_2 \cdots a_r \prod_{k=r+1}^{n} a_k = \left(\prod_{k=1}^{n-r} a_k \right) a_{n-r+1} a_{n-r+2} \cdots a_n,$$

for all $1 \leqslant r \leqslant n-1$.

(iii) *Shifting indices:*

$$\sum_{k=1}^{n} a_k = \sum_{k=r+1}^{n+r} a_{k-r}, \qquad \prod_{k=1}^{n} a_k = \prod_{k=r+1}^{n+r} a_{k-r},$$

for all $r \in \mathbb{Z}$.

(iv) *Reversal:*

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a_{n-k+1}, \qquad \prod_{k=1}^{n} a_k = \prod_{k=1}^{n} a_{n-k+1}.$$

Other properties are specific for sums and for products:

Properties of symbolic sums:

(i) *Additive property:*

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$$

(ii) *Homogeneous property:*

$$\sum_{k=1}^{n} c a_k = c \sum_{k=1}^{n} a_k, \qquad c \in \mathbb{R}.$$

(iii) *Telescoping property:*

$$\sum_{k=1}^{n} (a_k - a_{k-1}) = a_n - a_0, \qquad \sum_{k=1}^{n} (a_k - a_{k+1}) = a_1 - a_{n+1}.$$

(iv) *Distributive property:*

$$\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{k=1}^{m} b_k\right) = \sum_{j=1}^{n}\sum_{k=1}^{m} a_j b_k.$$

(v) *Sum of a constant:*

$$\sum_{k=1}^{n} c = nc, \qquad c \in \mathbb{R}.$$

Properties of symbolic products:

(i) *Multiplicative property:*

$$\prod_{k=1}^{n} a_k b_k = \left(\prod_{k=1}^{n} a_k\right)\left(\prod_{k=1}^{n} b_k\right)$$

(ii) *Homogeneous property:*

$$\prod_{k=1}^{n} a_k^{\gamma} = \left(\prod_{k=1}^{n} a_k\right)^{\gamma}, \qquad \gamma \in \mathbb{R}.$$

(iii) *Telescoping property:*

$$\prod_{k=1}^{n} \frac{a_k}{a_{k-1}} = \frac{a_n}{a_0}, \qquad \prod_{k=1}^{n} \frac{a_k}{a_{k+1}} = \frac{a_1}{a_{n+1}}.$$

(iv) *Product of a constant:*

$$\prod_{k=1}^{n} c = c^n, \qquad c \in \mathbb{R}.$$

■ **Example A.1** In order to calculate

$$1 + 2 + \cdots + n = \sum_{k=1}^{n} k$$

we can simply apply the reversal property of sums:

$$\sum_{k=1}^{n} k = \sum_{k=1}^{n} (n+1-k) = \sum_{k=1}^{n} (n+1) - \sum_{k=1}^{n} k.$$

This provides an equation for our sum, which leads to

$$2\sum_{k=1}^{n} k = \sum_{k=1}^{n} (n+1) = n(n+1),$$

because $(n+1)$ is a constant (i.e., does not dependent on k). Hence the well-known result

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

■

■ **Example A.2** Let us now calculate

$$1 + 3 + 5 + \cdots + (2n-1) = \sum_{k=1}^{n} (2k-1).$$

We will do that in two ways. The first one amounts to splitting the sum as

$$\sum_{k=1}^{n} (2k-1) = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = 2\frac{n(n+1)}{2} - n = n(n+1) - n = n^2.$$

For the second one we "complete" the sum with the even numbers, i.e., $2 + 4 + \cdots + 2n$, so that

$$\begin{aligned}\sum_{k=1}^{n} (2k-1) + \sum_{k=1}^{n} 2k &= \left[1 + 3 + \cdots + (2n-1)\right] + \left[2 + 4 + \cdots + 2n\right] \\ &= 1 + 2 + 3 + 4 + \cdots + (2n-1) + 2n = \sum_{k=1}^{2n} k = \frac{2n(2n+1)}{2} = 2n^2 + n.\end{aligned}$$

But it is very easy to calculate the sum of the even numbers:

$$\sum_{k=1}^{n} 2k = 2\sum_{k=1}^{n} k = 2\frac{n(n+1)}{2} = n^2 + n,$$

So we obtain

$$\sum_{k=1}^{n} (2k-1) = 2n^2 + n - (n^2 + n) = n^2.$$

■

Exercise A.1 Write down as symbolic sums:

(a) $3+6+9+\cdots+60$, and calculate it;

(b) the polynomial $3x-6x^2+9x^3-12x^4+\cdots+57x^{19}-60x^{20}$.

■

■ **Example A.3** The simplest way to calculate the sum

$$S=\sum_{k=1}^{3}\left(\sum_{m=k}^{3}mk\right).$$

is to write the first sum down explicitly, like

$$S=\sum_{m=1}^{3}m+\sum_{m=2}^{3}2m+\sum_{m=3}^{3}3m=(1+2+3)+2(2+3)+3\cdot 3=6+10+9=25.$$

■

■ **Example A.4** Let us calculate

$$S=\sum_{k=51}^{100}(2k-1)=101+103+105+\cdots+199.$$

To do that we write

$$S=\sum_{k=1}^{100}(2k-1)-\sum_{k=1}^{50}(2k-1),$$

where we have added an substracted all the terms from $k=1$ to $k=50$. But we now know that

$$\sum_{k=1}^{100}(2k-1)=100^2=10000,\qquad \sum_{k=1}^{50}(2k-1)=50^2=2500,$$

therefore

$$S=10000-2500=7500.$$

■

■ **Example A.5** The apparently complicated sum

$$S=\sum_{j=4}^{99}\sqrt{j}-\sum_{j=6}^{101}\sqrt{j-1}$$

is actually very easily done with a few manipulations. First of all, we can shift the index in the second sum as,

$$S=\sum_{j=4}^{99}\sqrt{j}-\sum_{j=5}^{100}\sqrt{j},$$

and secondly,

$$\sum_{j=4}^{99}\sqrt{j}=\sqrt{4}+\sum_{j=5}^{99}\sqrt{j}=2+\sum_{j=5}^{99}\sqrt{j},$$

$$\sum_{j=5}^{100}\sqrt{j}=\sum_{j=5}^{99}\sqrt{j}+\sqrt{100}=\sum_{j=5}^{99}\sqrt{j}+10,$$

so

$$S=2+\cancel{\sum_{j=5}^{99}\sqrt{j}}-\cancel{\sum_{j=5}^{99}\sqrt{j}}-10=-8.$$

■

B. Binomial formula

B.1 Binomial coefficients

Consider the following combinatorial problem. We have two letters, a and b and we want to form sequences of length n using k letters a and $n-k$ letters b. For the sake of clarity, suppose that $n = 5$ and $k = 3$. We can form 10 different sequences, namely

$$aaabb, \quad aabab, \quad abaab, \quad baaab, \quad aabba, \quad ababa, \quad baaba, \quad abbaa, \quad babaa, \quad bbaaa.$$

This procedure becomes inpractical if n and k are large, so we need an alternative way of counting these expression without explicitly writing them down. For the moment, we will introduce a symbol to express this number: the *combinatorial coefficient* $\binom{n}{k}$ (read it "n choose k").

Imagine that we want to evaluate $\binom{n}{k}$. We divide the sequences into a block with the first $n-1$ letters, and a block with the last letter. Sequences are of two types: those having an a in the second block, and those having a b. How many are there of the first type? Well, we have a block formed by all possible sequences of length $n-1$ containing $k-1$ letters a and $n-k$ letters b. We do not know how many of these sequences are there but we know the number must be $\binom{n-1}{k-1}$. And what about the second type? In this case the first block are all possible sequences of length $n-1$ containing k letters a and $n-k-1$ letters b, and this number is $\binom{n-1}{k}$. Then the total number of sequences of length n we want to know is the sum of the two types. In other words,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}. \tag{B.1}$$

This is a recurrence equation, but in order to iterate and obtain the numerical values we need some starting value.

Some combinatorial numbers are easy to obtain. For instance, if $k = 0$ or n all letters in the sequence are identical, so only one sequence is possible. Hence

$$\binom{n}{0} = \binom{n}{n} = 1. \tag{B.2}$$

Besides, the number of sequences with k letters a and $n-k$ letters b is the same as the number of sequences with k letters b and $n-k$ letters a, because this number does not depend on the identity of the letters. In terms of binomial coefficients,

$$\binom{n}{k} = \binom{n}{n-k}. \tag{B.3}$$

Now, equation (B.2) provides all binomial coefficients for $n = 1$. For $n = 2$ we can apply the recurrence (B.1) and obtain the one we still don't know,

$$\binom{2}{1} = \binom{1}{0} + \binom{1}{1} = 1 + 1 = 2.$$

There are two sequences with one a and one b (obviously, ab and ba). If we continue calculating now the coefficients for $n = 3, 4, \dots$ we will obtain the table

$$\begin{array}{lccccccccccccc}
n=1 & & & & & & 1 & & 1 & & & & & \\
n=2 & & & & & 1 & & 2 & & 1 & & & & \\
n=3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
n=4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
n=5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
n=5 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
\vdots & & & & & & & \vdots & & & & & &
\end{array}$$

which is known as *Pascal's triangle*. It is very easy to write a new row because they start and end with 1 and every position is obtained by adding up the two coefficients it has on top (this is what equation (B.1) actually says).

There is a closed formula for $\binom{n}{k}$ which we can derive using combinatorics. Imagine that the k letters a are all distinct (e.g., because the carry a subscript, $a_1, a_2, \dots, a_k$) and the same holds for the $n-k$ letters b. The total number of different sequences is not the permutation of n elements, because now all of them are different. This number is $n! = n \cdot (n-1) \cdots 2 \cdot 1$ (we can take any of the n letters for the first position, then any of the $n-1$ remaining ones for the second position, then any of the $n-2$ remaining ones for the third position, etc.).

But there is an alternative way of making all these sequences, namely we take the $\binom{n}{k}$ sequences we had when all a's and all b's were identical, but now make all possible substitutions of them by the nonidentical letters. There are $k!$ different arrangements of the letters $a_1, a_2, \dots, a_k$ in the stubs left by the letters a, and $(n-k)!$ different arrangements of the letters $b_1, b_2, \dots, b_{n-k}$ in the stubs left by the letters b. This makes a total of $k!(n-k)!\binom{n}{k}$ sequences. Since both countings must yield the same result,

$$k!(n-k)!\binom{n}{k} = n!.$$

In other words,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}, \tag{B.4}$$

where in the second version we have simply cancelled common factors in the numerator and the denominator.

Exercise B.1 Check recursion (B.1) explicitly using formula (B.4). ■

Exercise B.2 Prove the identities

$$k\binom{n}{k} = n\binom{n-1}{k-1}, \qquad (n-k)\binom{n}{k} = n\binom{n-1}{k}.$$

■

B.2 Binomial formula

We now whan to derive a formula for $(a+b)^n$ as a sum of powers of a and b. Take, for instance, the case $n=2$:

$$(a+b)^2 = (a+b)(a+b) = aa+ab+ba+bb = a^2+2ab+b^2.$$

Or the case $n=3$:

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2 = (a+b)(aa+ab+ba+bb)\\ &= aaa+aab+aba+abb+baa+bab+bba+bbb = a^3+3a^2b+3ab^2+b^3.\end{aligned}$$

In the explicit multiplication we can see how all sequences of letters a and b appear. As a matter of fact, the last step is just grouping those having k letters a and $n-k$ letters b and adding a coefficient that counts how many such sequences there are. So it's not surprising that the coefficients that we see appearing are the binommial coefficients in the nth row of Pascal's triangle.

We can therefore guess the *binomial formula*

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}. \tag{B.5}$$

We will prove this formula using induction.

For $n=1$ the formula reads

$$a+b = \binom{1}{0}a + \binom{1}{1}b,$$

which is trivially true.

Now, let us assume that the formula holds for n. Then we can obtain the formula for $n+1$ as

$$\begin{aligned}(a+b)^{n+1} &= (a+b)(a+b)^n = a(a+b)^n + b(a+b)^n = a\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} + b\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}\\ &= \sum_{k=0}^{n}\binom{n}{k}a^{k+1}b^{n-k} + \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k+1}.\end{aligned}$$

In the first sum we arrange the expression so that it explicitly depends on $k+1$ (by adding and substracting 1 as appropriate):

$$(a+b)^{n+1} = \sum_{k=0}^{n}\binom{n}{(k+1)-1}a^{k+1}b^{n+1-(k+1)} + \sum_{k=0}^{n}\binom{n}{k}a^k b^{n+1-k},$$

and now shift the index from $k+1$ to k (see Appendix A):

$$(a+b)^{n+1} = \sum_{k=1}^{n+1}\binom{n}{k-1}a^k b^{n+1-k} + \sum_{k=0}^{n}\binom{n}{k}a^k b^{n+1-k}.$$

Next we single out the last element ($k = n+1$) of the first sum and the first element ($k = 0$) of the second sum and merge both sums into one:

$$(a+b)^{n+1} = \underbrace{\binom{n}{n}}_{=1} a^{n+1} + \sum_{k=1}^{n} \binom{n}{k-1} a^k b^{n+1-k} + \sum_{k=1}^{n} \binom{n}{k} a^k b^{n+1-k} + \underbrace{\binom{n}{0}}_{=1} b^{n+1}$$

$$= a^{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k-1} + \binom{n}{k} \right] a^k b^{n+1-k} + b^{n+1}.$$

Finally we use the recurrence (B.1):

$$(a+b)^{n+1} = a^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^k b^{n+1-k} + b^{n+1},$$

and identify a^{n+1} and b^{n+1} with the terms $k = n+1$ and $k = 0$ of the sum, respectively:

$$(a+b)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k}.$$

This is formula (B.5) with n replaced by $n+1$, so assuming that the formula holds for n implies that it also holds for $n+1$. This completes the proof.

Exercise B.3 Using the binomial formula (B.5) prove the identities

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n, \qquad \sum_{k \text{ odd}}^{n} \binom{n}{k} = \sum_{k \text{ even}}^{n} \binom{n}{k}.$$

HINT: choose appropriate values for a and b. ■

C. Euler's number

A usurer lends money at an Annual Percentage Rate (APR) of 100%. This means that if you needed 1000 euros, you would have to return 2000 euros after a year (the 'principal' plus 1000 euros of interest). But this guy realises he is loosing money with this procedure, for he can lend you the 1000 euros for 6 months, after which you must return 1500 euros, and if you still need the money for another 6 months you can keep it, but the interest rate during the second 6 months is applied to the new principal, 1500 euros. So at the end you must pay 2250 euros, instead of 2000. You end up paying a factor $1.5 \times 1.5 = 2.25$ the initial principal (i.e., 125% APR).

But why stopping there? What if he repeats this procedure every three months? After each three-month period you owe the usurer 1.25 the previous principal. This means you owe 1250 euros after 3 months, 1562.50 after 6 months, 1953.125 euros after 9 months, and you must return 2441.40625 euros after one year. In total you pay $1.25^4 = 2.44140625$ times the principal ($\approx 144\%$ APR).

The logical step is obvious: if the usurer divides the year in n periods, after one year you end up paying

$$\left(1+\frac{1}{n}\right)^n$$

times the principal. The optimal for the usurer is to make as many divisions of the year as possible, and it's clear that there is no limit to the number of such divisions. So, according to this logical argument, the usurer feels entitled to multiply the principal by a factor

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = 2.71828182845904\ldots$$

Rounding up, you owe the usurer 2718.28 euros ($\approx 172\%$ APR). Perhaps the surprise here is that—to the usurer's disappointment—this amount is not infinite!

As a matter of fact this is what modern banks do—only they do not apply usurious annual interest rates. Thus, if the interest rate is $0 < r < 1$ per year, the amount you must return after one year is a factor over the principal that is calculated as

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r.$$

For instance, a 5% interest rate means $r = 0.05$ and $e^r \approx 1.051$, i.e., a 5.1% APR. If you return the money after, say, 10 years instead, the factor is $e^{10r} \approx 1.649$, i.e., a $64.9/10 \approx 6.5\%$ APR. And so on. The longer the return period the higher the APR (with an exponential increase). This is the reason why banks are so fond of lending mortgages.

It is not our concern here to discuss the fairness of financial practices, but to prove that the above limit exists, as well as its irrational nature.

C.1 Existence

The way we are going to prove the existence of e as a genuine real number is by proving that its defining sequence is monotonically increasing and bounded from above.

Proposition C.1.1 The sequence $\{a_n\}_{n=1}^{\infty}$ defined by

$$a_n = \left(1+\frac{1}{n}\right)^n$$

is monotonically increasing

Proof. Let us apply the binomial formula to a_n,

$$a_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k = \sum_{k=0}^{n}\frac{n!}{(n-k)!k!n^k} = 2+\sum_{k=2}^{n}\frac{n!}{(n-k)!k!n^k} = 2+\sum_{k=2}^{n}\frac{A_k(n)}{k!},$$

where

$$A_k(n) = \frac{n!}{(n-k)!n^k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{n^k} = \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right).$$

If we do the same with a_{n+1},

$$a_{n+1} = 2+\sum_{k=2}^{n+1}\frac{A_k(n+1)}{k!}, \qquad A_k(n+1) = \left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right).$$

Now, it is clear that for any $j \in \mathbb{N}$

$$\frac{j}{n+1} < \frac{j}{n} \quad \Rightarrow \quad 1-\frac{j}{n+1} > 1-\frac{j}{n},$$

therefore $A_k(n+1) > A_k(n)$ for all k. Also $A_n(n) > 0$. Accordingly,

$$a_{n+1} = 2+\sum_{k=2}^{n+1}\frac{A_k(n+1)}{k!} > 2+\sum_{k=2}^{n}\frac{A_k(n+1)}{k!} > 2+\sum_{k=2}^{n}\frac{A_k(n)}{k!} = a_n,$$

which proofs the claim. ■

To proceed we are going to need two trivial observations. The first one is that $n! \geqslant 2^{n-1}$ for all $n \in \mathbb{N}$ (the equality holding only for $n \leqslant 2$). This is clear because

$$n! = \underbrace{2\cdot 3\cdots(n-1)n}_{n-1 \text{ factors}}, \qquad 2^{n-1} = \underbrace{2\cdot 2\cdots 2\cdot 2}_{n-1 \text{ factors}},$$

and every factor of $n!$ is larger than (or, in the case of 2, equal to) 2. An immediate consequence is that

$$\frac{1}{n!} \leqslant \frac{1}{2^{n-1}}, \quad n > 2$$

(the equality holding only for $n \leqslant 2$).

The second observation is that $A_k(n) < 1$ for every $2 \leqslant k \leqslant n$, the reason being that every factor forming $A_k(n)$ is strictly smaller than 1.

With these two observations we can prove the second proposition:

Proposition C.1.2

$$a_n = \left(1+\frac{1}{n}\right)^n < 3.$$

Proof. As we have just seen,

$$a_n = 2+\sum_{k=2}^{n}\frac{A_k(n)}{k!} < 2+\sum_{k=2}^{n}\frac{1}{k!} < 2+\sum_{k=2}^{n}\frac{1}{2^{k-1}} = 2+\sum_{k=1}^{n-1}\frac{1}{2^k} = 2+\frac{\frac{1}{2}-\frac{1}{2^n}}{1-\frac{1}{2}} = 3-\frac{1}{2^{n-1}},$$

from which $a_n < 3$ for all $n \in \mathbb{N}$. ■

C.2 Irrationality

Before we consider the rational or irrational nature of e we are going to prove its most famous—and computationally efficient—representation.

Theorem C.2.1

$$e = \sum_{n=0}^{\infty}\frac{1}{n!}.$$

Proof. Establishing the convergence of this series is straightforward. As a matter of fact, in the proof of Proposition C.1.2 we have shown that

$$a_n = \left(1+\frac{1}{n}\right)^n < \sum_{k=0}^{n}\frac{1}{k!} < 3,$$

from which convergence follows immediately. Now, taking $m \in \mathbb{N}$ fixed, for every $n > m$ we have

$$a_n = 2+\sum_{k=2}^{n}\frac{A_k(n)}{k!} > 2+\sum_{k=2}^{m}\frac{A_k(n)}{k!}$$

because $A_k(n) > 0$ for every $2 \leqslant k \leqslant n$. Hence the bracketing

$$2+\sum_{k=2}^{m}\frac{A_k(n)}{k!} < a_n < \sum_{k=0}^{n}\frac{1}{k!}$$

holds true for every $n > m$. On the other hand, since

$$\lim_{n\to\infty} A_k(n) = 1, \quad k \geqslant 2,$$

taking limits in the bracketing we get

$$\sum_{k=0}^{m}\frac{1}{k!} \leqslant e \leqslant \sum_{k=0}^{\infty}\frac{1}{k!}.$$

This new braketing holds for every $m \in \mathbb{N}$, therefore

$$e = \sum_{k=0}^{\infty}\frac{1}{k!}.$$ ■

The proof the e is irrational makes use of this representation, and it's a proof by contradiction. But before we get into it we are going to obtain a bracketing of e that both provides an excellent approximation to calculate e as well as paves the way to prove its irrationality.

Lemma C.2.2

$$x_n < e < x_n + \frac{1}{n!n}, \qquad x_n \equiv \sum_{k=0}^{n} \frac{1}{k!}.$$

Proof. As all terms in the series defining e are positive, the first inequality, $x_n < e$ is straightforward. As for the second,

$$\begin{aligned}
e &= x_n + \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots \\
&= x_n + \frac{1}{(n+1)!}\left[1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \frac{1}{(n+2)(n+3)(n+4)} + \cdots\right] \\
&< x_n + \frac{1}{(n+1)!}\left[1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots\right] = x_n + \frac{1}{(n+1)!}\frac{1}{1-\frac{1}{n+1}} \\
&= x_n + \frac{1}{(n+1)!}\frac{n+1}{n} = x_n + \frac{1}{n!n}.
\end{aligned}$$

■

This bracketing states that the error we make approximating e by x_n is smaller than $1/n!n$—quite a small number for not too large values of n. For instance, if we take $n = 9$, the error is smaller than 3×10^{-7}, and indeed

$$x_9 = \frac{98641}{36288} = \mathbf{2.7182815}255\ldots, \qquad e = \mathbf{2.7182818}284\ldots$$

And here comes the irrationality proof, due to Fourier:

Theorem C.2.3 — Fourier, 1815. e is irrational.

Proof. Suppose otherwise that $e = p/q$, an irreducible fraction with $q > 1$ (e is not an integer). Then,

$$x_q < e < x_q + \frac{1}{q!q} \quad \Rightarrow \quad q!qx_q < (q-1)!p < q!qx_q + 1.$$

But $q!qx_q = N \in \mathbb{N}$, and also $(q-1)!p \in \mathbb{N}$, and yet $N < (q-1)!p < N+1$, which is impossible (there is no natural number between two consecutive natural numbers). Hence the rationality assumption for e leads to a contradiction. ■

D. Solutions to exercises

D.1 The Real Line

Problem 1.1

(a) First of all, if $0 < a < b$, then $0 < \sqrt{a} < \sqrt{b}$ (for suppose it were false and $\sqrt{b} \leqslant \sqrt{a}$; then squaring we would have $b \leqslant a$, which we know is false). Now, multiplying this last inequality by $\sqrt{a} > 0$ we get

$$\left(\sqrt{a}\right)^2 < \sqrt{a}\sqrt{b} \qquad \Leftrightarrow \qquad a < \sqrt{ab}.$$

For the second inequality, we start off from the fact that $\left(\sqrt{a} - \sqrt{b}\right)^2 > 0$ and then expand the binomial to obtain $a - 2\sqrt{ab} + b > 0$. Adding $2\sqrt{ab}$ to the inequality we get $a + b > 2\sqrt{ab}$, and finally multiplying by $1/2$ we obtain

$$\frac{a+b}{2} > \sqrt{ab}.$$

The last inequality is obtained by adding b to $a < b$ to obtain

$$a + b < b + b = 2b \qquad \Leftrightarrow \qquad \frac{a+b}{2} < b.$$

(b) Since $0 < a < b$ and $c > 0$, then $ac < bc$. Now we add ab to the inequality and obtain $ab + ac < ab + bc$. Factoring out the common factor in each side of it,

$$a(b+c) < b(a+c) \qquad \Leftrightarrow \qquad \frac{a}{b} < \frac{a+c}{b+c}$$

because dividing by $a > 0$ or $b > 0$ does not change the inequality.

Problem 1.2 Proving this amounts to proving two statements: (i) that $|a+b| = |a| + |b|$ implies $ab \geqslant 0$, and (ii) that $ab \geqslant 0$ implies $|a+b| = |a| + |b|$.

Let us start with (i). If we square the expression we get $|a+b|^2 = (|a|+|b|)^2$. But $|a+b|^2 = (a+b)^2$. Now expanding both binomials we obtain

$$a^2+2ab+b^2 = |a|^2+2|ab|+|b|^2,$$

and cancelling common terms in both sides we end up with $2ab = 2|ab| \geqslant 0$.

As for (ii), $ab \geqslant 0$ means that a and b have both the same sign. Suppose $a \geqslant 0$ and $b \geqslant 0$. Then $|a+b| = a+b = |a|+|b|$. Suppose now that $a \leqslant 0$ and $b \leqslant 0$. Then we can write $a = -|a|$ and $b = -|b|$, and therefore

$$|a+b| = \big|-|a|-|b|\big| = \big|-(|a|+|b|)\big| = |a|+|b|.$$

Problem 1.3

(a) Suppose $x \geqslant y$. Then $\max\{x,y\} = x$ and $|x-y| = x-y$, so

$$\frac{x+y+|x-y|}{2} = \frac{x+y+x-y}{2} = \frac{2x}{2} = x.$$

Suppose now that $x < y$. Then $\max\{x,y\} = y$ and $|x-y| = y-x$, so

$$\frac{x+y+|x-y|}{2} = \frac{x+y+y-x}{2} = \frac{2y}{2} = y.$$

In both cases the two sides of the equality yield the same result.

(b) Suppose $x \geqslant y$. Then $\min\{x,y\} = y$ and $|x-y| = x-y$, so

$$\frac{x+y-|x-y|}{2} = \frac{x+y-x+y}{2} = \frac{2y}{2} = y.$$

Suppose now that $x < y$. Then $\max\{x,y\} = x$ and $|x-y| = y-x$, so

$$\frac{x+y-|x-y|}{2} = \frac{x+y-y+x}{2} = \frac{2x}{2} = x.$$

Again, whichever the case, both sides of the equality yield the same result.

Problem 1.4 Clearly $\varphi(x) = \max\{x,0\}$, so using the formulas from the previous exercise

$$\varphi(x) = \frac{x+|x|}{2}.$$

Problem 1.5

(a) $n^2-n = n(n-1)$, which is even because it is the product of two consecutive numbers —one of which must be even.

(b) $n^3-n = n(n^2-1) = (n-1)n(n+1)$, hence must be a multiple of 3 because it is the product of three consecutive numbers —one of which must be a multiple of 3—, but is also a multiple of 2 because in every three consequtive numbers at least one is even —and possibly two. Thus it is a multiple of both, 2 and 3, therefore is a multiple of 6.

(c) Odd numbers are written as $n = 2k-1$, with $k \in \mathbb{N}$. Hence $n^2-1 = (2k-1)^2-1 = 4k^2-4k+1-1 = 4k^2-4k = 4k(k-1)$. It is clearly a multiple of 4, but since one of the other two factors must be even, it is a multiple of 8.

Problem 1.6

(a) Let us check that the identity holds for $n = 1$. The left-hand side is clearly $a-b$. As for the right-hand side,

$$\sum_{k=1}^{1} a^{1-k}b^{k-1} = a^0b^0 = 1,$$

so the right-hand side is also $a-b$.
Now let us assume that for a particular n the formula holds. Then

$$a^n = b^n + (a-b)\sum_{k=1}^{n} a^{n-k}b^{k-1}.$$

Multiplying this equation by a we get

$$a^{n+1} = ab^n + (a-b)\sum_{k=1}^{n} a\cdot a^{n-k}b^{k-1} = ab^n + (a-b)\sum_{k=1}^{n} a^{n+1-k}b^{k-1}.$$

Let us substract b^{n+1} from both side of the equation:

$$a^{n+1} - b^{n+1} = ab^n - b^{n+1} + (a-b)\sum_{k=1}^{n} a^{n+1-k}b^{k-1}.$$

In the first two terms of the right-hand side b^n is a common factor, so we can write

$$a^{n+1} - b^{n+1} = b^n(a-b) + (a-b)\sum_{k=1}^{n} a^{n+1-k}b^{k-1},$$

and now $(a-b)$ is a common factor of both term in the right-hand side, so

$$a^{n+1} - b^{n+1} = (a-b)\left[b^n + \sum_{k=1}^{n} a^{n+1-k}b^{k-1}\right] = (a-b)\sum_{k=1}^{n+1} a^{n+1-k}b^{k-1},$$

and we are done, because we have just proven that the formula is also valid for the next natural number $n+1$.

(b) For $n = 1$ we get $n^5 - n = 1 - 1 = 0$, which is trivially a multiple of 5. Now, assuming $n^5 - n$ is a multiple of 5, we can expand

$$(n+1)^5 - (n+1) = n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + \cancel{1} - n - \cancel{1} = (n^5 - n) + 5(n^4 + 2n^3 + 2n^2 + n).$$

The first term of this sum is a multiple of 5 by assumption, and the second one is obviously a multiple of 5 because of the factor 5 in front of it. Therefore $(n+1)^5 - (n+1)$ will also be a multiple of 5.

(c) For $n = 1$ we have $1 + x \geqslant 1 + x$, which is obviously true. Let us assume that for a certain n it holds $(1+x)^n \geqslant 1 + nx$. Since $x \geqslant -1$, we know that $1 + x \geqslant 0$, so if we multiply the inequality by $(1+x)$ we obtain

$$(1+x)^{n+1} \geqslant (1+x)(1+nx) = 1 + x + nx + nx^2 = 1 + (n+1)x + nx^2.$$

Now, $nx^2 \geqslant 0$ for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$, so $1 + (n+1)x + nx^2 \geqslant 1 + (n+1)x$. Therefore

$$(1+x)^{n+1} \geqslant 1 + (n+1)x.$$

Problem 1.7 In all cases, the first case we must check is $n = 2$, since the inequalities are valid for $n > 1$.

(a) For $n=2$ the left-hand side is $2!=2$ and the right-hand side is $(3/2)^2=9/4=2.25$, so the inequality is true. Now let us prove that

$$n! < \left(\frac{n+1}{2}\right)^n \qquad \Rightarrow \qquad (n+1)! < \left(\frac{n+2}{2}\right)^{n+1}.$$

To that purpose, we multiply the inequality by $n+1$ and get

$$(n+1)! < \frac{(n+1)^{n+1}}{2^n} = 2\left(\frac{n+1}{2}\right)^{n+1}.$$

Then, using the hint,

$$(n+1)! < 2\left(\frac{n+1}{2}\right)^{n+1} < \left(1+\frac{1}{n+1}\right)^{n+1}\left(\frac{n+1}{2}\right)^{n+1}.$$

But

$$\left(1+\frac{1}{n+1}\right)^{n+1} = \left(\frac{n+2}{n+1}\right)^{n+1},$$

so

$$(n+1)! < 2\left(\frac{n+1}{2}\right)^{n+1} < \left(\frac{n+2}{n+1}\right)^{n+1}\left(\frac{n+1}{2}\right)^{n+1} = \left(\frac{n+2}{2}\right)^{n+1},$$

which proves the inequality we were looking for.

(b) Let us first prove that $(2n+2)! > (n+2)^n(n+2)!$, or equivalently that

$$\frac{(2n+2)!}{(n+2)!} > (n+2)^n.$$

To do that we use the definition of factorial and cancel all common factors in the fraction:

$$\begin{aligned}\frac{(2n+2)!}{(n+2)!} &= \frac{(2n+2)(2n+1)2n\cdots(n+3)\,\cancel{(n+2)(n+1)n\cdots 3\cdot 2\cdot 1}}{\cancel{(n+2)(n+1)n\cdots 3\cdot 2\cdot 1}}\\ &= (2n+2)(2n+1)\cdots(n+3).\end{aligned}$$

If we now replace all factors by $n+2$, which is smaller than any of them, we get the lower bound

$$\frac{(2n+2)!}{(n+2)!} > \underbrace{(n+2)(n+2)\cdots(n+2)}_{n\text{ times}} = (n+2)^n,$$

which is the equality we wanted to prove.

Let us now prove $2!\cdot 4!\cdots(2n)! > \big[(n+1)!\big]^n$ using induction. Take $n=2$ —the first value for which the inequality is supposed to work. The left-hand side is $2!\cdot 4! = 2\cdot 24 = 48$, while the right-hand side is $(3!)^2 = 6^2 = 36$, so the inequality holds for $n=2$.

Assume now that $2!\cdot 4!\cdots(2n)! > \big[(n+1)!\big]^n$ holds, multiply both sides by $(2n+2)!$ and use the inequality just proven:

$$\begin{aligned}2!\cdot 4!\cdots(2n)!(2n+2)! &> \big[(n+1)!\big]^n(2n+2)! > \big[(n+1)!\big]^n(n+2)^n(n+2)!\\ &= \big[(n+1)!(n+2)\big]^n(n+2)! = \big[(n+2)!\big]^n(n+2)! = \big[(n+2)!\big]^{n+1},\end{aligned}$$

and so the inequality also holds for $n+1$.

(c) Take $n=2$. The left-hand side is $1+1/\sqrt{2}\approx 1.7$, whereas the right-hand side is $\sqrt{2}\approx 1.4$. Hence the inequality holds for the first value of n. Assume now that

$$1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}>\sqrt{n}$$

and add $1/\sqrt{n+1}$ to both sides of the inequality. We end up with

$$1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n+1}}>\sqrt{n}+\frac{1}{\sqrt{n+1}}=\frac{\sqrt{n}\sqrt{n+1}+1}{\sqrt{n+1}}=\frac{\sqrt{n(n+1)}+1}{\sqrt{n+1}}.$$

Now, $n(n+1)>n^2$, therefore

$$1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n+1}}>\frac{\sqrt{n^2}+1}{\sqrt{n+1}}=\frac{n+1}{\sqrt{n+1}}=\sqrt{n+1}$$

and the inequality is proven.

Notice that in this case there is a simpler way to obtain the inequality *without* using induction. All we need to do to find a lower bound to the sum in the left-hand side is to replace every term by the smallest one, namely $1/\sqrt{n}$. If there are two or more terms in the sum (i.e. if $n>1$) then we obtain a strict lower bound with this operation. So

$$1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}>\underbrace{\frac{1}{\sqrt{n}}+\cdots+\frac{1}{\sqrt{n}}}_{n\text{ times}}=\frac{n}{\sqrt{n}}=\sqrt{n}.$$

Problem 1.8

(a) $1+\sqrt{2}$ and $1-\sqrt{2}$ are clearly irrational numbers (their decimal expressions have the same decimal part than $\sqrt{2}$), but if we add them up we obtain 2, which is a rational number.

(b) If we multiply $\sqrt{2}$ by itself we obtain 2.

(c) Consider the number $a=\sqrt{2}^{\sqrt{2}}$, and let us compute

$$a^{\sqrt{2}}=\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}=\left(\sqrt{2}\right)^{\sqrt{2}\cdot\sqrt{2}}=\left(\sqrt{2}\right)^{2}=2.$$

Therefore, either $a\in\mathbb{Q}$ —in which case $x=y=\sqrt{2}$ is the example we are seeking— or $a\notin\mathbb{Q}$ —in which case $x=a$ and $y=\sqrt{2}$ is that example.

Problem 1.9

(a) Suppose that $\sqrt{2}+\sqrt{3}=r\in\mathbb{Q}$. If we square both sides,

$$\left(\sqrt{2}+\sqrt{3}\right)^2=r^2 \quad\Leftrightarrow\quad 2+3+2\sqrt{2}\sqrt{3}=r^2 \quad\Leftrightarrow\quad \sqrt{6}=\frac{r^2-5}{2}.$$

The right-hand side in the last expression is a rational number, so $\sqrt{6}$ must be a rational number. Suppose there is an irreducible fraction p/q such that $\sqrt{6}=p/q$. Squaring, $6q^2=p^2$, so p must be even, i.e., $p=2k$. Substituting $6q^2=4k^2$, which simplifies to $3q^2=2k^2$. Thus q must be even too, but that is not possible because q and p do not have common factors. Hence $\sqrt{6}\notin\mathbb{Q}$ and therefore $\sqrt{2}+\sqrt{3}\notin\mathbb{Q}$.

(b) Suppose $\sqrt{n}=k\sqrt{r}=p/q$, an irreducible fraction. Squaring this equation $rk^2q^2=p^2$. But this is impossible because r does not contain any square factor whereas all factors in the right-hand side are squared. Hence $\sqrt{n}\notin\mathbb{Q}$.

(c) Suppose $\sqrt{n-1}+\sqrt{n+1}=r\in\mathbb{Q}$. Squaring

$$n-1+n+1+2\sqrt{n-1}\sqrt{n+1}=r^2 \qquad \Leftrightarrow \qquad \sqrt{n^2-1}=\frac{r^2-2n}{2}.$$

The right-hand side is a rational number, but the left-hand side cannot be because n^2-1 is not a perfect square (two consecutive numbers cannot be both perfect squares, and n^2 is). Thus $\sqrt{n-1}+\sqrt{n+1}\notin\mathbb{Q}$.

Problem 1.10 All we need to do is to expand the binomials in the left-hand side of the equation:

$$\left(\frac{x+|x|}{2}\right)^2+\left(\frac{x-|x|}{2}\right)^2=\frac{1}{4}\big[(x+|x|)^2+(x-|x|)^2\big]=\frac{1}{4}[x^2+|x|^2+\cancel{2x|x|}+x^2+|x|^2-\cancel{2x|x|}]$$
$$=\frac{1}{\cancel{4}}\cancel{4}x^2=x^2.$$

Problem 1.11

(i) Since $|z|\leqslant a$ is the same as $-a\leqslant z\leqslant a$,

$$|x-3|\leqslant 8 \quad\Leftrightarrow\quad -8\leqslant x-3\leqslant 8 \quad\Leftrightarrow\quad 3-8\leqslant x\leqslant 3+8 \quad\Leftrightarrow\quad -5\leqslant x\leqslant 11.$$

Hence $A=[-5,11]$.

(ii) On the one hand

$$|x-2|<\frac{1}{2} \quad\Leftrightarrow\quad -\frac{1}{2}<x-2<\frac{1}{2} \quad\Leftrightarrow\quad \frac{3}{2}<x<\frac{5}{2}.$$

On the other hand $0<|x-2|$ holds if, and only if, $x\neq 2$. Therefore $B=\left(\frac{3}{2},2\right)\cup\left(2,\frac{5}{2}\right)$.

(iii) We can factor out $x^2-5x+6=(x-2)(x-3)$, therefore $C=\{x\in\mathbb{R}:\,(x-2)(x-3)\geqslant 0\}$. So C contains those x for which both factors are either nonnegative or nonpositive, i.e., $x\geqslant 3$ and $x\leqslant 2$. Hence $C=(-\infty,2]\cup[3,\infty)$.

(iv) D contains those x for which an odd numbers of the three factors in the inequality are negative, i.e. $x<-3$ and $0<x<5$. Thus $D=(-\infty,-3)\cup(0,5)$.

(v) Factoring out $x^2+8x+7=(x+1)(x+7)$ we can rewrite

$$E=\left\{x\in\mathbb{R}:\,\frac{2(x+4)}{(x+1)(x+7)}>0\right\}.$$

Thus either all factors in the fraction must be positive or one positive and the other two negatives. This holds for $x>-1$ and $-7<x<-4$, hence $E=(-7,-4)\cup(-1,\infty)$.

(vi) Since

$$\frac{4}{x}<x \quad\Leftrightarrow\quad 0<x-\frac{4}{x} \quad\Leftrightarrow\quad 0<\frac{x^2-4}{x} \quad\Leftrightarrow\quad 0<\frac{(x-2)(x+2)}{x},$$

F will contain those x for which either all three factors in the fraction are positive or one is positive and two negatives, i.e., $2<x$ and $-2<x<0$. Hence $F=(-2,0)\cup(2,\infty)$.

(vii) The inequality $4x<2x+1$ is equivalent to $2x<1$, i.e., $x<\frac{1}{2}$. The inequality $2x+1\leqslant 3x+2$ is equivalent to $0\leqslant x+1$, i.e., $-1\leqslant x$. Hence $G=[-1,\frac{1}{2})$.

(viii) $|x^2-2x|<1$ means $-1<x^2-2x<1$. The inequality $-1<x^2-2x$ means $0<x^2-2x+1=(x-1)^2$, which only holds for $x\neq 1$. On the other hand, the inequality $x^2-2x<1$ means $x^2-2x-1<0$, which holds for all x within the two roots of $x^2-2x-1=0$. These two roots are $1+\sqrt{2}>0$ and $1-\sqrt{2}<0$. Therefore $H=(1-\sqrt{2},1)\cup(1,1+\sqrt{2})$.

(ix) The equation

$$|x-1||x+2| = 10 \quad \Leftrightarrow \quad |(x-1)(x+2)| = 10 \quad \Leftrightarrow \quad |x^2+x-2| = 10$$

is actually two in one, namely

$$x^2+x-2 = 10, \qquad x^2+x-2 = -10.$$

The solutions of the first one are the solutions of $x^2+x-12=0$, i.e., $x=-4$ and $x=3$. On the other hand, the second equation becomes $x^2+x+8=0$, which has no real solutions. Thus $I = \{-4\} \cup \{3\}$.

(x) The inequality $|x-1|+|x+2| > 1$ has to be discussed in three regions: (a) $x \geqslant 1$, (b) $-2 \leqslant x < 1$, and (c) $x \leqslant -2$.
(a) $x \geqslant 1$. The inequality becomes $x-1+x+2 > 1$ because the numbers within the absolute values are both nonnegatives. This is equivalent to $2x+1 > 1$, i.e., $x > 0$. Since we are assuming that $x \geqslant 1$, all numbers in this region satisfy the inequality.
(b) $-2 \leqslant x < 1$. The inequality becomes $1-x+x+2 > 1$ since $x-1 < 0$ but $x+2 \geqslant 0$. This inequality turns out to be $3 > 1$, which is obviously true, so all numbers in this region satisfy the inequality.
(c) $x < -2$. The inequality becomes $1-x-x-2 > 1$ since both $x-1 < 0$ and $x+2 < 0$. This inequality becomes $-2x-1 > 1$, i.e., $2+2x < 0$ or $x < -1$. But we are in a region where $x < -2$, so all numbers in this region satisfy $x < -1$.
Consequently $J = \mathbb{R}$.

Problem 1.12

(i) $x(0) = a$, $x(1) = b$, $x(1/2) = (a+b)/2$.

(ii) $B = (a,b)$.

(iii) $C = (-\infty, a)$.

(iv) $D = (b, \infty)$.

Problem 1.13

(i) $\sup A = 3 \neq \max A$; $\inf A = -1 = \min A$.

(ii) $\sup B = 3 = \max B$; $\inf B = -1 = \min B$.

(iii) $\sup C = 3 = \max C$; $\inf C = 2 \neq \min C$.

(iv) Writing $(n^2+1)/n$ as $n+1/n$ is clear that $\sup D = \infty$; $\inf D = 2 = \min D$.

(v) The two roots of the parabola are $x=3$ and $x=1/3$. Since the coefficient of x^2 is positive, the parabola is negative between the two roots. Hence $E = (1/3, 3)$ and $\sup E = 3 \neq \max E$; $\inf E = 1/3 \neq \min E$.

(vi) F contains those numbers for which an odd number of factors are negative. Thus $F = (a,b) \cup (c,d)$ and $\sup F = d \neq \max F$; $\inf F = a \neq \min F$.

(vii) $\sup G = \frac{1}{2} + \frac{1}{5} = \frac{7}{10} = \max G$; $\inf G = 0 \neq \min G$.

(viii) We can express $H = H_+ \cup H_-$, where $H_+ = \{1+1/m : m \in \mathbb{N}\}$ and $H_- = \{-1+1/m : m \in \mathbb{N}\}$. Since all numbers in H_- are smaller than all numbers in H_+, $\sup H = \sup H_+ = 2 = \max H$, whereas $\inf H = \inf H_- = -1 \neq \min H$.

D.2 Real Functions

Problem 2.1

(i) We can factor out the denominator as $x^2-5x+6=(x-2)(x-3)$; therefore, the domain is $\mathbb{R}-\{2,3\}$.

(ii) There are two conditions for $f(x)$ to exist: $1-x^2\geqslant 0$ and $x^2-1\geqslant 0$. Together they imply $1-x^2=0$. Therefore the domain is just the set $\{-1,1\}$.

(iii) There are two conditions to be met for x to be in the domain: first, $1-x^2\geqslant 0$; second, $x\neq\sqrt{1-x^2}$. The first condition implies $x^2\leqslant 1$, or equivalently, $-1\leqslant x\leqslant 1$. The second condition is not fulfilled if $x=\sqrt{1-x^2}$. Squaring this equation we obtain $x^2=1-x^2$, which is equivalent to $x^2=1/2$. The two solutions of this equation are $x=\pm 1/\sqrt{2}$, but of them two, only the positive one is a solution of the original equation $x=\sqrt{1-x^2}$. Thus the domain is $[-1,1/\sqrt{2})\cup(1/\sqrt{2},1]$.

(iv) The two coditions to be met for x to be in the domain are $4-x^2\geqslant 0$ and $1-\sqrt{4-x^2}\geqslant 0$. The first one reads $x^2\leqslant 4$, i.e., $-2\leqslant x\leqslant 2$. The second one implies $\sqrt{4-x^2}\leqslant 1$. Both sides of this inequality are positive, so we can square it to obtain $4-x^2\leqslant 1$, i.e., $x^2\geqslant 3$. This holds either if $x\geqslant\sqrt{3}$ or $x\leqslant-\sqrt{3}$. Therefore, the domain is $[-2,-\sqrt{3}]\cup[\sqrt{3},2]$.

(v) The denominator vanishes if $\log x=1$, i.e., if $x=e$. Since the logarithm requires $x>0$, the domain is $(0,e)\cup(e,\infty)$.

(vi) The condition to be met now is $x-x^2>0$. We can factor $x-x^2=x(1-x)$, so the roots of the parabola are $x=0$ and $x=1$. Since the coefficient of x^2 is negative, the parabola is positive provided $0<x<1$. The domain is then $(0,1)$.

(vii) Three conditions need to be met: first, $x>0$ because x is the argument of a logarithm; second, $\log x\neq 0$ because it is the denominator; and third, $5-x\geqslant 0$ because it is the argument of a square root. The second condition implies $x\neq 1$, whereas the third one implies $x\leqslant 5$. Thus the domain is $(0,1)\cup(1,5]$.

(viii) The first requirement is $x>0$ for the logarithm to be meaningful. The second one is $-1\leqslant\log x\leqslant 1$ because the domain of the arcsin is the interval $[-1,1]$. Since $\log x$ is monotonically increasing, this inequality is equivalent to $e^{-1}\leqslant x\leqslant e$, so the domain is the interval $[e^{-1},e]$.

Problem 2.2

(a) We know that $f(-x)=-f(x)$ and $g(-x)=-g(x)$. Then

$$(f+g)(-x)=f(-x)+g(-x)=-f(x)-g(-x)=-(f+g)(x),$$

so $f+g$ is *odd*. Now,

$$(fg)(-x)=f(-x)g(-x)=[-f(x)][-g(x)]=f(x)g(x)=(fg)(x),$$

so fg is *even*. Finally,

$$(f\circ g)(-x)=f\big(g(-x)\big)=f\big(-g(x)\big)=-f\big(g(x)\big)=-(f\circ g)(x).$$

Thus $f\circ g$ is *odd*.

(b) Now $f(-x)=f(x)$ and $g(-x)=-g(x)$. Then

$$(f+g)(-x)=f(-x)+g(-x)=f(x)-g(-x),$$

so $f+g$ is neither even nor odd. As for the product,

$$(fg)(-x)=f(-x)g(-x)=f(x)[-g(x)]=-f(x)g(x)=-(fg)(x),$$

so fg is *odd*. Finally,

$$(f \circ g)(-x) = f\big(g(-x)\big) = f\big(-g(x)\big) = f\big(g(x)\big) = (f \circ g)(x).$$

Thus $f \circ g$ is *even*.

Problem 2.3 (i)

$$f(-x) = \frac{-x}{(-x)^2+1} = -f(x).$$

The function is *odd*.

(ii)

$$f(-x) = \frac{(-x)^2-(-x)}{(-x)^2+1} = \frac{x^2+x}{x^2+1} \neq \pm f(x),$$

so the function is neither.

(iii)

$$f(-x) = \frac{\sin(-x)}{-x} = \frac{-\sin x}{-x} = \frac{\sin x}{x} = f(x).$$

The function is *even*.

(iv)

$$f(-x) = \cos\big((-x)^3\big)\sin\big((-x)^2\big)e^{-(-x)^4} = \cos(-x^3)\sin(x^2)e^{-x^4} = \cos(x^3)\sin(x^2)e^{-x^4} = f(x).$$

The function is *even*.

(v)

$$f(-x) = \frac{1}{\sqrt{(-x)^2+1}-(-x)} = \frac{1}{\sqrt{x^2+1}+x},$$

so the function is neither.

(vi) This function is the logarithm of the function in the previous item, so it seems that it has no defined parity because

$$f(-x) = \log\left(\sqrt{x^2+1}+x\right).$$

However,

$$\sqrt{x^2+1}+x = \frac{\left(\sqrt{x^2+1}+x\right)\left(\sqrt{x^2+1}-x\right)}{\sqrt{x^2+1}-x} = \frac{x^2+1-x^2}{\sqrt{x^2+1}-x} = \frac{1}{\sqrt{x^2+1}-x},$$

so

$$f(-x) = \log\left(\sqrt{x^2+1}+x\right) = \log\left(\frac{1}{\sqrt{x^2+1}-x}\right) = -\log\left(\sqrt{x^2+1}-x\right) = -f(x).$$

The function is *odd*.

Problem 2.4 The equation defining the inverse function is $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$. If the function is to be its own inverse it must satisfy the equation $(f \circ f)(x) = x$. In other words,

$$(f \circ f)(x) = \frac{af(x)+b}{cf(x)+d} = \frac{a\left(\frac{ax+b}{cx+d}\right)+b}{c\left(\frac{ax+b}{cx+d}\right)+d} = \frac{a(ax+b)+b(cx+d)}{c(ax+b)+d(cx+d)} = \frac{(a^2+bc)x+(a+d)b}{(a+d)cx+(bc+d^2)} = x.$$

For the last two expressions to be the same we must have

$$(a+d)c = (a+d)b = 0,$$
$$a^2 = d^2.$$

There are two ways in which the top equations can be fulfilled. The first one is $c = b = 0$. Since the second equation implies $a = \pm d$, the two resulting functions are $f_1(x) = x$ and $f_2(x) = -x$. The second possibility is that $a+d = 0$, or $d = -a$. Then all three equations hold. This corresponds to the function

$$f(x) = \frac{ax+b}{cx-a},$$

whose only constraint is that c and a cannot be both zero.

Problem 2.5 The statement of the problem is that $f(x)$, understood as a mapping $f : \mathbb{R} - \{-1/2\} \mapsto \mathbb{R} - \{1/2\}$, is bijective. A simple way to see that the domain of f is $\mathbb{R} - \{-1/2\}$, that it can be inverted in its domain, and that the domain of f^{-1} is $\mathbb{R} - \{1/2\}$.

That the domain is $\mathbb{R} - \{-1/2\}$ is obvious because $x = -1/2$ is the only zero of the denominator. That f can be inverted is a matter of solving x as a function of y in the equation

$$y = \frac{x+3}{1+2x} \quad \Rightarrow \quad y(1+2x) = x+3 \quad \Rightarrow \quad y-3 = x(1-2y) \quad \Rightarrow \quad x = \frac{y-3}{1-2y}.$$

The inverse function is then

$$f^{-1}(x) = \frac{x-3}{1-2x}$$

and its domain is clearly $\mathbb{R} - \{1/2\}$.

Problem 2.6

(a) An easy way to check for injectivity is to determine whether the equation $y = f(x)$ has a unique solution for those y for which it can be solved.

(i) For every $y \in \mathbb{R}$,

$$y = 7x - 4 \quad \Rightarrow \quad x = \frac{y+4}{7}.$$

So there is a unique solution no matter y, which means that the function is injective.

(ii) Only if $-1 \leqslant y \leqslant 1$ the equation

$$y = \sin(7x-4)$$

can have a solution. On the other hand, two points x_1 and x_2 such that $7x_2 - 4 = 7x_1 - 4 + 2n\pi$, with $n \in \mathbb{Z}$, are both solutions of the same y. Clearly $x_2 = x_1 + 2n\pi/7$. Therefore there are infinitely many solutions for each $-1 \leqslant y \leqslant 1$, which means that the function is not injective.

(iii) For any $y \in \mathbb{R}$,

$$y = (x+1)^3 + 2 \quad \Rightarrow \quad x = (y-2)^{1/3} - 1,$$

so the solution is unique and the function is injective.

(iv) Take y so that

$$y = \frac{x+2}{x+1}.$$

Then

$$y(x+1) = x+2 \quad \Rightarrow \quad y-2 = x(1-y).$$

Thus, provided $y \neq 1$, we obtain

$$x = \frac{y-2}{1-y}$$

and the solution is unique. The function is injective.

(v) Take y and solve for $y = x^2 - 3x + 2$, or $x^2 - 3x + 2 - y = 0$. Then

$$x = \frac{3 \pm \sqrt{9+4(y-2)}}{2} = \frac{3 \pm \sqrt{4y+1}}{2}.$$

The equation has a solution only if $y \geqslant -1/4$. But for all $y > -1/4$ there are two different solutions. Therefore the function is not injective.

(vi) Consider the equation

$$y = \frac{x}{x^2+1}.$$

If $y = 0$ the only solution is $x = 0$. If $y \neq 0$ it can be transformed into

$$y(x^2+1) = x \quad \Rightarrow \quad yx^2 - x + y = 0.$$

The solutions of this quadratic equation are

$$x = \frac{1 \pm \sqrt{1-4y^2}}{2y}.$$

There is solution only if $y^2 \leqslant 1/4$, i.e., $-1/2 \leqslant y \leqslant 1/2$, but for every $-1/2 < y < 1/2$ there are two different solutions for the same y, hence the function is not injective.

(vii) For every $y > 0$,

$$y = e^{-x} \quad \Rightarrow \quad \log y = -x \quad \Rightarrow \quad x = -\log y.$$

The solution is unique and the function is injective.

(viii) For every $y \in \mathbb{R}$,

$$y = \log(x+1) \quad \Rightarrow \quad e^y = x+1 \quad \Rightarrow \quad x = e^y - 1.$$

The solution is unique and the function is injective.

(b) The solutions of the equation $y = x^2 - 3x + 2$ are (see previous item)

$$x = \frac{3 \pm \sqrt{4y+1}}{2}.$$

Clearly one solution is larger than $3/2$ and the other is smaller than $3/2$. Therefore, if we limit the domain to those x larger than $3/2$ only one solution survives and the function becomes injective.

(c) Take two values $1 < x_1 < x_2$ and calculate the difference

$$f(x_1) - f(x_2) = \frac{x_1}{x_1^2+1} - \frac{x_2}{x_2^2+1} = \frac{x_1(x_2^2+1) - x_2(x_1^2+1)}{(x_1^2+1)(x_2^2+1)} = \frac{(x_2-x_1)(x_1x_2-1)}{(x_1^2+1)(x_2^2+1)} > 0.$$

So $f(x)$ is monotonically decreasing for $x > 1$ and therefore injective.
Now, since the solution of $y = f(x)$ is (see previous item)

$$x = \frac{1 \pm \sqrt{1-4y^2}}{2y},$$

for $y = \sqrt{2}/3$,

$$x = \frac{1 \pm \sqrt{1-8/9}}{2\sqrt{2}/3} = \frac{3 \pm 1}{2\sqrt{2}} = \begin{cases} \sqrt{2}, \\ \dfrac{1}{\sqrt{2}}. \end{cases}$$

Since the domain of the function is $(1, \infty)$, only the top solution is in the domain; thus $f^{-1}(\sqrt{2}/3) = \sqrt{2}$.

(d)

(i) There is a unique solution for every $y \in \mathbb{R}$, therefore the function is surjective, hence bijective.

(ii) Not surjective because the range is $[-1, 1]$.

(iii) Surjective and bijective.

(iv) Not surjective because $y = 1$ is not in the range of the function.

(v) Not surjective because the range is $[-1/4, \infty)$.

(vi) Not surjective because the range is $[-1/2, 1/2]$.

(vii) Not surjective because the range is $(0, \infty)$.

(viii) Surjective and bijective.

Problem 2.7

(i) Let us denote $\theta_1 = \arctan \frac{1}{2}$, $\theta_2 = \arctan \frac{1}{3}$, and $\theta = \theta_1 + \theta_2$. Firts of all, $0 < \theta_{1,2} < \pi/4$, so $0 < \theta < \pi/2$. Now,

$$\tan\theta = \tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} = \frac{5/6}{5/6} = 1.$$

Therefore $\theta = \pi/4$.

(ii) Now $\theta_1 = \arctan 2$ and $\theta_2 = \arctan 3$, and $\pi/4 < \tan\theta_{1,2} < \pi/2$, so $\pi/2 < \tan\theta < \pi$. Then

$$\tan\theta = \tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2} = \frac{2+3}{1-6} = \frac{5}{-5} = -1.$$

Therefore $\theta = 3\pi/4$.

(iii) Denote $\theta_1 = \arctan\frac{1}{2}$, $\theta_2 = \arctan\frac{1}{5}$, $\theta_3 = \arctan\frac{1}{8}$, and $\theta = \theta_1 + \theta_2 + \theta_3$. Since $0 < \tan\theta_i < \pi/4$ then $0 < \tan\theta < 3\pi/4$. Accordingly $\tan\theta > 0$ if $0 < \theta < \pi/2$ and $\tan\theta < 0$ if $\pi/2 < \theta < 3\pi/4$.
First of all we need to work out a formula for $\tan(\theta_1 + \theta_2 + \theta_3)$. For the sake of simplicity we will denote $\tau_i = \tan\theta_i$ and $\tau_{ij} = \tan(\theta_i + \theta_j)$. Thus,

$$\tan(\theta_1 + \theta_2 + \theta_3) = \frac{\tau_{12} + \tau_3}{1 - \tau_{12}\tau_3} = \frac{\frac{\tau_1+\tau_2}{1-\tau_1\tau_2} + \tau_3}{1 - \frac{\tau_1+\tau_2}{1-\tau_1\tau_2}\tau_3} = \frac{\tau_1 + \tau_2 + \tau_3 - \tau_1\tau_2\tau_3}{1 - \tau_1\tau_2 - \tau_2\tau_3 - \tau_3\tau_1}.$$

The formula is

$$\tan\theta = \frac{\tan\theta_1 + \tan\theta_2 + \tan\theta_3 - \tan\theta_1\tan\theta_2\tan\theta_3}{1 - \tan\theta_1\tan\theta_2 - \tan\theta_2\tan\theta_3 - \tan\theta_3\tan\theta_1}.$$

substituting,

$$\tan\theta = \frac{\frac{1}{2} + \frac{1}{5} + \frac{1}{8} - \frac{1}{80}}{1 - \frac{1}{10} - \frac{1}{40} - \frac{1}{16}} = \frac{40 + 16 + 10 - 1}{80 - 8 - 2 - 5} = \frac{65}{65} = 1.$$

Thus $\theta = \pi/4$.

Problem 2.8

(i) Denote $\theta = \arccos x$, so that $\cos\theta = x$. Then

$$f(x) = \sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - x^2}.$$

(ii) Denote $\theta = \arcsin x$, so that $\sin\theta = x$. Then,

$$f(x) = \sin(2\theta) = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1 - \sin^2\theta} = 2x\sqrt{1 - x^2}.$$

(iii) Denote $\theta = \arccos x$, so that $\cos\theta = x$. Then

$$f(x) = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1 - \cos^2\theta}}{\cos\theta} = \frac{\sqrt{1 - x^2}}{x}.$$

(iv) Denote $\theta = \arctan x$, so that $\tan\theta = x$. Now,

$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} - 1,$$

so

$$\cos\theta = \frac{1}{\sqrt{1 + \tan^2\theta}}.$$

And since $\sin\theta = \cos\theta\tan\theta$,

$$\sin\theta = \frac{\tan\theta}{\sqrt{1 + \tan^2\theta}}.$$

Then

$$f(x) = \sin(2\theta) = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1 + \tan^2\theta} = \frac{2x}{1 + x^2}.$$

(v) Denote $\theta = \arctan x$, so that $\tan\theta = x$. Then

$$f(x) = \cos(2\theta) = \cos^2\theta - \sin^2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta} = \frac{1 - x^2}{1 + x^2}.$$

(vi) Since $4\log x = \log(x^4)$, then $f(x) = e^{\log(x^4)} = x^4$.

Problem 2.9 If we take logarithms in the first of these equations we obtain

$$y\log x = x\log y.$$

The second equation is $y = 3x$, so substituting for y in the previous equation we end up with

$$3x\log x = x\log(3x) = x\log 3 + x\log x.$$

Since $x > 0$ (i.e., $x \neq 0$) we can cancel one factor x from the equation,

$$3\log x = \log 3 + \log x \quad \Rightarrow \quad 2\log x = \log 3 \quad \Rightarrow \quad \log x = \frac{1}{2}\log 3 \quad \Rightarrow \quad \log x = \log\sqrt{3},$$

therefore $x = \sqrt{3}$.

Problem 2.10 Use GeoGebra to help you with this exercise.

Problem 2.11 Here are some hints to help you plot these functions:

(i) Start off with the plot of $g(x) = x^2$; function $f(x) = g(x+2) - 1$, so shift the plot two units to the left and one unit down.

(ii) Start off with the plot of $g(x) = \sqrt{x}$ and then tranform it into that of $h(x) = \sqrt{-x}$ by reflecting it on the Y axis. Then $f(x) = h(x-4)$, so shift this plot four units to the right.

(iii) Start off from the plots of $g_1(x) = x^2$ and $g_2(x) = 1/x$. Near $x = 0$ g_1 is negligible with respect to g_2 —which diverges to $\pm\infty$ at $x = 0$. Far from $x = 0$ it is g_2 that is negligible with respect to g_1, which grows indefinitely. So $f(x)$ is close to $g_2(x)$ as x 'moves' toward 0, and close to $g_1(x)$ as x goes far awat from $x = 0$. Sketch the plot of $f(x)$ using this information.

(iv) Start off with the plot of $g(x) = x^2$ and shift it up one unit to get that of $h(x) = x^2 + 1$. Then $f(x) = 1/h(x)$. Since $h(x) > 1$ for all $x \neq 0$ and $h(0) = 1$, then $f(x) < 1$ for all $x \neq 0$ and $f(0) = 1$. Besides, $h(x)$ grows indefinitely as x goes away from the origin, so $f(x)$ has to approach 0.

(v) $g(x) = x - x^2 = x(1-x)$, so $g(x) > 0$ if $0 < x < 1$ and $g(x) < 0$ if $x < 0$ or $x > 1$. Therefore

$$f(x) = \begin{cases} x^2, & \text{if } 0 \leqslant x \leqslant 1, \\ x, & \text{otherwise.} \end{cases}$$

(vi) e^x is monotonically increasing and croses 1 at $x = 0$. Therefore

$$f(x) = \begin{cases} e^x - 1, & \text{if } x \geqslant 0, \\ 1 - e^x, & \text{if } x < 0. \end{cases}$$

All that needs to be done is to reflect the graph of $e^x - 1$ (equal to that of e^x but shifted down one unit) for $x < 0$ on the X axis.

(vii) If $x \geqslant 0$ then $|x| - x = 0$, but if $x < 0$ then $|x| - x = -2x$. So

$$f(x) = \begin{cases} 0, & \text{if } x \geqslant 0, \\ \sqrt{-2x}, & \text{if } x < 0. \end{cases}$$

(viii) $\lfloor x \rfloor$ is the smallest integer not larger than x. So for instance, $\lfloor 3.14 \rfloor = 3$, $\lfloor 0.5 \rfloor = 0$, but $\lfloor -1.58 \rfloor = -2$. So the function $f(x)$ is going to be piecewise constant, equal to $1/n$ for some integer n, in given intervals. All we need to do is to determine those intervals. Of course, $f(x)$ is only defined if $x \neq 0$ and if $\lfloor \frac{1}{x} \rfloor \neq 0$.

Let n be an integer and let us try to figure out where

$$\left\lfloor \frac{1}{x} \right\rfloor = n.$$

By definition

$$n \leqslant \frac{1}{x} < n+1. \tag{D.1}$$

As we have mentioned above, $f(x)$ will not be defined if $n = 0$. This means all x such that

$$0 \leqslant \frac{1}{x} < 1.$$

The left inequality implies $x > 0$. The right inequality implies $x > 1$. Therefore the domain of f is $(-\infty, 0) \cup (0, 1]$.
Consider first $x \in (0, 1]$. Then, according to (D.1) $n > 0$. From the left inequality $x \leqslant 1/n$, and from the right one $x > 1/(n+1)$. Thus

$$f(x) = \frac{1}{n} \quad \text{for all } x \in \left(\frac{1}{n+1}, \frac{1}{n} \right], \qquad n \in \mathbb{N}.$$

In other words, $f(x) = 1$ for $x \in (1/2, 1]$, $f(x) = 1/2$ for $x \in (1/3, 1/2]$, $f(x) = 1/3$ for $x \in (1/4, 1/3]$, etc. This covers the plot of $f(x)$ within the interval $(0, 1]$. By the way, the function gets closer and closer to 0 as x approaches 0.
Consider now the interval $(-\infty, 0)$. Then n in (D.1) must be negative. Then the left inequality again implies $x \leqslant 1/n$ and the right one $x > 1/(n+1)$. The result is the same:

$$f(x) = \frac{1}{n} \quad \text{for all } x \in \left(\frac{1}{n+1}, \frac{1}{n} \right], \qquad n \in -\mathbb{N}.$$

So we have $f(x) = -1$ if $x \in (-\infty, -1]$, $f(x) = -1/2$ if $x \in (-1, -1/2]$, $f(x) = -1/3$ if $x \in (-1/2, -1/3]$, etc. This covers the whole interval $(-\infty, 0)$.

(ix) Function $g(x) = x^2 - 1 < 0$ if $-1 < x < 1$ and $g(x) > 0$ otherwise, so

$$f(x) = \begin{cases} 1 - x^2, & \text{if } -1 < x < 1, \\ x^2 - 1, & \text{otherwise.} \end{cases}$$

All that one has to do is to reflect the portion of the graph of $x^2 - 1$ in the interval $(-1, 1)$ on the X axis.

(x) Plot $g(x) = e^x$. The plot of $g(-x)$ is just the mirror image with respect to the Y axis. And that of $-g(-x)$ is a new reflection with respect to the X axis. Shift the whole plot one unit upward and you will get the plot of $f(x) = -g(-x) + 1 = 1 - e^{-x}$.

(xi) The function is defined only if $|x| \geqslant 1$. Besides, it is an even function, so it will be symmetric with respect to the Y axis. Let us then focus on the positive interval $[1, \infty)$. Notice that $f(x) = \log(x-1) + \log(x+1)$. These are two graphs of $\log x$, the first one shifted one unit to the right and the second one shifted one unit to the left. Since $\log x$ grows very slowly but diverges at $x = 0$, near the point $x = 1$ function $\log(x-1)$ will diverge and $\log(x+1)$ will then be negligible. In oher words, $f(x) \approx \log(x-1)$. On the other hand, when x is large $x \pm 1 \approx x$, so $f(x) \approx 2\log x$. Plot $f(x)$ using this information.

(xii) As x grows far away from the origin (positive or negative) $1/x$ becomes very small, so $\sin(1/x)$ approaches $1/x$, and therefore $f(x)$ approaches 1. On the other hand, $\sin(1/x)$ oscillates wildly as x gets near the origin, but x modulates the amplitude (making it smaller the closer to the origin).

Problem 2.12

(a) Since $\cosh x = (e^x + e^{-x})/2$, $\sinh x = (e^x - e^{-x})/2$,

$$\cosh(-x) = \frac{e^{-x} + e^x}{2} = \cosh x, \qquad \sinh(-x) = \frac{e^{-x} - e^x}{2} = -\sinh x.$$

(b) First identity:

$$\cosh^2 x - \sinh^2 x = \frac{1}{4}\left(e^{2x} + e^{-2x} + 2\right) - \frac{1}{4}\left(e^{2x} + e^{-2x} - 2\right) = 1.$$

Second identity:

$$2\sinh x \cosh x = \frac{1}{2}\left(e^x - e^{-x}\right)\left(e^x + e^{-x}\right) = \frac{1}{2}\left(e^{2x} - 1 + 1 - e^{-2x}\right) = \sinh(2x).$$

D.3 Sequences

Problem 3.1

(a) If $\lim_{n\to\infty} x_n = 0$ then it is a $0\cdot\infty$ indeterminacy and the result will depend on the sequences involved. For instance, if $x_n = 1/n$ and $y_n = n$, then $x_n y_n = 1$ is a convergent series. Or if $y_n = n^2$ instead, then $x_n y_n = n$ diverges. Or if $x_n = (-1)^n/n$ and $y_n = n$ then $x_n y_n = (-1)^n$ which is neither convergent nor divergent.
However, if $\lim_{n\to\infty} x_n = a \neq 0$, then we are certain that $x_n y_n$ will diverge. To prove that suppose that $a > 0$ (if $a < 0$ the argument is analogous). Take some $0 < \varepsilon < a$ and chose $C > 0$ arbitrarily large. For sufficently large n we will have at the same time $a - \varepsilon < x_n < a + \varepsilon$ and $C < y_n$. Thus $0 < (a-\varepsilon)C < x_n y_n$, and it is clear that $(a-\varepsilon)C$ can be made arbitrarily large by suitably choosing C.

(b) There will be an index $N \in \mathbb{N}$ such that the sequence is constant for all $n > N$. To prove it just choose a very small $\varepsilon > 0$. If the limit of the sequence is ℓ, then $\ell - \varepsilon < x_n < \ell + \varepsilon$ for all $n > N$. But $x_n \in \mathbb{Z}$, so the only way that thisn inequality can hold for very small ε is that $\ell \in \mathbb{Z}$ and that $x_n = \ell$ for all $n > N$.

(c) By definition, if we take some $\varepsilon > 0$ there will be an $N \in \mathbb{N}$ such that $\ell - \varepsilon < x_n < \ell + \varepsilon$ for all $n > N$ (ℓ is the limit). Then the sequence is bounded for $n > N$. There remain $\{x_1, x_2, \cdots, x_N\}$ outside that interval. But there is a finite number of these numbers, so one of them is the largest (say x_j) and another one is the smallest (say x_k). Define $a = \min\{x_k, \ell - \varepsilon\}$ and $b = \max\{x_j, \ell + \varepsilon\}$. Then for all $n \in \mathbb{N}$ we have $a \leqslant x_n \leqslant b$, which proves that the sequence is bounded.

Problem 3.2

(i) Let us compute a few terms of the sequence:

$$\left\{0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \dots\right\}.$$

The pattern is clearly

$$a_n = \frac{2^n - 1}{2^n} = 1 - 2^{-n}.$$

Clearly this holds for $a_0 = 1 - 1 = 0$, and if we introduce this formula in the recurrence we obtain

$$a_{n+1} = \frac{a_n + 1}{2} = \frac{1 - 2^{-n} + 1}{2} = \frac{2 - 2^{-n}}{2} = 1 - 2^{-(n+1)}.$$

This proves the guessed formula by induction.
The limit of the sequence a_n is clearly 1.

(ii) Define $a_n = \log_c b_n$, where c is the base of the logarithm —to be specified. If we take $\log_c$ in both sides of the equation we obtain

$$\log_c b_{n+1} = \log_c\left(\sqrt{2b_n}\right) = \frac{1}{2}(\log_c 2 + \log_c b_n) \quad \Leftrightarrow \quad a_{n+1} = \frac{\log_c 2 + a_n}{2},$$

and obviously $a_0 = \log_c b_0 = \log_c 1 = 0$. Now, if we take $c = 2$ (binary logarithms) then $\log_2 2 = 1$ and recurrence we obtain is

$$a_{n+1} = \frac{a_n + 1}{2}, \qquad a_0 = 0,$$

exactly the same as in item (a). The solution will therefore be the same, and

$$b_n = 2^{a_n} = 2^{1-2^{-n}}.$$

The limit of b_n is clearly 2.

Problem 3.3

(i) Applying Corollary 3.4.4 to Stolz's theorem, instead of calculating the limit of $\sqrt[n]{a_n}$ we will calculate the limit of a_n/a_{n-1}. Thus we want to obtain the limit

$$\lim_{n\to\infty} \frac{a^n + b^n}{a^{n-1} + b^{n-1}} = \ell.$$

Suppose $a = b$. Then

$$\ell = \lim_{n\to\infty} \frac{2a^n}{2a^{n-1}} = a.$$

Suppose that $a > b$. Then

$$\ell = \lim_{n\to\infty} \frac{a^n\left(1+\frac{b^n}{a^n}\right)}{a^{n-1}\left(1+\frac{b^{n-1}}{a^{n-1}}\right)} = a \lim_{n\to\infty} \frac{1+\left(\frac{b}{a}\right)^n}{1+\left(\frac{b}{a}\right)^{n-1}} = a\frac{1}{1} = a.$$

By symmetry the limit will be b if $a < b$.
Summarising all cases in a single expression, $\ell = \max\{a, b\}$.

(ii) Since

$$\lim_{n\to\infty} \frac{\sqrt[n]{a} + \sqrt[n]{b}}{2} = 1$$

the limit we want to calculate is an indeterminacy 1^∞. Hence

$$\lim_{n\to\infty} \left(\frac{\sqrt[n]{a} + \sqrt[n]{b}}{2}\right)^n = e^c,$$

where

$$c = \lim_{n\to\infty} n\left(\frac{\sqrt[n]{a} + \sqrt[n]{b}}{2} - 1\right) = \lim_{n\to\infty} n\frac{\sqrt[n]{a} + \sqrt[n]{b} - 2}{2} = \frac{1}{2}\lim_{n\to\infty} n\left(\sqrt[n]{a} - 1 + \sqrt[n]{b} - 1\right).$$

So we need to calculate

$$\ell(a) = \lim_{n\to\infty} n\left(\sqrt[n]{a} - 1\right)$$

and then $c = \frac{1}{2}[\ell(a) + \ell(b)]$. But

$$\ell(a) = \lim_{n\to\infty} \frac{a^{1/n} - 1}{1/n} = \lim_{n\to\infty} \frac{e^{(\log a/n)} - 1}{1/n} = \lim_{n\to\infty} \frac{e^{(\log a/n)} - 1}{\log a/n} \log a,$$

where we have first used the identity $a^x = e^{x\log a}$ and then multiplied and divided by $\log a$. Denoting $\varepsilon_n = \log a/n$

$$\ell(a) = \log a \lim_{n\to\infty} \frac{e^{\varepsilon_n} - 1}{\varepsilon_n},$$

and since $\varepsilon_n \underset{n\to\infty}{\longrightarrow} 0$ and we have the equivalence $e^{\varepsilon_n} - 1 \sim \varepsilon_n$ we conclude $\ell(a) = \log a$. Thus

$$c = \frac{1}{2}(\log a + \log b) = \frac{1}{2}\log(ab) = \log\sqrt{ab}$$

and finally

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n = e^{\log\sqrt{ab}} = \sqrt{ab}.$$

(iii) Using the identity $x^2 - y^2 = (x-y)(x+y)$ in the form $x-y = (x^2-y^2)/(x+y)$ (equivalently, multiplying and dividing by the conjugate of $x-y$),

$$\lim_{n\to\infty} n\left(\sqrt{n^2+1}-n\right) = \lim_{n\to\infty} n\frac{\left(\sqrt{n^2+1}\right)^2 - n^2}{\sqrt{n^2+1}+n} = \lim_{n\to\infty} n\frac{\cancel{n^2}+1-\cancel{n^2}}{\sqrt{n^2+1}+n} = \lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}+n}.$$

Now,

$$\sqrt{n^2+1} = \sqrt{n^2\left(1+\frac{1}{n^2}\right)} = n\sqrt{1+\frac{1}{n^2}},$$

so

$$\lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}+n} = \lim_{n\to\infty}\frac{\cancel{n}}{\cancel{n}\sqrt{1+\frac{1}{n^2}}+\cancel{n}\cdot 1} = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n^2}}+1} = \frac{1}{2}.$$

(iv) This time we need to use the identity $x^4 - y^4 = (x-y)(x^3+x^2y+xy^2+y^3)$ as $x-y = (x^4-y^4)/(x^3+x^2y+xy^2+y^3)$. Thus, if we denote

$$\ell = \lim_{n\to\infty}\sqrt{n}\left(\sqrt[4]{n^2+1}-\sqrt{n+1}\right),$$

then

$$\ell = \lim_{n\to\infty}\sqrt{n}\frac{n^2+1-(n+1)^2}{(n^2+1)^{3/4}+(n^2+1)^{1/2}(n+1)^{1/2}+(n^2+1)^{1/4}(n+1)+(n+1)^{3/2}}.$$

But $n^2+1-(n+1)^2 = \cancel{n^2}+\cancel{1}-\cancel{n^2}-2n-\cancel{1} = -2n$, and

$$(n^2+1)^{3/4} = n^{3/2}\left(1+\frac{1}{n^2}\right)^{3/4} \sim n^{3/2},$$
$$(n^2+1)^{1/2}(n+1)^{1/2} = n\left(1+\frac{1}{n^2}\right)^{1/2} n^{1/2}\left(1+\frac{1}{n}\right)^{1/2} \sim n^{3/2},$$
$$(n^2+1)^{1/4}(n+1) = n^{1/2}\left(1+\frac{1}{n^2}\right)^{1/4} n\left(1+\frac{1}{n}\right) \sim n^{3/2},$$
$$(n+1)^{3/2} = n^{3/2}\left(1+\frac{1}{n}\right)^{3/2} \sim n^{3/2},$$

so the denominator is $\sim 4n^{3/2}$. Accordingly,

$$\ell = \lim_{n\to\infty}\frac{-2n\sqrt{n}}{4n^{3/2}} = \lim_{n\to\infty}\frac{-2n^{3/2}}{4n^{3/2}} = -\frac{1}{2}.$$

(v) Factoring out the largest term of the numerator and of the denominator,

$$\lim_{n\to\infty}\frac{2^{n+1}+3^{n+1}}{2^n+3^n}=\lim_{n\to\infty}\frac{3^{n+1}\left[1+(2/3)^{n+1}\right]}{3^n\left[1+(2/3)^n\right]}=\lim_{n\to\infty}3\frac{1+(2/3)^{n+1}}{1+(2/3)^n}=3.$$

(vi) This is a 1^∞ indeterminacy, so

$$\ell=\lim_{n\to\infty}\left(\frac{n^2+1}{n^2-3n}\right)^{\frac{n^2-1}{2n}}=e^c,$$

where

$$c=\lim_{n\to\infty}\frac{n^2-1}{2n}\left(\frac{n^2+1}{n^2-3n}-1\right)=\lim_{n\to\infty}\frac{n^2-1}{2n}\cdot\frac{\cancel{n^2}+1-\cancel{n^2}+3n}{n^2-3n}=\lim_{n\to\infty}\frac{n^2-1}{n^2-3n}\cdot\frac{1+3n}{2n}=\frac{3}{2}.$$

Thus $\ell=e^{3/2}$.

Problem 3.4

(i) Since $\sin n\pi=0$ for every $n\in\mathbb{N}$ the sequence is identically 0, therefore the limit is 0.

(ii) We rewrite the expression as

$$\ell=\lim_{n\to\infty}\frac{n\left(e^{\frac{1}{n}}-e^{\sin\frac{1}{n}}\right)}{1-n\sin(1/n)}=\lim_{n\to\infty}\frac{e^{\frac{1}{n}}-e^{\sin\frac{1}{n}}}{\frac{1}{n}-\sin\frac{1}{n}}=\lim_{n\to\infty}e^{\sin\frac{1}{n}}\cdot\frac{e^{\frac{1}{n}-\sin\frac{1}{n}}-1}{\frac{1}{n}-\sin\frac{1}{n}}.$$

If we now denote $\varepsilon_n=\frac{1}{n}-\sin\frac{1}{n}\xrightarrow[n\to\infty]{}0$, then

$$\ell=\lim_{n\to\infty}e^{\sin\frac{1}{n}}\cdot\frac{e^{\varepsilon_n}-1}{\varepsilon_n}=1,$$

because $e^{\varepsilon_n}-1\sim\varepsilon_n$.

(iii) We can rewrite

$$\ell=\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{n!}},$$

and now apply Corollary 3.4.4 to Stolz's theorem to calculate the limit of a_n/a_{n-1} instead of the limit of $\sqrt[n]{a_n}$. Thus

$$\ell=\lim_{n\to\infty}\frac{n^n}{n!}\cdot\frac{(n-1)!}{(n-1)^{n-1}}.$$

But $n!=n\cdot(n-1)!$, so

$$\ell=\lim_{n\to\infty}\frac{n^n}{n(n-1)^{n-1}}=\lim_{n\to\infty}\frac{n^{n-1}}{(n-1)^{n-1}}=\lim_{n\to\infty}\left(\frac{n}{n-1}\right)^{n-1}=\lim_{n\to\infty}\left(\frac{n-1+1}{n-1}\right)^{n-1}$$
$$=\lim_{n\to\infty}\left(1+\frac{1}{n-1}\right)^{n-1}=e.$$

Alternatively, we can use the equivalence $n!\sim(2\pi n)^{1/2}n^ne^{-n}$ and subtitute to obtain

$$\ell=\lim_{n\to\infty}\frac{n}{(n!)^{1/n}}=\lim_{n\to\infty}\frac{n}{\left((2\pi n)^{1/2}n^ne^{-n}\right)^{1/n}}=\lim_{n\to\infty}\frac{n}{(2\pi n)^{1/2n}ne^{-1}}=\lim_{n\to\infty}\frac{e}{(2\pi n)^{1/2n}}=e,$$

given that

$$\lim_{n\to\infty}(2\pi n)^{1/2n}=\lim_{n\to\infty}(\sqrt{2\pi})^{1/n}\left(\sqrt[n]{n}\right)^{1/2}=1.$$

(iv)

$$\lim_{n\to\infty} n^{-3/n} = \lim_{n\to\infty} \left(\sqrt[n]{n}\right)^{-3} = 1.$$

(v) The limit is 0 because $2^n \ll n!$; nevertheless we will prove the same limit in an alternative way, just for the sake of illustration. Let us expand

$$0 < \frac{2^n}{n!} = \frac{\overbrace{2\cdot 2\cdots 2\cdot 2\cdot 2}^{n \text{ times}}}{n\cdot(n-1)\cdots 3\cdot 2\cdot 1} = \frac{2}{n}\cdot\frac{2}{n-1}\cdots\frac{2}{3}\cdot\underbrace{\frac{2}{2}}_{=1}\cdot\underbrace{\frac{2}{1}}_{=2}.$$

Now, among all the fractions appearing in this expression (excluding the factors 1 and 2) the largest one is $2/3$. Therefore we obtain an upper bound if we replace all fractions by this one, i.e.,

$$0 < \frac{2^n}{n!} < \left(\frac{2}{3}\right)^{n-2} 2.$$

Since the rightmost side of this inequality goes to 0 as $n\to\infty$, by the sandwich rule

$$\lim_{n\to\infty} \frac{2^n}{n!} = 0.$$

(vi) Likewise, this limit is 0 because $n^2 \ll 2^n$. But there is an alternative way to prove this. For that, we apply Stolz's theorem (the denominator is a monotonically increasing function that diverges to ∞) and calculate

$$\lim_{n\to\infty} \frac{n^2-(n-1)^2}{2^n-2^{n-1}} = \lim_{n\to\infty} \frac{2n-1}{2^{n-1}(2-1)} = \lim_{n\to\infty} \frac{2n-1}{2^{n-1}}.$$

And now apply Stolz's theorem again and calculate

$$\lim_{n\to\infty} \frac{2n-1-(2n-3)}{2^{n-1}-2^{n-2}} = \lim_{n\to\infty} \frac{2}{2^{n-2}(2-1)} = \lim_{n\to\infty} \frac{2}{2^{n-2}} = 0.$$

Therefore

$$\lim_{n\to\infty} \frac{n^2}{2^n} = 0.$$

(vii)

$$\lim_{n\to\infty} \frac{n^{n-1}}{(n-1)^n} = \lim_{n\to\infty} \frac{n^n}{n(n-1)^n} = \lim_{n\to\infty} \frac{1}{n}\cdot\left(\frac{n}{n-1}\right)^n = 0\cdot e = 0.$$

(viii) Applying Stolz's theorem

$$\ell = \lim_{n\to\infty} \frac{1+2\sqrt{2}+3\sqrt[3]{3}+\cdots+n\sqrt[n]{n}}{n^2}$$

(notice that the denominator is monotonically increasing and divergent) can be obtained as

$$\ell = \lim_{n\to\infty} \frac{n\sqrt[n]{n}}{n^2-(n-1)^2} = \lim_{n\to\infty} \frac{n\sqrt[n]{n}}{2n-1} = \lim_{n\to\infty} \frac{n}{2n-1}\cdot\sqrt[n]{n} = \frac{1}{2}\cdot 1 = \frac{1}{2}.$$

Problem 3.5

(i) This is a 1^∞ indeterminacy, thus

$$\ell = \lim_{n\to\infty}\left(\cos\frac{b}{n} + a\sin\frac{b}{n}\right)^n = e^c,$$

where

$$c = \lim_{n\to\infty} n\left(\cos\frac{b}{n} + a\sin\frac{b}{n} - 1\right) = \lim_{n\to\infty} n\left(\cos\frac{b}{n} - 1\right) + a\lim_{n\to\infty} n\sin\frac{b}{n}.$$

But $1-\cos\frac{b}{n} \sim \frac{b^2}{2n^2}$ and $\sin\frac{b}{n} \sim \frac{b}{n}$, thus

$$c = \lim_{n\to\infty} n\left(-\frac{b^2}{2n^2}\right) + a\lim_{n\to\infty} n\frac{b}{n} = 0 + ab = ab.$$

Therefore $\ell = e^{ab}$.

(ii) Here is another 1^∞ indeterminacy, so

$$\ell = \lim_{n\to\infty}\sqrt[u_n]{\frac{a-bu_n}{a+bu_n}} = e^c,$$

where

$$c = \lim_{n\to\infty}\frac{1}{u_n}\left(\frac{a-bu_n}{a+bu_n} - 1\right) = \lim_{n\to\infty}\frac{1}{u_n}\cdot\frac{\cancel{a} - bu_n - \cancel{a} - bu_n}{a+bu_n} = \lim_{n\to\infty}\frac{-2bu_n}{u_n(a+bu_n)}$$
$$= \lim_{n\to\infty}\frac{-2b}{a+bu_n} = -\frac{2b}{a}.$$

Therefore $\ell = e^{-2b/a}$.

Problem 3.6 We solve all these limits using Stolz's theorem.

(i) The denominator is $\log n$, a monotonically increasin sequence that diverges. Thus, if we call the limit ℓ,

$$\ell = \lim_{n\to\infty}\frac{\sin(\pi/n)}{\log n - \log(n-1)} = \lim_{n\to\infty}\frac{\sin(\pi/n)}{\log\left(\frac{n}{n-1}\right)} = \lim_{n\to\infty}\frac{\sin(\pi/n)}{\log\left(\frac{n-1+1}{n-1}\right)} = \lim_{n\to\infty}\frac{\sin(\pi/n)}{\log\left(1+\frac{1}{n-1}\right)}$$
$$= \lim_{n\to\infty}\frac{\pi/n}{1/(n-1)} = \pi\lim_{n\to\infty}\frac{n-1}{n} = \pi.$$

(ii) If we denote the limit by ℓ, then

$$\log\ell = \lim_{n\to\infty}\log\left(\prod_{k=1}^n (2k-1)^{1/n^2}\right) = \lim_{n\to\infty}\sum_{k=1}^n \log\left((2k-1)^{1/n^2}\right) = \lim_{n\to\infty}\sum_{k=1}^n\frac{1}{n^2}\log(2k-1)$$
$$= \lim_{n\to\infty}\frac{\sum_{k=1}^n \log(2k-1)}{n^2}.$$

Now we apply Stolz's theorem and compute

$$\log\ell = \lim_{n\to\infty}\frac{\log(2n-1)}{n^2-(n-1)^2} = \lim_{n\to\infty}\frac{\log(2n-1)}{2n-1} = 0.$$

Therefore $\ell = 1$.

(iii) Taking the n^2 out of the sum

$$\ell = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^2} \sin\frac{1}{k} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} k^2 \sin\frac{1}{k}}{n^2},$$

thus applying Stolz's theorem

$$\ell = \lim_{n\to\infty} \frac{n^2 \sin\frac{1}{n}}{n^2-(n-1)^2} = \lim_{n\to\infty} \frac{n^2\frac{1}{n}}{2n-1} = \lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2},$$

where we have used $\sin\frac{1}{n} \sim \frac{1}{n}$.

Problem 3.7 If we denote ℓ the limit we want to calculate and apply Stolz's theorem (the denominator, $\log(n+1)$, is a monotonically and divergent sequence) we get

$$\ell = \lim_{n\to\infty} \frac{a_n/n}{\log(n+1)-\log n} = \lim_{n\to\infty} \frac{a_n}{n\log\left(\frac{n+1}{n}\right)} = \lim_{n\to\infty} \frac{a_n}{n\log\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{a_n}{n\frac{1}{n}} = \lim_{n\to\infty} a_n = a,$$

using the equivalence $\log\left(1+\frac{1}{n}\right) \sim \frac{1}{n}$.

Problem 3.8 The smallest term in the sum is $\frac{1}{\sqrt{n^2+3n}}$ (the one with the largest denominator) and the largest term is $\frac{1}{\sqrt{n^2+1}}$ (the one with the smallest denominator). Since there are $3n$ terms in the sum

$$\frac{3n}{\sqrt{n^2+3n}} < \sum_{k=1}^{3n} \frac{1}{\sqrt{n^2+k}} < \frac{3n}{\sqrt{n^2+1}}.$$

For the two bounding sequences we have

$$\lim_{n\to\infty} \frac{3n}{\sqrt{n^2+3n}} = 3, \qquad \lim_{n\to\infty} \frac{3n}{\sqrt{n^2+1}} = 3,$$

therefore applying the sandwich rule we conclude

$$\lim_{n\to\infty} \sum_{k=1}^{3n} \frac{1}{\sqrt{n^2+k}} = 3.$$

Problem 3.9

(a) $\displaystyle\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \frac{a_n-n+n}{n} = \lim_{n\to\infty} \frac{a_n-n}{n} + 1 = 1$ because $\displaystyle\frac{a_n-n}{n} \sim \frac{\ell}{n}$.

(b) $\displaystyle\lim_{n\to\infty} n\log\left(\frac{a_n}{n}\right) = \lim_{n\to\infty} n\log\left(1+\frac{a_n-n}{n}\right) = \lim_{n\to\infty} n\cdot\frac{a_n-n}{n} = \lim_{n\to\infty}(a_n-n) = \ell$, where we have used the equivalence $\log(1+\varepsilon_n) \sim \varepsilon_n$ for any sequence $\varepsilon_n \xrightarrow[n\to\infty]{} 0$.

Problem 3.10 Assume that

$$\lim_{n\to\infty} \sqrt[n^2]{\frac{a_n^n}{a_1 a_2 \cdots a_n}} = x$$

Then

$$\log x = \lim_{n\to\infty} \frac{1}{n^2}\left(n\log a_n - \sum_{k=1}^{n} \log a_k\right).$$

Since n^2 is a monotonically increasing, divergent sequence we can apply Stolz's theorem with $b_n = n^2$ and

$$a_n = n \log a_n - \sum_{k=1}^{n} \log a_k,$$

and calculate instead

$$\log x = \lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}}.$$

On the one hand $n^2 - (n-1)^2 = n^2 - n^2 + 2n - 1 = 2n - 1$. On the other hand,

$$\begin{aligned} a_n - a_{n-1} &= n \log a_n - \sum_{k=1}^{n} \log a_k - (n-1)\log a_{n-1} + \sum_{k=1}^{n-1} \log a_k = n \log a_n - (n-1)\log a_{n-1} - \log a_n \\ &= (n-1)\log a_n - (n-1)\log a_{n-1} = (n-1)\log\left(\frac{a_n}{a_{n-1}}\right). \end{aligned}$$

Therefore

$$\log x = \lim_{n\to\infty} \frac{n-1}{2n-1} \log\left(\frac{a_n}{a_{n-1}}\right) = \frac{\log \ell}{2} = \log\sqrt{\ell} \qquad \Rightarrow \qquad x = \sqrt{\ell}.$$

Problem 3.11

(i) We can write the sequence as $x_{n+1} = \sqrt{2x_n}$, with $x_0 = 1$. In order to know whether the sequence is monotonically increasing or decreasing we need to assess the sign of

$$x_{n+1} - x_n = \sqrt{2x_n} - x_n = \frac{2x_n - x_n^2}{\sqrt{2x_n} + x_n} = \frac{(2 - x_n)x_n}{\sqrt{2x_n} + x_n}.$$

(We have used the identity $x - y = (x^2 - y^2)/(x + y)$.) The sequence is clearly posivite, because $x_0 = 1 > 0$ and $x_{n+1} = \sqrt{2x_n} > 0$ whenever $x_n > 0$. Thus the sign of $x_{n+1} - x_n$ will be the sign of the factor $2 - x_n$.
We are going to prove now that $x_n < 2$ for all $n \in \mathbb{N}$. First, $x_0 = 1 < 2$; second, if $x_n < 2$ then $x_{n+1} = \sqrt{2x_n} < \sqrt{2 \cdot 2} = 2$. Hence it is proven by induction. But with this we have simultaneously proven

$$x_n < 2, \qquad x_{n+1} - x_n > 0,$$

so the sequence increases monotonically and is bounded from above —hence converges.
The limit can be obtained by taking limits in the recurrence equation:

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2x_n},$$

and if we denote $\lim_{n\to\infty} x_n = x$, this equation becomes

$$x = \sqrt{2x} \qquad \Rightarrow \qquad x^2 = 2x \qquad \Rightarrow \qquad x(x-2) = 0.$$

Of the two solutions of this equation, $x = 0$ and $x = 2$, the latter has to be the solution because the sequence begins at $x_0 = 1$ and increases.

(ii) We describe de sequence as $x_{n+1} = \sqrt{2+x_n}$, with $x_0 = 0$, and proceed as in (i):

$$x_{n+1} - x_n = \sqrt{2+x_n} - x_n = \frac{2+x_n-x_n^2}{\sqrt{2+x_n}+x_n}.$$

The equation $x^2 - x - 2 = 0$ has two roots, namely $x = -1$ and $x = 2$, and therefore $2+x_n-x_n^2 = (x_n+1)(2-x_n)$. Thus,

$$x_{n+1} - x_n = \frac{(x_n+1)(2-x_n)}{\sqrt{2+x_n}+x_n}.$$

The sign of $x_{n+1} - x_n$ is the sign of $2 - x_n$ —because, as in (i), $x_n > 0$ for all $n \in \mathbb{N}$.
Let us now prove —using induction— that $x_n < 2$ for all $n \in \mathbb{N}$. Clearly $x_0 = 0 < 2$, and if $x_n < 2$ then $x_{n+1} = \sqrt{2+x_n} < \sqrt{2+2} = 2$. So we have just proven that

$$x_n < 2, \qquad x_{n+1} - x_n > 0,$$

and thefore the sequence converges. Denoting $\lim\limits_{n\to\infty} x_n = x$ and taking limits in the recurrence equation

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2+x_n} \qquad \Rightarrow x = \sqrt{2+x} \qquad \Rightarrow x^2 = 2+x \qquad \Rightarrow x^2 - x - 2 = 0,$$

whose roots are $x = -1$ and $x = 2$. Of them, the latter is the limit because the whole sequence is positive.

(iii) The difference of two consecutive terms is

$$u_{n+1} - u_n = 3 + \frac{u_n}{2} - u_n = 3 - \frac{u_n}{2} = \frac{6-u_n}{2},$$

so its sign will depend on whether $u_n < 6$ or $u_n > 6$. Let us prove by induction that it is the former. To begin with $u_0 = 0 < 6$. Now let us assume that $u_n < 6$. Then

$$u_{n+1} = 3 + \frac{u_n}{2} < 3 + \frac{6}{2} = 3 + 3 = 6,$$

so $u_n < 6$ implies $u_{n+1} < 6$, and this completes the proof. Then we can conclude that $u_{n+1} - u_n > 0$, hence the sequence is monotonically increasing. On the other hand 6 is an upper bound, so it is convergent. To calculate the limit we need to take limits in both sides of the recurrence equation. If we denote $\lim\limits_{n\to\infty} u_n = \ell$, then

$$\ell = 3 + \frac{\ell}{2} \qquad \Rightarrow \qquad \ell = 6.$$

(iv) In this case $u_1 = 3$ and $u_{n+1} = 3 + 2u_n > 2u_n$. In particular this implies $u_2 > 2u_1 = 2 \cdot 3$; also $u_3 > 2u_2 = 2^2 \cdot 3$; $u_4 > 2u_3 = 2^3 \cdot 3$; etc. In general

$$u_n > 2^{n-1} \cdot 3.$$

But $2^{n-1} \xrightarrow[n\to\infty]{} \infty$, so $\lim\limits_{n\to\infty} u_n = \infty$.

(v) We calculate the difference

$$u_{n+1} - u_n = \frac{u_n^3 + 6}{7} - u_n = \frac{u_n^3 - 7u_n + 6}{7}.$$

But $x^3 - 7x + 6 = (x-1)(x^2+x-6) = (x-1)(x-2)(x+3)$, therefore

$$u_{n+1} - u_n = \frac{1}{7}(u_n - 1)(u_n - 2)(u_n + 3).$$

(a) Since $u_0 = 1/2$ we have $0 < u_0 < 1$, so let us try to prove by induction that the whole sequence is in $(0,1)$. Assume $0 < u_n < 1$. Then

$$0 < u_n^3 < 1 \qquad \Rightarrow \qquad 0 < \frac{u_n^3+6}{7} < \frac{1+6}{7} = 1,$$

therefore $0 < u_{n+1} < 1$. And since the whole sequence is in $(0,1)$ we have

$$u_n - 1 < 0, \qquad u_n - 2 < 0, \qquad u_n + 3 > 0,$$

and therefore u_n is monotonically increasing and bounded from above by 1 —hence convergent. We can calculate the limit, ℓ, by solving the equation

$$\ell = \frac{\ell^3+6}{7} \qquad \Rightarrow \qquad \ell^3 - 7\ell + 6 = 0.$$

There are three solutions, namely $\ell = -3$, 1, and 2. Of those three the limit must be 1 because $0 < u_n < 1$, $u_0 = 1/2$, and u_n increases monotonically.

(b) Since $u_0 = 3/2$ we have $1 < u_0 < 2$, so let us try to prove by induction that the whole sequence is in $(1,2)$. Assume $1 < u_n < 2$. Then

$$1 < u_n^3 < 8 \qquad \Rightarrow \qquad \frac{1+7}{7} < \frac{u_n^3+6}{7} < \frac{8+6}{7} \qquad \Rightarrow \qquad 1 < \frac{u_n^3+6}{7} < 2,$$

therefore $1 < u_{n+1} < 2$. And since the whole sequence is in $(1,2)$ we have

$$u_n - 1 > 0, \qquad u_n - 2 < 0, \qquad u_n + 3 > 0,$$

and therefore u_n is monotonically decreasing and bounded from below by 1 —hence convergent. Again the limit can only be $\ell = -3$, 0, and 1. Of those three the limit must be 1 because $1 < u_n < 2$, $u_0 = 3/2$, and u_n decreases monotonically.

(c) Since $u_0 = 3$ we have $2 < u_0$, so let us try to prove by induction that the whole sequence is bounded from below by 2. Assume $u_n > 2$. Then

$$u_n^3 > 8 \qquad \Rightarrow \qquad \frac{u_n^3+6}{7} > \frac{8+6}{7} = 2,$$

therefore $u_{n+1} > 2$. This completes the proof. Now,

$$u_n - 1 > 0, \qquad u_n - 2 > 0, \qquad u_n + 3 > 0,$$

and therefore u_n is monotonically increasing —but there is no upper bound that we know of. Suppose there is such an upper bound —even though we do not know which one—; it that case the sequence would have a limit. But the limit can only be $\ell = -3$, 1, or 2. And neither of them can be because $u_0 = 3$ and the sequence is monotonically *increasing*. Therefore there is no upper bound for this sequence —which means that it *diverges*.

Problem 3.12

(a) Let us calculate the difference

$$\begin{aligned} a_{n+1} - a_n = \sqrt{1+3a_n} - 1 - a_n &= \frac{1+3a_n-(1+a_n)^2}{\sqrt{1+3a_n}+1+a_n} = \frac{\cancel{1}+\cancel{3}a_n-\cancel{1}-\cancel{2a_n}-a_n^2}{\sqrt{1+3a_n}+1+a_n} \\ &= \frac{a_n - a_n^2}{\sqrt{1+3a_n}+1+a_n} = \frac{a_n(1-a_n)}{\sqrt{1+3a_n}+1+a_n}. \end{aligned}$$

The sequence is $a_n > 0$. The reason is that $a_0 = 1/2 > 0$ and if $a_n > 0$ then $a_{n+1} = \sqrt{1+3a_n} - 1 > \sqrt{1} - 1 = 0$. Accordingly both the denominator of the fraction and the first factor in the numerator are positive. On the other hand, $a_0 = 1/2 < 1$, and if $a_n < 1$ then $a_{n+1} = \sqrt{1+3a_n} - 1 < \sqrt{1+3} - 1 = 2 - 1 = 1$, so 1 is an upper bound for the sequence —which means that the second factor in the numerator is also positive. Hence $a_{n+1} - a_n > 0$ and the sequence is monotonically increasing. Since is is also bounded from above —by 1— it has a limit ℓ —to be determined.
If we now take limits in the recurrence equation we obtain

$$\begin{aligned} \ell = \sqrt{1+3\ell} - 1 \qquad &\Rightarrow \qquad \ell + 1 = \sqrt{1+3\ell} \qquad \Rightarrow \qquad (\ell+1)^2 = 1 + 3\ell \\ &\Rightarrow \qquad \ell^2 + \cancel{2}\ell + \cancel{1} = \cancel{1} + \cancel{3}\ell \qquad \Rightarrow \qquad \ell^2 = \ell. \end{aligned}$$

The only solutions are $\ell = 0$ and 1, but since $a_0 = 1/2$ and a_n is monotonically increasing, the limit must be $\ell = 1$.

(b) Both the numerator and the denominator go to 0 as $n \to \infty$, so we face a $0/0$ indeterminacy. Substituting the recurrence, and using the identity $x - y = (x^2 - y^2)/(x+y)$,

$$\begin{aligned} \lim_{n\to\infty} \frac{a_{n+1} - 1}{a_n - 1} &= \lim_{n\to\infty} \frac{\sqrt{1+3a_n} - 2}{a_n - 1} = \lim_{n\to\infty} \frac{1 + 3a_n - 4}{(\sqrt{1+3a_n} + 2)(a_n - 1)} \\ &= \lim_{n\to\infty} \frac{3a_n - 3}{(\sqrt{1+3a_n} + 2)(a_n - 1)} = \lim_{n\to\infty} \frac{3\cancel{(a_n - 1)}}{(\sqrt{1+3a_n} + 2)\cancel{(a_n - 1)}} \\ &= \frac{3}{\sqrt{1+3} + 2} = \frac{3}{4}. \end{aligned}$$

Problem 3.13

(a) We need to compute the product $(b_{n+1} - b_n)(b_n - b_{n-1})$ and prove that it is negative. In order to do that a good strategy is to write down this expression as a function of b_n alone. The term b_{n+1} is directly provided by the recurrence equation

$$b_{n+1} = 1 - \frac{b_n}{2}.$$

On the other hand, the same recurrence implies

$$b_n = 1 - \frac{b_{n-1}}{2} \qquad \Rightarrow \qquad b_{n-1} = 2(1 - b_n).$$

Therefore

$$(b_{n+1} - b_n)(b_n - b_{n-1}) = \left(1 - \frac{b_n}{2} - b_n\right)(b_n - 2 + 2b_n) = \frac{2 - 3b_n}{2}(3b_n - 2) = -\frac{(3b_n - 2)^2}{2}.$$

The last expression is always negative unless $b_n = 2/3$. So as long as $b_n \neq 2/3$ the sequence keeps alternating. On the other hand, if $b_n = 2/3$ then

$$b_{n+1} = 1 - \frac{b_n}{2} = 1 - \frac{1}{3} = \frac{2}{3},$$

so the rest of the sequence remains fixed at $2/3$.

(b) If $\lim_{n\to\infty} b_n = \ell$ then, taking limits in the recurrence equation we obtain

$$\ell = 1 - \frac{\ell}{2} \qquad \Rightarrow \qquad \ell = \frac{2}{3}.$$

(c) Substituting $\ell = 2/3$,

$$b_{n+1} - \frac{2}{3} = 1 - \frac{b_n}{2} - \frac{2}{3} = \frac{1}{3} - \frac{b_n}{2} = \frac{1}{2}\left(\frac{2}{3} - b_n\right),$$

factoring out a $1/2$ factor. Taking absolute values in both sides of the obtained equation we get to the desired result.

(d) If $\left|b_n - \frac{2}{3}\right| = \frac{1}{2}\left|b_{n-1} - \frac{2}{3}\right|$, then, setting $n = 1$ we obtain

$$\left|b_1 - \frac{2}{3}\right| = \frac{1}{2}\left|b_0 - \frac{2}{3}\right| = \frac{1}{2} \cdot \frac{2}{3}.$$

Setting $n = 2$,

$$\left|b_2 - \frac{2}{3}\right| = \frac{1}{2}\left|b_1 - \frac{2}{3}\right| = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{2^2} \cdot \frac{2}{3}.$$

Setting $n = 3$,

$$\left|b_3 - \frac{2}{3}\right| = \frac{1}{2}\left|b_2 - \frac{2}{3}\right| = \frac{1}{2} \cdot \frac{1}{2^2} \cdot \frac{2}{3} = \frac{1}{2^3} \cdot \frac{2}{3}.$$

And we can continue this way to obtain the general expression

$$\left|b_n - \frac{2}{3}\right| = \frac{1}{2^n} \cdot \frac{2}{3}$$

valid for all $n \in \mathbb{N}$. If we take limits,

$$\lim_{n\to\infty}\left|b_n - \frac{2}{3}\right| = \frac{2}{3}\lim_{n\to\infty}\frac{1}{2^n} = 0 \quad \Rightarrow \quad \lim_{n\to\infty}\left(b_n - \frac{2}{3}\right) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} b_n = \frac{2}{3}.$$

Problem 3.14

(a) Clearly $x_1 = 1 > 0$. Now, if we assume $x_n > 0$ then also $1 + x_n > 0$ and $1 + 2x_n > 0$; therefore

$$x_{n+1} = \frac{x_n(1+x_n)}{1+2x_n} > 0,$$

and the results is proven by induction.

(b) Let us calculate

$$x_{n+1} - x_n = \frac{x_n(1+x_n)}{1+2x_n} - x_n = \frac{\cancel{x_n} + \cancel{x_n^2} - \cancel{x_n} - \cancel{2}x_n^2}{1+2x_n} = -\frac{x_n^2}{1+2x_n} < 0$$

because the denominator is positive and $x_n^2 > 0$. Hence the sequence decreases monotonically.

(c) Denoning ℓ the limit of x_n and taking limits in the recurrence we obtain

$$\ell = \frac{\ell(1+\ell)}{1+2\ell} \quad \Rightarrow \quad \ell(1+2\ell) = \ell + \ell^2 \quad \Rightarrow \quad \cancel{\ell} + \cancel{2}\ell^2 = \cancel{\ell} + \cancel{\ell^2} \quad \Rightarrow \quad \ell = 0.$$

D.4 Series

Problem 4.1

(i) Converges according to the root test:

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\frac{n+1}{2n-1}=\frac{1}{2}<1.$$

(ii) Converges because $\frac{1}{(3n-1)^2}\sim\frac{1}{9n^2}$ and $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}<\infty$.

(iii) Converges because $\frac{1}{\sqrt{2n^4+1}}\sim\frac{1}{\sqrt{2}n^2}$ and $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}<\infty$.

(iv) Diverges because $\frac{1}{\sqrt{n(n+1)}}\sim\frac{1}{n}$ and $\sum\limits_{n=1}^{\infty}\frac{1}{n}=\infty$.

(v) Since $|\sin n|\leqslant 1$ we can write

$$\frac{|\sin n|}{n^2+n}<\frac{1}{n^2+n}.$$

Since $\frac{1}{n^2+n}\sim\frac{1}{n^2}$ and $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}<\infty$ then

$$\sum_{n=1}^{\infty}\frac{1}{n^2+n}<\infty \qquad\Rightarrow\qquad \sum_{n=1}^{\infty}\frac{|\sin n|}{n^2+n}<\infty$$

by the comparison test.

(vi) Converges because $\sin\left(\frac{1}{n^2}\right)\sim\frac{1}{n^2}$ and $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}<\infty$.

(vii) Diverges because $\arcsin\left(\frac{1}{\sqrt{n}}\right)\sim\frac{1}{\sqrt{n}}=\frac{1}{n^{1/2}}$ and $\sum\limits_{n=1}^{\infty}\frac{1}{n^{1/2}}=\infty$.

(viii) Converges according to the root test:

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\frac{\sqrt[n]{3n-1}}{\sqrt{2}}=\frac{1}{\sqrt{2}}<1.$$

Alternatively,

$$\sum_{n=1}^{\infty}\frac{3n-1}{(\sqrt{2})^n}=3\sum_{n=1}^{\infty}n\left(\frac{1}{\sqrt{2}}\right)^n-\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n.$$

The first series is arithmetic-geometric and the second is geometric, both with argument $\frac{1}{\sqrt{2}}<1$.

(ix) Coverges according to the quotient test:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)^{n+1}}{3^{n+1}(n+1)!}\cdot\frac{3^n n!}{n^n}.$$

But $\frac{n!}{(n+1)!}=\frac{1}{n+1}$ and $\frac{3^n}{3^{n+1}}=\frac{1}{3}$, so

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)^{n+1}}{3(n+1)n^n}=\frac{1}{3}\lim_{n\to\infty}\frac{(n+1)^n}{n^n}=\frac{1}{3}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\frac{e}{3}<1.$$

(x) Converges according to the root test:

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\left(\sqrt[n]{n}-1\right)=0<1.$$

(xi) Converges according to the root test:

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\frac{1}{3}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\frac{e}{3}<1.$$

(xii) Converges according to the root test:

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\frac{1}{\log n}=0<1.$$

(xiii) Converges according to the root test:

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\frac{\sqrt[n]{n^2}}{\log n}=0<1.$$

(xiv) If we rewrite

$$\sqrt{n^2+1}-n=\frac{\cancel{n^2}+1-\cancel{n^2}}{\sqrt{n^2+1}+n}=\frac{1}{\sqrt{n^2+1}+n}\sim\frac{1}{2n},$$

we easily conclude that the series diverges because $\sum\limits_{n=1}^{\infty}\frac{1}{n}=\infty$.

(xv) $\log n$ is an ever increasing function of n, so eventually $\log n>2$. Then $n^{\log n}>n^2$ and therefore

$$\frac{1}{n^{\log n}}<\frac{1}{n^2}.$$

Since $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}<\infty$ we conclude that the series converges by the comparison test.

(xvi) Let us apply Cauchy's condensation test: the convergence of the series will be equivalent to that of

$$\sum_{k=1}^{\infty}\frac{2^k}{(\log 2^k)^{\log 2^k}}=\sum_{k=1}^{\infty}\frac{2^k}{(k\log 2)^{k\log 2}}=\sum_{k=1}^{\infty}\underbrace{\frac{1}{k^{k\log 2}}\left(\frac{2}{(\log 2)^{\log 2}}\right)^k}_{=c_k}.$$

If we apply the root test,

$$\lim_{k\to\infty}\sqrt[k]{c_k}=\lim_{k\to\infty}\frac{1}{k^{\log 2}}\left(\frac{2}{(\log 2)^{\log 2}}\right)=0,$$

so the series converges.

(xvii) Since

$$\lim_{n\to\infty}\sqrt[n]{n}=1,$$

then

$$\frac{1}{n\sqrt[n]{n}}\sim\frac{1}{n}.$$

Therefore the series diverges because $\sum\limits_{n=1}^{\infty}\frac{1}{n}=\infty$.

(xviii) Since

$$\lim_{n\to\infty}\left(\frac{n}{n-1}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n-1}\right)^n = e,$$

the series diverges (in any convergent series the general term must tend to zero).

Problem 4.2 Reducing the general term to a unique fraction,

$$\frac{a}{2n-1} - \frac{b}{2n+1} = \frac{2(a-b)n+a+b}{4n^2-1} \sim \begin{cases} \dfrac{2(a-b)}{n}, & \text{if } a \neq b, \\ \dfrac{a+b}{n^2}, & \text{if } a = b. \end{cases}$$

Clearly the series converges if, and only if, $a = b$.

Now, in the case $a = b$, the series is

$$S = \sum_{n=1}^{\infty}\left(\frac{a}{2n-1} - \frac{a}{2n+1}\right) = \sum_{n=1}^{\infty}(u_n - u_{n+1}), \qquad u_n = \frac{a}{2n-1}.$$

Then

$$S = u_1 - \lim_{n\to\infty} u_n = a.$$

Problem 4.3

(i) Applying the root test,

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{n}(1+a)e^{-a} = (1+a)e^{-a}.$$

But $e^a > 1+a$ for all $a \neq 0$, so $(1+a)e^{-a} < 1$ for all $a \neq 0$ and the series converges. For $a = 0$ we have $(1+a)e^{-a} = 1$ and the series becomes

$$\sum_{n=1}^{\infty} n = \infty.$$

(ii) Using the quotient test,

$$\lim_{n\to\infty}\frac{a_n}{a_{n-1}} = \lim_{n\to\infty}\frac{n^n}{a^n n!}\cdot\frac{a^{n-1}(n-1)!}{(n-1)^{n-1}} = \frac{1}{a}\lim_{n\to\infty}\left(\frac{n}{n-1}\right)^{n-1} = \frac{e}{a},$$

so the series converges for $a > e$ and diverges for $a < e$. What happens for $a = e$ must be decided with a different argument. But using Stirling,

$$\frac{n^n}{e^n n!} \sim \frac{n^n}{e^n\sqrt{2\pi n}n^n e^{-n}} \sim \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{n^{1/2}},$$

so the series diverges for $a = e$ because $\sum_{n=1}^{\infty}\frac{1}{n^\alpha} = \infty$ for any $\alpha \leqslant 1$.

(iii) It seems we might use the quotient test, but

$$\lim_{n\to\infty}\frac{a_n}{a_{n-1}} = \lim_{n\to\infty}\frac{n!e^n}{n^{n+a}}\cdot\frac{(n-1)^{n-1+a}}{(n-1)!e^{n-1}} = e\lim_{n\to\infty}\left(\frac{n-1}{n}\right)^{n-1+a} = e\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n+a-1} = ee^{-1} = 1,$$

so it does not decide in this case. Let us use Stirling instead,

$$\frac{n!e^n}{n^{n+a}} \sim \frac{\sqrt{2\pi n}n^n e^{-n}e^n}{n^{n+a}} = \frac{\sqrt{2\pi n}}{n^a} = \sqrt{2\pi}\frac{1}{n^{a-1/2}},$$

so the series converges for $a - 1/2 > 1$ (i.e. $a > 3/2$) and diverges otherwise.

(iv) First of all, the series converges trivially for $a = 0$, because all terms are zero in this case. For $a > 0$ we have

$$a_n = \frac{a^n}{(1+a)(1+a^2)\cdots(1+a^n)} > 0,$$

so we can try the quotien test. This amounts to computing the limit

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \frac{a^{n+1}}{(1+a)(1+a^2)\cdots(1+a^n)(1+a^{n+1})} \frac{(1+a)(1+a^2)\cdots(1+a^n)}{a^n} \\ &= \lim_{n\to\infty} \frac{a}{(1+a^{n+1})} = \begin{cases} a, & \text{if } a < 1, \\ \frac{1}{2}, & \text{if } a = 1, \\ 0, & \text{if } a > 1. \end{cases}\end{aligned}$$

Whichever the case, this limit is always smaller than 1, hence the series coverges for any $a \geqslant 0$.

Problem 4.4

(i) The series does not converge conditionally because $\log n < n$, so $1/\log n > 1/n$, and the harmonic series diverges. Thus

$$\sum_{n=2}^{\infty} \frac{1}{\log n} = \infty$$

by the comparison test.
It does converge conditionally though, as the Leibniz's test proves: $1/\log n$ is a monotonically decreasing sequence that tends to zero.

(ii) Let us first expand

$$\sin\left(n\pi + \frac{1}{n}\right) = \underbrace{\sin n\pi}_{=0} \cos\frac{1}{n} + \underbrace{\cos n\pi}_{=(-1)^n} \sin\frac{1}{n} = (-1)^n \sin\frac{1}{n}.$$

Thus this is an alternating series of general term $\sin(1/n)$. It is not absolutely convergent because $\sin(1/n) \sim 1/n$ and the harmonic series diverges. It is conditionally convergent though, because $\sin(1/n)$ is monotonically decreasing toward zero (Leibniz's test).

(iii) Since

$$\left(\arctan\frac{1}{n}\right)^2 \sim \frac{1}{n^2},$$

the series converges absolutely.

(iv) Since

$$\lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$$

the series does not converge (not even conditionally).

(v) We can write

$$\sqrt{n^2-1} - n = \frac{\left(\sqrt{n^2-1}-n\right)\left(\sqrt{n^2-1}+n\right)}{\sqrt{n^2-1}+n} = \frac{n^2-1-n^2}{\sqrt{n^2-1}+n} = \frac{-1}{\sqrt{n^2-1}+n},$$

so

$$\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{n^2-1}-n\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n^2-1}+n}.$$

Now,

$$\frac{1}{\sqrt{n^2-1}+n}\sim\frac{1}{2n},$$

hence the series does not converge absolutely, but it does conditionally because the general term (without sign) is monotonically decreasing toward zero (Leibniz's test).

(vi) Let us rewrite

$$\sum_{n=1}^{\infty}(-1)^n\log\left(\frac{n}{n+1}\right)=\sum_{n=1}^{\infty}(-1)^{n+1}\log\left(\frac{n+1}{n}\right)=\sum_{n=1}^{\infty}(-1)^{n+1}\log\left(1+\frac{1}{n}\right).$$

As

$$\log\left(1+\frac{1}{n}\right)\sim\frac{1}{n}$$

the series does not converge absolutely, but it does conditionally because the general term (without sign) is monotonically decreasing toward zero (Leibniz's test).

(vii) We know that $1-\cos\varepsilon_n\sim\varepsilon_n^2/2$ for any vanishing sequence ε_n. Then,

$$1-\cos\frac{1}{n}\sim\frac{1}{2n^2}$$

and the series converges absolutely.

(viii) First of all,

$$e^n+e^{-n}\sim e^n\quad\Rightarrow\quad\log(e^n+e^-n)\sim\log(e^n)=n.$$

Therefore

$$\frac{1}{\log(e^n+e^-n)}\sim\frac{1}{n}$$

and the series does not converge absolutely because the harmonic series diverges. It doe conditionally though, because the general term (without sign) is monotonically decreasing toward zero (Leibniz's test).

Problem 4.5 There are only three types of series of which we know the sum: geometric, arithmetic-geometric, and telescoping series. So the point of this exercise is to identify these three types of series within the proposed ones.

(i) We can write

$$\frac{3^{n+1}-2^{n-3}}{4^n}=3\frac{3^n}{4^n}-\frac{1}{8}\frac{2^n}{4^n}=3\left(\frac{3}{4}\right)^n-\frac{1}{8}\left(\frac{1}{2}\right)^n.$$

Therefore

$$\sum_{n=0}^{\infty}\frac{3^{n+1}-2^{n-3}}{4^n}=3\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n-\frac{1}{8}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n=3\frac{1}{1-3/4}-\frac{1}{8}\frac{1}{1-1/2}=12-\frac{1}{4}=\frac{47}{4}.$$

(ii) This is an arithmetic-geometric series of argument $1/2$, so

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = 2.$$

(iii) This one is the sum of an arithmetic-geometric and a plain geometric series of argument $1/3$, so

$$\sum_{n=0}^{\infty} \frac{4n+1}{3^n} = 4\sum_{n=0}^{\infty} n\left(\frac{1}{3}\right)^n + \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n = 4\frac{1/3}{(1-1/3)^2} + \frac{1}{1-1/3} = 3+\frac{3}{2} = \frac{9}{2}.$$

Notice that in the arithmetic-geometric series $\sum_{n=0}^{\infty} nx^n = \sum_{n=1}^{\infty} nx^n$ because the $n=0$ term is zero.

(iv) If we denote $u_n = \sqrt{n} - \sqrt{n+1}$, then

$$\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n} = \sqrt{n} - \sqrt{n+1} - \left(\sqrt{n+1} - \sqrt{n+2}\right) = u_n - u_{n+1}.$$

Thus,

$$\sum_{n=1}^{\infty}\left(\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n}\right) = u_1 - \lim_{n\to\infty} u_n.$$

Since $u_1 = 1-\sqrt{2}$ and

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\sqrt{n} - \sqrt{n+1}\right) = \lim_{n\to\infty} \frac{-1}{\sqrt{n}+\sqrt{n+1}} = 0,$$

the sum

$$\sum_{n=1}^{\infty}\left(\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n}\right) = 1-\sqrt{2}.$$

(v) We can split

$$\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n(n+1)}} = \frac{\cancel{\sqrt{n+1}}}{\sqrt{n\cancel{(n+1)}}} - \frac{\cancel{\sqrt{n}}}{\sqrt{\cancel{n}(n+1)}} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = u_n - u_{n+1}, \qquad u_n = \frac{1}{\sqrt{n}}.$$

Thus

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n(n+1)}} = u_1 - \lim_{n\to\infty} u_n = 1.$$

(vi) Splitting

$$\log\left[\frac{n(n+2)}{(n+1)^2}\right] = \log\left(\frac{n}{n+1}\right) + \log\left(\frac{n+2}{n+1}\right) = \log\left(\frac{n}{n+1}\right) - \log\left(\frac{n+1}{n+2}\right) = u_n - u_{n+1},$$

where we can identify

$$u_n = \log\left(\frac{n}{n+1}\right),$$

we obtain

$$\sum_{n=1}^{\infty} \log\left[\frac{n(n+2)}{(n+1)^2}\right] = u_1 - \lim_{n\to\infty} u_n = -\log 2.$$

(vii) We can split the terms of the series into even and odd:

$$\sum_{n=0}^{\infty} x^{\lfloor \frac{n}{2} \rfloor} y^{\lfloor \frac{n+1}{2} \rfloor} = \sum_{k=0}^{\infty} x^{\lfloor \frac{2k}{2} \rfloor} y^{\lfloor \frac{2k+1}{2} \rfloor} + \sum_{k=0}^{\infty} x^{\lfloor \frac{2k+1}{2} \rfloor} y^{\lfloor \frac{2k+2}{2} \rfloor} = \sum_{k=0}^{\infty} x^k y^k + \sum_{k=0}^{\infty} x^k y^{k+1}$$
$$= \sum_{k=0}^{\infty} (xy)^k + y \sum_{k=0}^{\infty} (xy)^k = (1+y) \sum_{k=0}^{\infty} (xy)^k.$$

Since $|xy| < 1$, then

$$\sum_{k=0}^{\infty} (xy)^k = \frac{1}{1-xy}.$$

Therefore

$$\sum_{n=0}^{\infty} x^{\lfloor \frac{n}{2} \rfloor} y^{\lfloor \frac{n+1}{2} \rfloor} = \frac{1+y}{1-xy}.$$

(viii) If we give values to n and evaluate $\cos(2\pi n/3)$ we observe a repetitive pattern:

n	0	1	2	3	4	5	6	7	8	$\cdots$
$\cos(2\pi n/3)$	1	$-1/2$	$-1/2$	1	$-1/2$	$-1/2$	1	$-1/2$	$-1/2$	$\cdots$

If $n = 3k$ then $\cos(2\pi n/3) = 1$, otherwise $\cos(2\pi n/3) = -1/2$. Therefore

$$S = \sum_{n=0}^{\infty} \frac{1}{2^n} \cos\frac{2\pi n}{3} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^n} + \frac{3}{2} \sum_{k=0}^{\infty} \frac{1}{2^{3k}} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^n} + \frac{3}{2} \sum_{k=0}^{\infty} \frac{1}{8^k},$$

where we have summed up for all n as if $\cos(2\pi n/3) = -1/2$ always, and then we have corrected for the multiples of 3 so that the contribution of each one is $1(= -1/2 + 3/2)$. Then

$$S = -\frac{1}{2} \frac{1}{1-\frac{1}{2}} + \frac{3}{2} \frac{1}{1-\frac{1}{8}} = -1 + \frac{12}{7} = \frac{5}{7}.$$

Problem 4.6 The denominator factorises as $n(n+1)(n+2)$, and the elementary fractions expansion yields

$$\frac{1}{n^3+3n^2+2n} = \frac{1}{2}\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right) = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+1} + \frac{1}{n+2}\right)$$
$$= \underbrace{\frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right)}_{=u_n} - \underbrace{\frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)}_{=u_{n+1}}.$$

Since

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 0,$$

then

$$\sum_{n=1}^{\infty} \frac{1}{n^3+3n^2+2n} = u_1 = \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}.$$

Problem 4.7 Let r_n be the radius of the circle $\mathscr{C}_n$. The diagonal of $\mathscr{Q}_n$, the inscribed square will be $2r_n$, therefore its side will be $\sqrt{2}r_n$. This side is the diameter of $\mathscr{C}_{n+1}$, the circle inscribed in $\mathscr{Q}_n$, therefore

$$r_{n+1} = \frac{r_n}{\sqrt{2}}, \qquad r_0 = r,$$

and the areas of $\mathscr{C}_{n+1}$ and $\mathscr{C}_n$ will be related by

$$A_{n+1} = \pi r_{n+1}^2 = \pi \frac{r_n^2}{2} = \frac{A_n}{2}, \qquad A_0 = \pi r^2.$$

Solving this iterative equation is very easy, because applying successively the iteration,

$$A_1 = \frac{\pi r^2}{2}, \quad A_2 = \frac{\pi r^2}{4}, \quad A_3 = \frac{\pi r^2}{8}, \quad \cdots \quad A_n = \frac{\pi r^2}{2^n}, \quad \cdots$$

Thus,

$$\sum_{n=0}^{\infty} A_n = \pi r^2 \sum_{n=0}^{\infty} \frac{1}{2^n} = \pi r^2 \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \pi r^2 \frac{1}{1-1/2} = 2\pi r^2.$$

Problem 4.8 Taking the logarithm of the sequence

$$\log a_n = \frac{1}{2}\log 2 + \frac{1}{4}\log 2 + \frac{1}{8}\log 2 + \cdots + \frac{1}{2^n}\log 2 = \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}\right)\log 2,$$

therefore

$$\lim_{n\to\infty} \log a_n = \log 2 \sum_{n=1}^{\infty} \frac{1}{2^n} = \log 2 \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \log 2 \, \frac{\frac{1}{2}}{1-\frac{1}{2}} = \log 2,$$

which implies $\lim_{n\to\infty} a_n = 2$.

Problem 4.9

(a) The series can be written as

$$\sum_{n=0}^{\infty} \frac{b_n}{10^n} = b_0 + \sum_{n=1}^{\infty} \frac{b_n}{10^n},$$

so we will prove the convergence of the series on the right-hand side. The reason to separate out the term b_0 is because its nature is different from the rest of the coefficientes b_n. Now, this is a series of nonnegative terms and $b_n \leqslant 9$ for all $n \in \mathbb{N}$, so

$$\sum_{n=1}^{\infty} \frac{b_n}{10^n} \leqslant \sum_{n=1}^{\infty} \frac{9}{10^n} = 9 \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n = 9 \cdot \frac{1/10}{1-1/10} = 1.$$

The series converges by the comparison test. Not just that: its maximum value is 1, and it is reached when all $b_n = 9$.

(b) Take any real number, say $\pi = 3.141592653\ldots$. Its decimal expression is an integer number followed by a decimal point and an infinite sequence of digits (numbers in the set $\{0,1,2,\ldots,9\}$). The meaning of this positional expression is this:

$$\pi = 3 + 1\times 0.1 + 4\times 0.01 + 1\times 0.001 + 5\times 0.0001 + \cdots = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \cdots$$

In other words, the decimal expression of any real number is a series of the form of the one we are discussing. The fact that the series converges means that this representation is meaningful, and not just a formal description of the number. As a matter of fact, the proof that the series starting in $n = 1$ can be at most 1 means that the decimal part of the real number is always in the interval $[0,1]$, as it should.

(c) This question is asking for the value of the real number 9.999999... Written as a series,

$$9.999999\cdots = \sum_{n=0}^{\infty} \frac{9}{10^n} = 9 \cdot \frac{1}{1-1/10} = 10.$$

So $9.999999\cdots = 10$.

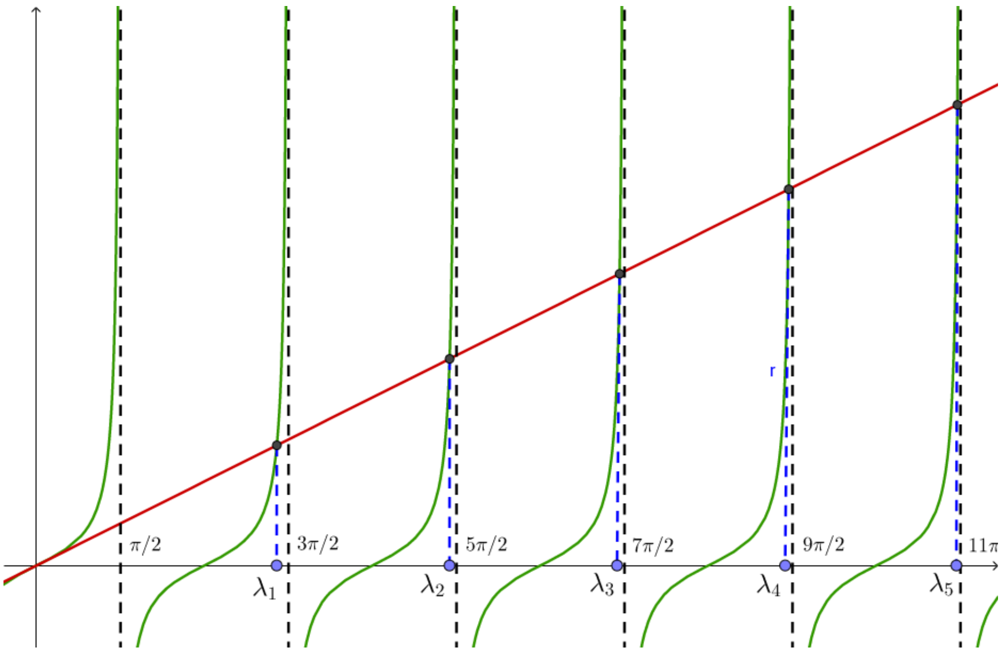
(d) Now we have to calculate the number 1.21212121...:

$$1.21212121\ldots = \sum_{n=0}^{\infty} \frac{b_n}{10^n} = \sum_{k=0}^{\infty} \frac{b_{2k}}{10^{2k}} + \sum_{k=0}^{\infty} \frac{b_{2k+1}}{10^{2k+1}} = \sum_{k=0}^{\infty} \frac{1}{10^{2k}} + \sum_{k=0}^{\infty} \frac{2}{10^{2k+1}}$$
$$= \sum_{k=0}^{\infty} \left(\frac{1}{100}\right)^k + \frac{2}{10} \sum_{k=0}^{\infty} \left(\frac{1}{100}\right)^k = \left(1+\frac{1}{5}\right) \frac{1}{1-1/100} = \frac{6}{5} \cdot \frac{100}{99} = \frac{120}{99}.$$

In other words, $1.21212121\cdots = 120/99$, a rational number (its decimal expression is periodic).

Problem 4.10

(a) The location of the points λ_n is illustrated in the following figure, which proves graphically the result:



(b) Since $\lambda_n > (2n-1)\pi/2$, then $\lambda_n^{-2} < 4/\pi^2(2n-1)^2$. But

$$\frac{4}{\pi^2(2n-1)^2} \sim \frac{1}{\pi^2}\frac{1}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

so the series $\sum_{n=1}^{\infty} \lambda_n^{-2} < \infty$ by the comparison test.

Problem 4.11

(a) The inequality $xy \leqslant (x^2+y^2)/2$ implies

$$\sqrt{a_n b_n} = \sqrt{a_n}\sqrt{b_n} \leqslant \frac{a_n+b_n}{2}.$$

The series

$$\sum_{n=1}^{\infty} \frac{a_n+b_n}{2} = \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n\right) < \infty$$

because the two series of the right-hand side converge. Therefore the series $\sum_{n=1}^{\infty} \sqrt{a_n b_n} < \infty$ by the comparison test.

(b) If we take $b_n = 1/n^2$, the result is a straightforward application of the previous result because $\sum_{n=1}^{\infty} n^{-2} < \infty$.

Problem 4.12

(a) Let $\mathscr{U}_k$ denote the set of positive integers with exactly k digits, none of which is zero. Clearly,

$$\sum_{n=1}^{\infty} \frac{1}{u_n} = \sum_{k=1}^{\infty} \sum_{u\in\mathscr{U}_k} \frac{1}{u},$$

which amounts to nothing but performing the sum grouping those terms corresponding to integers with the same number of digits. The advantage of doing this is that we know that the smallest integer in $\mathscr{U}_k$ will have all its digits equal to 1, and its largest all its digits equal to 9. Now, the smallest integer in $\mathscr{U}_k$ satisfies

$$\min \mathscr{U}_k = \underbrace{111\cdots 1}_{k \text{ digits}} > 1\underbrace{00\cdots 0}_{k-1 \text{ zeros}} = 10^{k-1},$$

therefore $u > 10^{k-1}$ for all $u \in \mathscr{U}_k$ and

$$\frac{1}{u} < \frac{1}{10^{k-1}}.$$

As there are 9^k integers in $\mathscr{U}_k$ (each of the k digits can be anything between 1 and 9), then

$$\sum_{u\in\mathscr{U}_k} \frac{1}{u} < \sum_{u\in\mathscr{U}_k} \frac{1}{10^{k-1}} = \frac{9^k}{10^{k-1}} = 9\left(\frac{9}{10}\right)^{k-1}.$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{u_n} < 9\sum_{k=1}^{\infty}\left(\frac{9}{10}\right)^{k-1} = 9\sum_{k=0}^{\infty}\left(\frac{9}{10}\right)^{k} = 9\frac{1}{1-9/10} = 90.$$

(b) We know that

$$\sum_{n=1}^{\infty} \frac{1}{n} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{u_n}}_{\text{integers with no zeros}} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{w_n}}_{\text{integers with zeros}}.$$

But the series of the left-hand side is the divergent harmonic series, and we have proven that the first series of the right-hand side is convergent, so the second one must be divergent.

Problem 4.13

(a) If we write $2 \cdot 4 \cdot 6 \cdots (2n) = (2 \cdot 1)(2 \cdot 2)(2 \cdot 3) \cdots (2 \cdot n)$ we see that there are n factors 2 in the product, and the remaining factors form the product $1 \cdot 2 \cdot 3 \cdots n = n!$, hence the result. As for the product $1 \cdot 3 \cdot 5 \cdots (2n-1)$, we need to realise that

$$(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots (2n-1) \cdot (2n) = 1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot 2 \cdot 4 \cdot 6 \cdots (2n);$$

hence

$$1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{(2n)!}{2 \cdot 4 \cdot 6 \cdots (2n)} = \frac{(2n)!}{n!2^n}.$$

(b) On the one hand,

$$2 \cdot 4 \cdot 6 \cdots (2n) = n!2^n \sim \sqrt{2\pi n}e^{-n}n^n 2^n = \sqrt{2\pi n}e^{-n}(2n)^n,$$

and on the other hand,

$$1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{(2n)!}{n!2^n} \sim \frac{\sqrt{4\pi n}e^{-2n}(2n)^{2n}}{\sqrt{2\pi n}e^{-n}(2n)^n} = \sqrt{2}e^{-n}(2n)^n.$$

(c) Let us calculate a few terms:

$$\begin{aligned}
S_2 &= -\log\frac{1}{2} + \log\frac{2}{3} = \log\left(\frac{2^2}{1 \cdot 3}\right), \\
S_4 &= -\log\frac{1}{2} + \log\frac{2}{3} - \log\frac{3}{4} + \frac{4}{5} = \log\left(\frac{2^2 \cdot 4^2}{1 \cdot 3^2 \cdot 5}\right), \\
S_6 &= -\log\frac{1}{2} + \log\frac{2}{3} - \log\frac{3}{4} + \frac{4}{5} - \log\frac{5}{6} + \log\frac{6}{7} = \log\left(\frac{2^2 \cdot 4^2 \cdot 6^2}{1 \cdot 3^2 \cdot 5^2 \cdot 7}\right),
\end{aligned}$$

so the pattern seems to hold. Let us prove the result by induction. Assume

$$S_{2k} = \log\left(\frac{2^2 \cdot 4^4 \cdot 6^2 \cdots (2k)^2}{1 \cdot 3^5 \cdot 5^2 \cdots (2k-1)^2(2k+1)}\right).$$

We must prove that

$$S_{2(k+1)} = \log\left(\frac{2^2 \cdot 4^4 \cdot 6^2 \cdots (2k)^2(2k+2)^2}{1 \cdot 3^5 \cdot 5^2 \cdots (2k-1)^2(2k+1)^2(2k+3)}\right).$$

In order to do it we notice that

$$\begin{aligned}
S_{2(k+1)} =& S_{2k} - \log\left(\frac{2k+1}{2k+2}\right) + \log\left(\frac{2k+2}{2k+3}\right) = \log\left(\frac{2^2 \cdot 4^4 \cdot 6^2 \cdots (2k)^2}{1 \cdot 3^5 \cdot 5^2 \cdots (2k-1)^2(2k+1)}\right) \\
&+ \log\left(\frac{(2k+2)^2}{(2k+1)(2k+3)}\right) = \log\left(\frac{2^2 \cdot 4^4 \cdot 6^2 \cdots (2k)^2(2k+2)^2}{1 \cdot 3^5 \cdot 5^2 \cdots (2k-1)^2(2k+1)^2(2k+3)}\right),
\end{aligned}$$

which is what we wanted to prove.

(d) We have

$$2^2 \cdot 4^4 \cdot 6^2 \cdots (2k)^2 \sim 2\pi k e^{-2k}(2k)^{2k}, \qquad 1 \cdot 3^5 \cdot 5^2 \cdots (2k-1)^2 \sim 2e^{-2k}(2k)^{2k},$$

therefore

$$\frac{2^2 \cdot 4^4 \cdot 6^2 \cdots (2k)^2}{1 \cdot 3^5 \cdot 5^2 \cdots (2k-1)^2(2k+1)} \sim \frac{\cancel{2}\pi k \cancel{e^{-2k}}\,\cancel{(2k)^{2k}}}{\cancel{2}\cancel{e^{-2k}}\,\cancel{(2k)^{2k}}} = \pi k,$$

so

$$S_{2k} \sim \log\left(\frac{\pi k}{2k+1}\right)$$

and therefore

$$\lim_{k\to\infty} S_{2k} = \log\left(\frac{\pi}{2}\right).$$

$\{S_{2k}\}_{k=1}^{\infty}$ is a subsequence of $\{S_k\}_{k=1}^{\infty}$. Since the latter converges the limit of the former must be the same. Hence

$$\lim_{k\to\infty} S_k = \sum_{n=1}^{\infty} (-1)^n \log\left(\frac{n}{n+1}\right) = \log\left(\frac{\pi}{2}\right).$$

Problem 4.14

(a) Following the hint, we can rewrite the general term of the series as

$$\begin{aligned} a_n &= \alpha_0 u_n + \alpha_1 u_{n+1} + \alpha_2 u_{n+2} = \alpha_0 u_n + (\alpha_0+\alpha_1)u_{n+1} - \alpha_0 u_{n+1} - (\alpha_0+\alpha_1)u_{n+2} \\ &= \left[\alpha_0 u_n + (\alpha_0+\alpha_1)u_{n+1}\right] - \left[\alpha_0 u_{n+1} + (\alpha_0+\alpha_1)u_{n+2}\right] = U_n - U_{n+1}, \end{aligned}$$

where $U_n \equiv \alpha_0 u_n + (\alpha_0+\alpha_1)u_{n+1}$. Thus

$$\sum_{n=1}^{\infty} (\alpha_0 u_n + \alpha_1 u_{n+1} + \alpha_2 u_{n+2}) = \alpha_0 u_1 + (\alpha_0+\alpha_1)u_2 - \lim_{n\to\infty}\left[\alpha_0 u_n + (\alpha_0+\alpha_1)u_{n+1}\right].$$

(b) Expanding the rational expression into elementary fractions we find

$$\frac{2n+1}{n(n+1)(n+2)} = \frac{1}{2}\left(\frac{1}{n} + \frac{2}{n+1} - \frac{3}{n+2}\right),$$

so $\alpha_0 = 1$, $\alpha_1 = 2$, $\alpha_2 = -3$ and $u_n = 1/n$. Therefore

$$\sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)(n+2)} = \frac{1}{2}(u_1 + 3u_2 - 0) = \frac{5}{4}.$$

(c) In the general case we can rewrite the general term as

$$\begin{aligned} a_n &= \alpha_0 u_n + \alpha_1 u_{n+1} + \alpha_2 u_{n+2} + \cdots + \alpha_{k-1}u_{n+k-1} + \alpha_k u_{n+k} \\ &= \alpha_0 u_n + (\boxed{\alpha_0} + \alpha_1)u_{n+1} + \alpha_2 u_{n+2} + \cdots + \alpha_{k-1}u_{n+k-1} + \alpha_k u_{n+k} \\ &\quad - \boxed{\alpha_0 u_{n+1}} \\ &= \alpha_0 u_n + (\alpha_0+\alpha_1)u_{n+1} + (\boxed{\alpha_0+\alpha_1} + \alpha_2)u_{n+2} + \cdots + \alpha_{k-1}u_{n+k-1} + \alpha_k u_{n+k} \\ &\quad - \alpha_0 u_{n+1} - \boxed{(\alpha_0+\alpha_1)u_{n+2}} \\ &= \cdots = \\ &= \alpha_0 u_n + (\alpha_0+\alpha_1)u_{n+1} + (\alpha_0+\alpha_1+\alpha_2)u_{n+2} + \cdots + (\boxed{\alpha_0+\cdots+\alpha_{k-2}} + \alpha_{k-1})u_{n+k-1} + \alpha_k u_{n+k} \\ &\quad - \alpha_0 u_{n+1} - (\alpha_0+\alpha_1)u_{n+2} - \cdots - \boxed{(\alpha_0+\alpha_1+\cdots+\alpha_{k-2})u_{n+k-1}}, \end{aligned}$$

and since $\alpha_k u_{n+k} = -(\alpha_0 + \cdots + \alpha_{k-2} + \alpha_{k-1})u_{n+k}$, we finally have $a_n = U_n - U_{n+1}$, where

$$U_n = \alpha_0 u_n + (\alpha_0+\alpha_1)u_{n+1} + (\alpha_0+\alpha_1+\alpha_2)u_{n+2} + \cdots + (\alpha_0 + \cdots + \alpha_{k-2} + \alpha_{k-1})u_{n+k-1}.$$

D.5 Limit of a Function

Problem 5.1

(i) We use the identity $x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1})$ and obtain

$$\lim_{x\to a} \frac{x^n - a^n}{x-a} = \lim_{x\to a} \frac{\cancel{(x-a)}(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1})}{\cancel{x-a}} = na^{n-1}.$$

(ii) We use the identity $x - a = (\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})$ and get

$$\lim_{x\to a} \frac{\sqrt{x} - \sqrt{a}}{x-a} = \lim_{x\to a} \frac{\cancel{\sqrt{x} - \sqrt{a}}}{\cancel{(\sqrt{x} - \sqrt{a})}(\sqrt{x} + \sqrt{a})} = \frac{1}{2\sqrt{a}}.$$

(iii) Here we need to use two identities. Since $64 = 8^2 = 4^3$,

$$x - 64 = (\sqrt{x} - 8)(\sqrt{x} + 8), \qquad x - 64 = (\sqrt[3]{x} - 4)\left(\sqrt[3]{x^2} + 4\sqrt[3]{x} + 16\right).$$

Then,

$$\lim_{x\to 64} \frac{\sqrt{x} - 8}{\sqrt[3]{x} - 4} = \lim_{x\to 64} \frac{\cancel{(x-64)}\left(\sqrt[3]{x^2} + 4\sqrt[3]{x} + 16\right)}{\cancel{(x-64)}(\sqrt{x} + 8)}$$

(iv) We can rewrite

$$1 - \sqrt{1-x^2} = \frac{\left(1 - \sqrt{1-x^2}\right)\left(1 + \sqrt{1-x^2}\right)}{1 + \sqrt{1-x^2}} = \frac{1 - (1-x^2)}{1 + \sqrt{1-x^2}} = \frac{x^2}{1 + \sqrt{1-x^2}}.$$

Therefore

$$\lim_{x\to 0} \frac{1 - \sqrt{1-x^2}}{x^2} = \lim_{x\to 0} \frac{\cancel{x^2}}{\cancel{x^2}\left(1 + \sqrt{1-x^2}\right)} = \lim_{x\to 0} \frac{1}{1 + \sqrt{1-x^2}} = \frac{1}{2}.$$

(v) We can rewrite

$$\frac{1}{(1-x)^3} - 1 = \frac{1 - (1-x)^3}{(1-x)^3} = \frac{1 - (1 - 3x + 3x^2 - x^3)}{(1-x)^3} = \frac{3x - 3x^2 + x^3}{(1-x)^3} = \frac{x(3 - 3x + x^2)}{(1-x)^3}.$$

Thus

$$\lim_{x\to 0} \frac{\frac{1}{(1-x)^3} - 1}{x} = \lim_{x\to 0} \frac{\cancel{x}(3 - 3x + x^2)}{\cancel{x}(1-x)^3} = \lim_{x\to 0} \frac{3 - 3x + x^2}{(1-x)^3} = 3.$$

(vi) We can rewrite

$$\frac{1}{\sqrt{x} - 1} = \frac{\sqrt{x} + 1}{(\sqrt{x} - 1)(\sqrt{x} + 1)} = \frac{\sqrt{x} + 1}{x - 1}.$$

Therefore

$$\begin{aligned}
\lim_{x\to 1} \left(\frac{1}{\sqrt{x} - 1} - \frac{2}{x-1}\right) &= \lim_{x\to 1} \left(\frac{\sqrt{x} + 1}{x-1} - \frac{2}{x-1}\right) = \lim_{x\to 1} \frac{\sqrt{x} + 1 - 2}{x-1} = \lim_{x\to 1} \frac{\sqrt{x} - 1}{x-1} \\
&= \lim_{x\to 1} \frac{\cancel{x-1}}{(\sqrt{x} + 1)\cancel{(x-1)}} = \lim_{x\to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{2}.
\end{aligned}$$

Problem 5.2

(i) If $x \to 0$ we know that $\sin 2x^3 \sim 2x^3$, so

$$\lim_{x\to 0} \frac{(\sin 2x^3)^2}{x^6} = \lim_{x\to 0} \frac{4\cancel{x^6}}{\cancel{x^6}} = 4.$$

(ii) We can divide numerator and denominator by x:

$$\ell = \lim_{x\to 0} \frac{\frac{\tan x^2}{x} + 2}{1+x} = \lim_{x\to 0} \frac{x\frac{\tan x^2}{x^2} + 2}{1+x}.$$

Now,

$$\lim_{x\to 0} \frac{\tan x^2}{x^2} = \lim_{x\to 0} \frac{1}{\cos x^2} \frac{\sin x^2}{x^2} = 1 \qquad \Rightarrow \qquad \lim_{x\to 0} x\frac{\tan x^2}{x^2} = 0.$$

Thus $\ell = 2$.

(iii) Expanding $\sin(x+a) = \sin x \cos a + \cos x \sin a$,

$$\begin{aligned}\ell &= \lim_{x\to 0} \frac{\sin(x+a) - \sin a}{x} = \lim_{x\to 0} \frac{\sin x \cos a - \sin a(1-\cos x)}{x} \\ &= \cos a \lim_{x\to 0} \frac{\sin x}{x} - \sin a \lim_{x\to 0} \frac{1-\cos x}{x} = \cos a \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}} - \sin a \underbrace{\lim_{x\to 0} \frac{x^2/2}{x}}_{=0} = \cos a,\end{aligned}$$

since $\sin x \sim x$ and $1 - \cos x \sim x^2/2$ when $x \to 0$.

(iv) This limit is an indeterminacy 1^∞, therefore

$$\ell = \lim_{x\to 0} (1+x)^{1/x} = e^c, \qquad c = \lim_{x\to 0} \frac{1}{x}(1+x-1) = \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}} = 1.$$

Thus $\ell = e$.

(v) Since $\log(1-2x) \sim -2x$ and $\sin x \sim x$ when $x \to 0$,

$$\lim_{x\to 0} \frac{\log(1-2x)}{\sin x} = \lim_{x\to 0} \frac{-2\cancel{x}}{\cancel{x}} = -2.$$

(vi) This limit is an indeterminacy 1^∞, therefore

$$\ell = \lim_{x\to 0} (1+\sin x)^{2/x} = e^c, \qquad c = \lim_{x\to 0} \frac{2}{x}(1+\sin x - 1) = \lim_{x\to 0} \frac{2\sin x}{x} = 2.$$

Thus $\ell = e^2$.

(vii) We can factor out $e^{\sin x}$ in the numerator to obtain

$$\ell = \lim_{x\to 0} \frac{e^x - e^{\sin x}}{x - \sin x} = \lim_{x\to 0} e^{\sin x} \frac{e^{x-\sin x} - 1}{x - \sin x} = \lim_{x\to 0} \frac{e^{x-\sin x} - 1}{x - \sin x},$$

because $\lim\limits_{x\to 0} e^{\sin x} = 1$. Now $x - \sin x \to 0$ as $x \to 0$, therefore $e^{x-\sin x} - 1 \sim x - \sin x$, so

$$\ell = \lim_{x\to 0} \frac{\cancel{x - \sin x}}{\cancel{x - \sin x}} = 1.$$

(viii) We can rewrite

$$\frac{\tan x - \sin x}{x^3} = \frac{\frac{\sin x}{\cos x} - \sin x}{x^3} = \frac{\sin x}{x} \cdot \frac{1 - \cos x}{x^2} \cdot \frac{1}{\cos x}.$$

But $\sin x \sim x$ and $1 - \cos x \sim x^2/2$ as $x \to 0$, so

$$\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}} \cdot \lim_{x\to 0} \frac{\cancel{x^2}/2}{\cancel{x^2}} \cdot \lim_{x\to 0} \frac{1}{\cos x} = 1 \cdot \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

(ix) First of all,

$$\lim_{x\to 0} \frac{x}{\sin x} = 1, \qquad \lim_{x\to 0} \frac{\sin x}{\sin x - x} = \lim_{x\to 0} \frac{\frac{\sin x}{x}}{\frac{\sin x}{x} - 1} = \infty,$$

so the limit is an indeterminacy 1^∞. Thus,

$$\ell = \lim_{x\to 0} \left(\frac{x}{\sin x}\right)^{\frac{\sin x}{\sin x - x}} = e^c, \qquad c = \lim_{x\to 0} \frac{\sin x}{\sin x - x}\left(\frac{x}{\sin x} - 1\right) = \lim_{x\to 0} \frac{\sin x}{\sin x - x} \cdot \frac{x - \sin x}{\sin x} = -1.$$

Therefore $\ell = 1/e$.

(x) Another indeterminacy 1^∞, so

$$\lim_{x\to 0} (\cos x)^{1/x^2} = e^c, \qquad c = \lim_{x\to 0} \frac{1}{x^2}(\cos x - 1) = \lim_{x\to 0} \frac{-\cancel{x^2}/2}{\cancel{x^2}} = -1/2.$$

Therefore $\ell = 1/\sqrt{e}$.

(xi) The best strategy here is to change the variable to $t = x - \pi$, so that $x \to \pi$ becomes $t \to 0$. Then

$$\sin(x/2) = \sin(\pi/2 + t/2) = \underbrace{\sin(\pi/2)}_{=1}\cos(t/2) + \underbrace{\cos(\pi/2)}_{=0}\sin(t/2) = \cos(t/2).$$

Then

$$\lim_{x\to\pi} \frac{1 - \sin(x/2)}{(x-\pi)^2} = \lim_{t\to 0} \frac{1 - \cos(t/2)}{t^2} = \lim_{t\to 0} \frac{\cancel{t^2}/8}{\cancel{t^2}} = \frac{1}{8},$$

where we have made use of the equivalence, valid for $t \to 0$,

$$1 - \cos(t/2) \sim \frac{(t/2)^2}{2} = \frac{t^2}{8}.$$

(xii) We first need to manipulate a little this expression. For that purpose we substract and add 1 to the numerator to write

$$\ell = \lim_{x\to 0} \frac{a^x - b^x}{x} = \lim_{x\to 0} \frac{a^x - 1 - b^x + 1}{x} = \lim_{x\to 0} \frac{a^x - 1}{x} - \lim_{x\to 0} \frac{b^x - 1}{x}.$$

We can calculate separately

$$\ell_a = \lim_{x\to 0} \frac{a^x - 1}{x}.$$

Whatever result is yields, the other limit will be the same replacing a by b. But first of all we must realise that $a^x = e^{x\log a}$, so that $a^x - 1 \sim e^{x\log a} - 1 \sim x\log a$ when $x \to 0$. Then

$$\ell_a = \lim_{x\to 0} \frac{e^{x\log a} - 1}{x} = \lim_{x\to 0} \frac{\cancel{x}\log a}{\cancel{x}} = \log a.$$

Therefore $\ell = \log a - \log b = \log(a/b)$.

Problem 5.3

(i) On the one hand, as $x \to \infty$,

$$x^3 + 4x - 7 = x^3 \left(1 + \frac{4}{x^2} - \frac{7}{x^3}\right) \sim x^3.$$

On the other hand,

$$7x^2 - \sqrt{2x^6 + x^5} = 7x^2 - x^3\sqrt{2 + \frac{1}{x}} = x^3 \left(\frac{7}{x} - \sqrt{2 + \frac{1}{x}}\right) \sim -\sqrt{2}x^3.$$

Therefore

$$\lim_{x\to\infty} \frac{x^3 + 4x - 7}{7x^2 - \sqrt{2x^6 + x^5}} = \lim_{x\to\infty} \frac{\cancel{x^3}}{-\sqrt{2}\cancel{x^3}} = -\frac{1}{\sqrt{2}}.$$

(ii) On the one hand, as $x \to \infty$,

$$x + \sin x^3 = x\left(1 + \frac{\sin x^3}{x}\right) \sim x$$

because $|\sin x^3| \leqslant 1$ for all $x \in \mathbb{R}$. On the other hand,

$$5x + 6 \sim 5x.$$

Therefore

$$\lim_{x\to\infty} \frac{x + \sin x^3}{5x + 6} = \lim_{x\to\infty} \frac{\cancel{x}}{5\cancel{x}} = \frac{1}{5}.$$

(iii) As $x \to \infty$,

$$\sqrt{x + \sqrt{x + \sqrt{x}}} = \sqrt{x}\sqrt{1 + \frac{1}{x}\sqrt{x + \sqrt{x}}} = \sqrt{x}\sqrt{1 + \sqrt{\frac{1}{x} + \frac{1}{x^{3/2}}}} \sim \sqrt{x},$$

thus

$$\lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x + \sqrt{x + \sqrt{x}}}} = \lim_{x\to\infty} \frac{\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}} = 1.$$

(iv) This is an indeterminacy $\infty - \infty$, so we must transform

$$\sqrt{x^2 + 4x} - x = \frac{\left(\sqrt{x^2 + 4x} - x\right)\left(\sqrt{x^2 + 4x} + x\right)}{\sqrt{x^2 + 4x} + x} = \frac{x^2 + 4x - x^2}{\sqrt{x^2 + 4x} + x} = \frac{4x}{\sqrt{x^2 + 4x} + x}.$$

Now, as $x \to \infty$,

$$\sqrt{x^2 + 4x} + x = x\left(\sqrt{1 + \frac{4}{x}} + 1\right) \sim 2x,$$

therefore

$$\lim_{x\to\infty} \left(\sqrt{x^2 + 4x} - x\right) = \lim_{x\to\infty} \frac{4x}{\sqrt{x^2 + 4x} + x} = \lim_{x\to\infty} \frac{4\cancel{x}}{2\cancel{x}} = 2.$$

(v) To begin with, $e^x - 1 \sim e^x$ as $x \to \infty$, so

$$\lim_{x\to\infty} \frac{e^x}{e^x - 1} = \lim_{x\to\infty} \frac{\cancel{e^x}}{\cancel{e^x}} = 1.$$

Now, since $\lim_{x\to-\infty} e^x = 0$,

$$\lim_{x\to-\infty} \frac{e^x}{e^x - 1} = 0.$$

(vi) When the sign of x is not defined —as in this case that we need to calculate the two limits when $x \to \pm\infty$— we can write

$$\sqrt{4x^2+1} = 2|x|\sqrt{1+\frac{1}{4x^2}}.$$

So as $x \to \pm\infty$

$$\sqrt{4x^2+1} \sim 2|x|.$$

Then,

$$\lim_{x\to\infty} \frac{x-2}{4x^2+1} = \lim_{x\to\pm\infty} \frac{\cancel{x}}{2\cancel{x}} = \frac{1}{2}, \qquad \lim_{x\to-\infty} \frac{x-2}{4x^2+1} = \lim_{x\to\pm\infty} \frac{\cancel{x}}{-2\cancel{x}} = -\frac{1}{2}.$$

(vii) We can express $\tanh x$ in different ways. Each one will be more suitable to calculate one specific limit. Thus, multiplying or dividing numerator and denominator by e^x,

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1} = \frac{1-e^{-2x}}{1+e^{-2x}}.$$

So

$$\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{e^{2x}-1}{e^{2x}+1} = \lim_{x\to\infty} \frac{\cancel{e^{2x}}}{\cancel{e^{2x}}} = 1,$$

and

$$\lim_{x\to-\infty} \tanh x = \lim_{x\to-\infty} \frac{1-e^{-2x}}{1+e^{-2x}} = \lim_{x\to-\infty} \frac{-\cancel{e^{-2x}}}{\cancel{e^{-2x}}} = -1,$$

(viii) We can rewrite

$$\frac{e^x}{\sinh x} = \frac{2e^x}{e^x - e^{-x}} = \frac{2e^{2x}}{e^{2x}-1} = \frac{2}{1-e^{-2x}},$$

so,

$$\lim_{x\to\infty} \frac{e^x}{\sinh x} = \lim_{x\to\infty} \frac{2e^{2x}}{e^{2x}-1} = \lim_{x\to\infty} \frac{2\cancel{e^{2x}}}{\cancel{e^{2x}}} = 2,$$

and

$$\lim_{x\to-\infty} \frac{e^x}{\sinh x} = \lim_{x\to-\infty} \frac{2}{1-e^{-2x}} = \lim_{x\to-\infty} \frac{2}{-e^{-2x}} = 0.$$

(ix) We are facing here an indeterminacy 1^∞, therefore

$$\lim_{x\to\pm\infty}\left(\frac{2x+7}{2x-6}\right)^{\sqrt{4x^2+x-3}} = e^c,$$

where

$$c = \lim_{x\to\pm\infty}\sqrt{4x^2+x-3}\left(\frac{2x+7}{2x-6}-1\right) = \lim_{x\to\pm\infty}\frac{13\sqrt{4x^2+x-3}}{2x-6}.$$

But $\sqrt{4x^2+x-3}\sim 2|x|$ as $x\to\pm\infty$, we have

$$\lim_{x\to\infty}\frac{13\sqrt{4x^2+x-3}}{2x-6} = \lim_{x\to\infty}\frac{26\cancel{x}}{2\cancel{x}} = 13, \qquad \lim_{x\to-\infty}\frac{13\sqrt{4x^2+x-3}}{2x-6} = \lim_{x\to-\infty}\frac{-26\cancel{x}}{2\cancel{x}} = -13.$$

Problem 5.4

(i) If x is a *positive* number close to 0 we have $\lfloor x\rfloor = 0$. If it is *negative,* $\lfloor x\rfloor = -1$. Therefore

$$\lim_{x\to0^+}\left(\frac{1}{x}\right)^{\lfloor x\rfloor} = \lim_{x\to0^+}\left(\frac{1}{x}\right)^{0} = \lim_{x\to0^+}1 = 1,$$

and

$$\lim_{x\to0^-}\left(\frac{1}{x}\right)^{\lfloor x\rfloor} = \lim_{x\to0^-}\left(\frac{1}{x}\right)^{-1} = \lim_{x\to0^-}x = 0.$$

(ii) We will change the variable x to $t = 1/x$:

$$\lim_{x\to0^+}e^{1/x} = \lim_{t\to+\infty}e^t = \infty, \qquad \lim_{x\to0^-}e^{1/x} = \lim_{t\to-\infty}e^t = 0.$$

(iii) We will change the variable x to $t = 1/x$:

$$\lim_{x\to0^+}\frac{1-e^{1/x}}{1+e^{1/x}} = \lim_{t\to+\infty}\frac{1-e^t}{1+e^t} = \lim_{t\to+\infty}\frac{-\cancel{e^t}}{\cancel{e^t}} = -1,$$

and

$$\lim_{x\to0^-}\frac{1-e^{1/x}}{1+e^{1/x}} = \lim_{t\to-\infty}\frac{1-e^t}{1+e^t} = 1.$$

D.6 Continuity

Problem 6.1

(a) The function $g(x) = |x|$ is a continuous function and $|f(x)| = (g \circ f)(x)$ is continuous because the composition of continuous functions is a continuous function.
As for the reciprocal, take

$$f(x) = \begin{cases} 1, & x \geqslant 0, \\ -1, & x < 0. \end{cases}$$

It is clearly a discontinuous function, however $|f(x)| = 1$ everywhere, which is continuous. This example illustrates that from the fact that $|f(x)|$ is continuous one cannot conclude that $f(x)$ itself is continuous.

(b) We are talking here about a function $f : \mathbb{R} \mapsto \mathbb{Q}$ that is continuous. One such function would necessarily be constant. Let us see why. Suppose that $f(x_1) = q_1$ and $f(x_2) = q_2 \neq q_1$. Since the function is continuous it must take all intermediate values between q_1 and q_2 within the interval $[x_1, x_2]$. But between any two rational numbers there are infinitely many irrational numbers, so there must exist $x \in (x_1, x_2)$ such that $f(x)$ is irrational. This is a contradiction and therefore $q_2 \neq q_1$ is not possible.

Problem 6.2

(a) The information that the function is surjective means that x_0 and x_1 in $[0,1]$ such that $f(x_0) = 0$ and $f(x_1) = 1$. Now, consider the interval $[x_0, x_1]$ (or $[x_1, x_0]$, depending on which one is bigger). The function $g(x) = f(x) - x$ is continuous (the sum of two continuous functions) and satisfies

$$g(x_0) = -x_0, \qquad g(x_1) = 1 - x_1.$$

If $x_0 = 0$ then $c = 0$ is the point we are looking for. If $x_1 = 1$ then $c = 1$ is that point. If none of these two things happen then $g(x_0) < 0$ and $g(x_1) > 0$ and we can apply Bolzano's theorem: there must exist $c \in (0,1)$ such that $g(c) = 0$ —which is equivalent to $f(c) = c$. Whichever the case, we can conclude that there exists $c \in [0,1]$ such that $f(c) = c$.

(b) Consider the number

$$\mu = \frac{1}{n} \sum_{k=1}^{n} f(x_k).$$

We can obtain a lower bound to μ by replacing in this expression all the $f(x_k)$ by the smallest one. Thus,

$$\mu \geqslant \min_{k=1,\dots,n} f(x_k).$$

Likewise, we can obtain an upper bound replacing them by the largest one:

$$\mu \leqslant \max_{k=1,\dots,n} f(x_k).$$

So μ is a value intermediate between two values that the function f takes in the interval $[a,b]$, therefore, since it is continuous, there must be a number $c \in [a,b]$ at which $f(c) = \mu$.

Problem 6.3 Since f is a rational function, all that it is required for it to be continuous is that the denominator does not vanish within the specified set.

(a) In this case the denominator must never vanish. If $\lambda = 0$ the function $f(x) = 1$ and trivially continuous in $\mathbb{R}$. Consider now $\lambda \neq 0$. Since in this case the denominator is a quadratic polinomial, the requirement that it never vanishes can be rephrase as its two roots being complex. The condition for that is that the discriminant is negative, so

$$4\lambda^2 - 4\lambda < 0 \quad \Leftrightarrow \quad \lambda(\lambda - 1) < 0.$$

This holds if each factor has a different sign, i.e., if $0 < \lambda < 1$. Therefore the function is continuous in $\mathbb{R}$ provided $\lambda \in [0, 1)$.

(b) Any of the values of λ found in the previous item make the function continuous in $\mathbb{R}$ —hence also in $[0,1]$—, so we just have to check what happens if $\lambda < 0$ or if $\lambda \geqslant 1$. In any of these two cases the denominator will have two real roots, so the key point is that none of them lies within the interval $[0,1]$ where we want $f(x)$ to be continuous.
By solving the quadratic equation we find the two roots as

$$x_1 = \frac{\lambda + \sqrt{\lambda(\lambda-1)}}{\lambda} = 1 + \sqrt{1-\lambda^{-1}}, \qquad x_2 = \frac{\lambda - \sqrt{\lambda(\lambda-1)}}{\lambda} = 1 - \sqrt{1-\lambda^{-1}}.$$

If $\lambda = 1$ both $x_1 = x_2 = 1$ and so f is not continuous at $x = 1$. Thus $\lambda \neq 1$ is required. In this case $x_1 > 1$, so it will always be outside the interval $[0,1]$. We can ignore it. On the contrary, $x_2 < 1$, so it will be also ouside the interval provided $x_2 < 0$. This condition implies $\sqrt{1-\lambda^{-1}} > 1$, which can only hold if $\lambda < 0$.
Summarising, $f(x)$ will be continuous in $[0,1]$ provided $\lambda < 1$.

Problem 6.4

(i) Numerator and denominator are continuous functions in $\mathbb{R}$, so this function will be continuous except when the denominator vanishes. It does when $x^2 - 8x + 12 = (x-6)(x-2) = 0$, so f is continuous in $\mathbb{R} - \{2, 6\}$.

(ii) The function is the sum of a plynomial (continuous in $\mathbb{R}$) and the function $e^{3/x}$. The exponential is continuous everywhere and the function $3/x$ too, except for $x = 0$. Besides,

$$\lim_{x \to 0^+} e^{3/x} = \infty,$$

so f is continous in $\mathbb{R} - \{0\}$.

(iii) Polynomials are continuous in $\mathbb{R}$ and so the tangent except when its argument is an odd multiple of $\pi/2$. This means the points

$$3x + 2 = n\pi + \frac{\pi}{2} \quad \Rightarrow x = \frac{n\pi - 2}{3} + \frac{\pi}{6}, \quad n \in \mathbb{Z}.$$

f is continuous except at these infinitely many points.

(iv) The polynomial is continuous in $\mathbb{R}$, so f is continuous wherever the argument of the square root is not negative. This means $x^2 - 5x + 6 = (x-3)(x-2) \geqslant 0$, which happens for $x \geqslant 3$ or $x \leqslant 2$. Thus f is continuous in $(-\infty, 2] \cup [3, \infty)$.

(v) $\arcsin x$ is only defined for $x \in [-1, 1]$, but in this region it is continuous because is the inverse of a continuous function. Thus f is continuous in $[-1, 1]$.

(vi) The polynomials are continuous everywhere, so the only requirement is that the argument of the logarithm is positive, i.e., $8x - 3 > 0$. Hence f is continuous in $(3/8, \infty)$.

(vii) This function represents the decimal part of x and is clearly discontinuous at the integers. Thus f is continuous in $\mathbb{R} - \mathbb{Z}$.

(viii) The polynomial and the sine function are both continuous everywhere, and so is $1/x$ except at $x = 0$. Function f is defined at $x = 0$ though, so we must check the definition of continuity at this specific point. Since $|x^2 \sin(1/x)| \leqslant x^2$ and $x^2 \to 0$ as $x \to 0$, then

$$\lim_{x\to 0} f(x) = 0 = f(0)$$

and f is continuous in $\mathbb{R}$.

(ix) For $x > 0$ the function is continuous except for $x = (2n-1)\pi/2$, $n \in \mathbb{N}$. For $x < 0$ the function is always continuous. We must compute the two one-sided limits at $x = 0$ to check for continuity at that point. Now,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\tan x}{\sqrt{x}} = \lim_{x\to 0^+} \frac{x}{\sqrt{x}} = \lim_{x\to 0^+} \sqrt{x} = 0.$$

And on the other side,

$$\lim_{x\to 0^-} e^{1/x} = \lim_{t\to -\infty} e^t = 0.$$

Thus,

$$\lim_{x\to 0} f(x) = 0 = f(0),$$

so f is continuous in $\mathbb{R} - \{(2n-1)\pi/2 : n \in \mathbb{N}\}$.

(x) As close as we like to a rational number there is always an irrational number. As close as we like to an irrational number there is always a rational number. So, f is discontinuous at every $x \neq 0$. At $x = 0$ function $f(x)$ is continuous though. The reason is that $|f(x)| = |x| \to 0$ as $x \to 0$, so

$$\lim_{x\to 0} f(x) = 0 = f(0).$$

(xi) Each piece of this piecwise function separately is a continuous function, so we just need to check what happens at the joints. Thus,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-1)^3 = 0, \qquad \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (|x| - x) = 0,$$

so

$$\lim_{x\to 1} f(x) = 0 = f(1).$$

And

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} (|x| - x) = 2, \qquad \lim_{x\to -1^+} f(x) = \lim_{x\to -1^-} \sin(\pi x) = 0,$$

so $f(x)$ is continuous in $\mathbb{R} - \{-1\}$.

(xii) The two polynomials defining the function for $|x| \geqslant 1$ are continuous function. In $(-1,1)$ the function is defined as $\operatorname{sgn} x + 1$, which is continuous except at $x = 0$. We now need to check the two joints. Thus,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x = 2, \qquad \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (\operatorname{sgn} x + 1) = 2,$$

so

$$\lim_{x\to 1} f(x) = 2 = f(1).$$

And

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} (\operatorname{sgn} x + 1) = 0, \qquad \lim_{x\to -1^+} f(x) = \lim_{x\to -1^-} (x+1)^2 = 0,$$

so

$$\lim_{x\to -1} f(x) = 0 = f(-1).$$

Summarising, $f(x)$ is continuous in $\mathbb{R} - \{0\}$.

(xiii) Each of the three pieces of this piecewise function is continuous (a polynomial or the absolute value of a polynomial), so we need to check just the joints. Thus,

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (4x-5) = 3, \qquad \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} |x^2-1| = 3,$$

so

$$\lim_{x\to 2} f(x) = 3 = f(2).$$

And

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} |x^2-1| = 3, \qquad \lim_{x\to -2^+} f(x) = \lim_{x\to -2^-} x^2 = 4,$$

so $f(x)$ is continuous in $\mathbb{R} - \{-2\}$.

(xiv) The functions defining $f(x)$ for $|x| > 1$ are both polynomials —hence continuous. Within $|x| \leqslant 1$ it is defined as $g(x) = x - \lfloor x \rfloor$. Now, $g(x) = x+1$ for all $-1 \leqslant x < 0$, $g(x) = x$ for all $0 \leqslant x < 1$, and $g(1) = 0$. Thus function $f(x)$ can be redefined as

$$f(x) = \begin{cases} (x-1)^2, & x \geqslant 1, \\ x, & 0 \leqslant x < 1, \\ x+1, & x < 0. \end{cases}$$

All three pieces are continuous (polynomials), so we must look at the joints. So,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-1)^2 = 0, \qquad \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1,$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0, \qquad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x+1) = 1.$$

Therefore the $f(x)$ is continuous in $\mathbb{R} - \{0,1\}$.

Problem 6.5

(i) Denoting $f(x) = x^2 - 18x + 2$, a continuous function in $\mathbb{R}$, we have $f(-1) = 21$, $f(1) = -15$, so Bolzano's theorem guarantees at least one zero in $[-1,1]$.

(ii) Denoting $f(x) = x - \sin x - 1$, a continuous function in $\mathbb{R}$, we have $f(0) = -1$ and $f(\pi) = \pi - 1 > 0$, so Bolzano's theorem guarantees at least one zero in $[0,\pi]$.

(iii) Since $e^x > 0$, we know that $e^x + 1 > 0$, so the equation cannot have any solution in $\mathbb{R}$.

(iv) Since $-1 \leqslant \cos x \leqslant 1$ for all $x \in \mathbb{R}$, the equation $\cos x = -2$ cannot have any solution in $\mathbb{R}$.

(v) $f(x) > 0$ for all $-2 \leqslant x < 0$ and $f(x) < 0$ for all $0 \leqslant x \leqslant 2$. If $f(x)$ where continuous this would imply that $f(0) = 0$. But the function is not continuous at $x = 0$ ($f(0^-) = 2$, $f(0^+) = -2$), so there is no solution to the equation $f(x) = 0$ in $[-2,2]$.

(vi) Denoting

$$f(x) = \frac{x^3}{4} - \sin(\pi x) + 3 - \frac{7}{3} = \frac{x^3}{4} - \sin(\pi x) + \frac{2}{3},$$

$f(-2) = -4/3$ and $f(2) = 8/3$, so Bolzano's theorem guarantees at least one zero in $[-2,2]$.

(vii) Clearly $|\sin x| - \sin x \leqslant 2$, so the equation $|\sin x| - \sin x = 3$ cannot have any solution in $\mathbb{R}$.

Problem 6.6 If $f(x) = a_{2n+1}x^{2n+1} + a_{2n}x^{2n} + \cdots + a_1x + a_0$ then, as $x \to \pm\infty$ we have $f(x) \sim a_{2n+1}x^{2n+1}$. Therefore the signs of $f(x)$ for large positive x and large negative x are opposite, so we can apply Bolzano and conclude that $f(x)$ must be zero at least at one point in $\mathbb{R}$.

D.7 Derivatives

Problem 7.1

(i)

$$h'(x) = \frac{f(x)f'(x)+g(x)g'(x)}{\sqrt{f(x)^2+g(x)^2}}.$$

(ii)

$$h'(x) = \frac{1}{1+\left(\frac{f(x)}{g(x)}\right)^2} \cdot \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2} = \frac{f(x)g'(x)-f'(x)g(x)}{f(x)^2+g(x)^2}.$$

(iii)

$$h'(x) = f'\big(g(x)\big)g'(x)e^{f(x)} + f\big(g(x)\big)f'(x)e^{f(x)} = \left[f'\big(g(x)\big)g'(x) + f\big(g(x)\big)f'(x)\right]e^{f(x)}.$$

(iv) First of all $h(x) = \log\big(g(x)\big) + \log\big(\sin f(x)\big)$, so

$$h'(x) = \frac{g'(x)}{g(x)} + \frac{f'(x)\cos f(x)}{\sin f(x)} = \frac{g'(x)}{g(x)} + f'(x)\cot f(x).$$

(v) We first write $f(x)^{g(x)} = \exp\{g(x)\log f(x)\}$. Then

$$\begin{aligned} h'(x) &= \left[g'(x)\log f(x) + \frac{g(x)f'(x)}{f(x)}\right]\exp\{g(x)\log f(x)\} \\ &= \left[g'(x)\log f(x) + \frac{g(x)f'(x)}{f(x)}\right] f(x)^{g(x)} \\ &= f(x)^{g(x)}g'(x)\log f(x) + g(x)f'(x)f(x)^{g(x)-1}. \end{aligned}$$

(vi)

$$h'(x) = -\frac{1}{\left[\log\big(f(x)+g(x)^2\big)\right]^2} \cdot \frac{f'(x)+2g(x)g'(x)}{f(x)+g(x)^2}.$$

Problem 7.2 In both items we are asked to figure out a function $g(x)$ such that

$$f(x) = \begin{cases} 1, & |x| \leqslant 1, \\ 0, & |x| \geqslant 2, \\ g(x), & 1 < x < 2, \\ g(-x), & -2 < x < -1, \end{cases}$$

is the requested function.

(a) For $f(x)$ to be continuous we need $g(x)$ to be continuous and fulfill the two conditions $g(1) = 1$, $g(2) = 0$. The simplest such function is the straight line $g(x) = ax + b$, for which these two conditions imply

$$\begin{cases} a+b = 1, \\ 2a+b = 0, \end{cases} \qquad \Leftrightarrow \qquad a = -1, \quad b = 2.$$

Thus $g(x) = -x + 2$.

(b) Since the derivative of f for $|x| < 1$ and $|x| > 2$ is 0, now we need $g(x)$ to satisfy also $g'(1) = g'(2) = 0$. These are four equations, so the simplests function would be a polynomial with four unknown coefficients, namely $g(x) = ax^3 + bx^2 + cx + d$. But we can determine the polynomial more precisely given the information we have. For instance, the fact that $g(1) = 0$ means that $g(x) = (x-1)q(x)$, with $q(x)$ is a second degree polynomial. Given this expression, $g'(x) = q(x) + (x-1)q'(x)$, so $0 = g'(1) = q(1)$, and this implies $q(x) = (x-1)r(x)$, with $r(x)$ a lineal polynomial. In other words,

$$g(x) = (x-1)^2(ax+b), \qquad g'(x) = 2(x-1)(ax+b) + a(x-1)^2.$$

We can now impose the constraints $g(2) = 1$, $g'(2) = 0$, and this leads to

$$\begin{cases} 2a+b=1, \\ 2(2a+b)+a=0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} 2a+b=1, \\ 5a+2b=0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} a=-2, \\ b=5. \end{cases}$$

Thus $g(x) = (x-1)^2(5-2x)$.

There is a simpler way to achieve the same result though. It amounts to finding a continuous and differentiable function with a local maximum and a local minimum. One such function is $\cos^2 a(x-b)$. This function reaches a maximum at $x = b$, where it is 1, and a minimum at $a(x-b) = \pi/2$, where it is 0. If we want the maximum to be at $x = 1$ then we must choose $b = 1$. If we want the minimum to be at $x = 2$ we must choose $a(2-1) = \pi/2$, i.e., $a = \pi/2$. Thus $g(x) = \cos^2 \frac{\pi}{2}(x-1)$.

Problem 7.3

(i) $f'(x) = -\dfrac{c}{x^2}$, therefore

$$xf' + f = -\frac{c}{x} + \frac{c}{x} = 0.$$

(ii) $f'(x) = \tan x + x(1+\tan^2 x)$, therefore

$$xf' - f - f^2 = x\tan x + x^2 - x^2\tan^2 x - x\tan x - x^2\tan^2 x = x^2.$$

(iii) $f'(x) = 3c_1\cos 3x - 3c_2\sin 3x$ and $f''(x) = -9c_1\sin 3x - 9c_2\cos 3x$, therefore

$$f'' + 9f = -9c_1\sin 3x - 9c_2\cos 3x + 9(c_1\sin 3x + c_2\cos 3x) = 0.$$

(iv) $f'(x) = 3c_1e^{3x} - 3c_2e^{-3x}$ and $f''(x) = 9c_1e^{3x} + 9c_2e^{-3x}$, therefore

$$f'' - 9f = 9c_1e^{3x} + 9c_2e^{-3x} - 9(c_1e^{3x} + c_2e^{-3x}) = 0.$$

(v) $f'(x) = 2c_1e^{2x} + 5c_2e^{5x}$ and $f''(x) = 4c_1e^{2x} + 25c_2e^{5x}$, therefore

$$\begin{aligned} f'' - 7f' + 10f &= 4c_1e^{2x} + 25c_2e^{5x} - 7(2c_1e^{2x} + 5c_2e^{5x}) + 10(c_1e^{2x} + c_2e^{5x}) \\ &= (4-14+10)e^{2x} + (25-35+10)e^{5x} = 0. \end{aligned}$$

(vi) $f'(x) = \dfrac{c_1e^x - e^{-x}}{c_1e^x + e^{-x}}$ and

$$f''(x) = \frac{(c_1e^x + e^{-x})^2 - (c_1e^x - e^{-x})^2}{(c_1e^x + e^{-x})^2} = 1 - \left(\frac{c_1e^x - e^{-x}}{c_1e^x + e^{-x}}\right)^2,$$

therefore

$$f'' - (f')^2 = 1 - \left(\frac{c_1e^x - e^{-x}}{c_1e^x + e^{-x}}\right)^2 + \left(\frac{c_1e^x - e^{-x}}{c_1e^x + e^{-x}}\right)^2 = 0.$$

Problem 7.4

(i) Differentiating $f(x) = \arctan x + \arctan \frac{1}{x}$,

$$\frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}}\left(-\frac{1}{x^2}\right) = \frac{1}{1+x^2} - \frac{1}{x^2+1} = 0.$$

Therefore $f(x) = c$, a constant. To find out which constant we must evaluate $f(x)$ at any point $x > 0$, say $x = 1$. Then $f(1) = c = \arctan 1 + \arctan 1 = 2\pi/4 = \pi/2$.

(ii) Differentiating $f(x) = \arctan \frac{1+x}{1-x} - \arctan x$,

$$f'(x) = \frac{1}{1+\left(\frac{1+x}{1-x}\right)^2} \frac{1-x+1+x}{(1-x)^2} - \frac{1}{1+x^2} = \frac{2}{(1-x)^2+(1+x)^2} - \frac{1}{1+x^2}$$
$$= \frac{2}{1-2x+x^2+1+2x+x^2} - \frac{1}{1+x^2} = \frac{2}{2+2x^2} - \frac{1}{1+x^2} = 0.$$

Therefore $f(x) = c$, a constant. To find out which constant we must evaluate $f(x)$ at any point $x < 1$, say $x = 0$. Then $f(0) = c = \arctan 1 + \arctan 0 = \pi/4$.

(iii) Differentiating $f(x) = 2\arctan x + \arcsin \frac{2x}{1+x^2}$,

$$f'(x) = \frac{2}{1+x^2} + \frac{1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}} \frac{2(1+x^2)-2x\cdot 2x}{(1+x^2)^2}$$
$$= \frac{2}{1+x^2} + \frac{1+x^2}{\sqrt{(1+x^2)^2-4x^2}} \frac{2(1-x^2)}{(1+x^2)^2} = \frac{2}{1+x^2} + \frac{2(1-x^2)}{(1+x^2)\sqrt{(1-x^2)^2}}$$
$$\underset{(*)}{=} \frac{2}{1+x^2} + \frac{2(1-x^2)}{(1+x^2)(x^2-1)} = \frac{2}{1+x^2} - \frac{2}{1+x^2} = 0,$$

where in (*) we have used the fact that $x \geqslant 1$ implies that $\sqrt{(1-x^2)^2} = x^2 - 1 \geqslant 0$. Therefore $f(x) = c$, a constant. To find out which constant we must evaluate $f(x)$ at any point $x \geqslant 1$, say $x = 1$. Then $f(1) = c = 2\arctan 1 + \arcsin 1 = 2\pi/4 + \pi/2 = \pi$.

Problem 7.5 If we calculate $f'(x) = 1 + \frac{1}{3}(\sin x)^{-2/3}\cos x$ we observe that this function diverges whenever $\sin x = 0$, i.e., for $x = n\pi$ with $n \in \mathbb{Z}$. Those are the points where the tangent straight line is vertical.

Problem 7.6 Let us calculate the derivative on the left, $f'(0^-)$ and on the right, $f'(0^+)$. Since $f(0) = 0$,

$$f'(0^-) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^-} \frac{1}{1+e^{1/x}} = \lim_{t\to -\infty} \frac{1}{1+e^t} = 1,$$
$$f'(0^+) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^+} \frac{1}{1+e^{1/x}} = \lim_{t\to \infty} \frac{1}{1+e^t} = 0.$$

So the slope of the tangent on the left is 1 —hence it forms an angle $\pi/4$ with the X axis— and that on the right is 0 —hence it is parallel to the X axis. Thus the angle between both tangents is $\pi/4$.

Problem 7.7 The domain of this function requires that $x+2 \geqslant 0$ and $-1 \leqslant x+2 \leqslant 1$ be satisfied simultaneously. This happens for x such that $0 \leqslant x+2 \leqslant 1$, in other words, for $x \in [-2,-1]$. Within this domain the function is continuous because so are $x+2$, $\sqrt{x}$, and $\cos x$ —hence its inverse— in their respective domains.

About differentiability,

$$f'(x) = \frac{\arccos(x+2)}{2\sqrt{x+2}} - \frac{\sqrt{x+2}}{\sqrt{1-(x+2)^2}} = \frac{\arccos(x+2)}{2\sqrt{x+2}} - \sqrt{\frac{x+2}{-3-4x-x^2}},$$

which diverges when $x=-2$ and is defined only if $x^2+4x+3=(x+1)(x+3)<0$. This happens for $x\in(-3,-1)$, an interval that overlaps with the domain excluding the point $x=-1$. Thus the derivative exits only for $x\in(-2,-1)$.

Problem 7.8 Function $f(x)$ will be differentiable if and only if $\alpha x^2-x+3\geqslant 0$ for all $x\in\mathbb{R}$ or $\alpha x^2-x+3\leqslant 0$ for all $x\in\mathbb{R}$. The reason is that in either of these two cases the parabola does not cross the X axis or it just touches the axis at one point (it is only if the parabola crosses the axis that its absolute value generates points with no derivative). The condition for this to happen is that the discriminant of the parabola be $\leqslant 0$, i.e., $1-12\alpha\leqslant 0$. Thus $\alpha\geqslant 1/12$.

Problem 7.9 Function $f(x)$ is even, so it is enough to make sure that it is continuous and differentiable at $x=c$. The function will be continuous at $x=c$ if

$$a+bc^2=\frac{1}{c}.$$

On the other hand, for $x\geqslant 0$ the function is

$$f(x)=\begin{cases} a+bx^2, & 0\leqslant x\leqslant c,\\ \dfrac{1}{x}, & x>c,\end{cases}$$

so its derivative will be

$$f'(x)=\begin{cases} 2bx, & 0\leqslant x<c,\\ -\dfrac{1}{x^2}, & x>c,\end{cases}$$

and therefore $f(x)$ will be differentiable at $x=c$ if

$$2bc=-\frac{1}{c^2} \qquad \Leftrightarrow \qquad b=-\frac{1}{2c^3}.$$

And from the previous equation we obtain

$$a=\frac{1}{c}-bc^2=\frac{1}{c}+\frac{1}{2c}=\frac{3}{2c}.$$

So for $|x|<c$ the function is defined as

$$f(x)=\frac{1}{2c}\left(3-\frac{x^2}{c^2}\right).$$

Problem 7.10

(a) The two pieces defining this function are continuous and differentiable within their respective sets, so the only critical point is $x=1$. Let us first check the continuity at this point. So

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}\frac{1}{x}=1, \qquad \lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\frac{3-x^2}{2}=1,$$

hence

$$\lim_{x\to 1} f(x) = 1 = f(1),$$

which proves that the function is continuous also at this point. As for differentiability,

$$f'(1^+) = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{\frac{1}{x}-1}{x-1} = \lim_{x\to 1^+} \frac{1-x}{x(x-1)} = -1,$$

$$\begin{aligned} f'(1^-) &= \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{\frac{3-x^2}{2}-1}{x-1} = \lim_{x\to 1^-} \frac{1-x^2}{2(x-1)} = \lim_{x\to 1^-} \frac{(1-x)(1+x)}{2(x-1)} \\ &= \lim_{x\to 1^-} \frac{-(1+x)}{2} = -1, \end{aligned}$$

so f is differentiable at this point and $f'(1) = -1$. Summarising, f is continuous and differentiable in $\mathbb{R}$.

(b) Given that f is differentiable in $\mathbb{R}$, there must exist $c \in (0,2)$ such that

$$f(2) - f(0) = f'(c)(2-0) \quad \Leftrightarrow \quad \frac{1}{2} - \frac{3}{2} = 2f'(c) \quad \Leftrightarrow \quad -\frac{1}{2} = f'(c).$$

We do not know whether c is in $(0,1)$ or in $[1,2)$, so we have to check both. We have

$$f'(x) = \begin{cases} -x, & x < 1, \\ -\dfrac{1}{x^2}, & x \geqslant 1. \end{cases}$$

Assuming $0 < c < 1$, the equation becomes

$$-\frac{1}{2} = -c \quad \Rightarrow \quad c = \frac{1}{2}.$$

Assuming $1 \leqslant c < 2$, the equation becomes

$$-\frac{1}{2} = -\frac{1}{c^2} \quad \Rightarrow \quad c = \sqrt{2}.$$

Problem 7.11 The derivative is

$$f'(x) = -\frac{2}{3x^{1/3}},$$

so f is not differentiable at $x = 0$. This is the hypothesis that is not met.

Problem 7.12

(i) Assume that $a \leqslant x_1 < x_2 < \cdots < x_{k-1} < x_k \leqslant b$ are the k points where f vanishes in $[a,b]$. In any of the $k-1$ intervals $[x_j, x_{j+1}]$, with $j = 1, 2, \ldots, k-1$, we can apply Rolle's theorem and conclude that there must be at least a point in each of them where f' vanishes. This means that f' vanishes at least $k-1$ times in (a,b) —hence in $[a,b]$.

(ii) We can recursively apply the previous result and obtain

$$\begin{aligned} f \text{ vanishes } n+1 \text{ times in } [a,b] \quad &\Rightarrow \quad f' \text{ vanishes } n \text{ times in } [a,b] \\ f' \text{ vanishes } n \text{ times in } [a,b] \quad &\Rightarrow \quad f'' \text{ vanishes } n-1 \text{ times in } [a,b] \\ f'' \text{ vanishes } n-1 \text{ times in } [a,b] \quad &\Rightarrow \quad f''' \text{ vanishes } n-2 \text{ times in } [a,b] \\ &\vdots \\ f^{(n-1)} \text{ vanishes 2 times in } [a,b] \quad &\Rightarrow \quad f^{(n)} \text{ vanishes 1 time in } [a,b]. \end{aligned}$$

Problem 7.13 Let us consider the function $f(x) = x^{2/3}$ in the interval $[26,27]$. By the mean value theorem

$$27^{2/3} - 26^{2/3} = \frac{2}{3c^{1/3}}, \quad 26<c<27,$$

so

$$26^{2/3} = 9 - \frac{2}{3c^{1/3}}, \quad 26<c<27.$$

Approximating $c \approx 27$ we obtain

$$26^{2/3} \approx 9 - \frac{2}{9} = \frac{79}{9} = 8.777777\ldots$$

The exact value is $8.776382955\ldots$

Taking now $g(x) = \log x$ in $[1,3/2]$ we can write

$$\log(3/2) = \frac{1}{c}\left(\frac{3}{2} - 1\right) = \frac{1}{2c}, \quad 1 < c < \frac{3}{2}.$$

From this we conclude

$$\frac{1}{3} < \log(3/2) < \frac{1}{2} \quad \Leftrightarrow \quad 0.3333333\cdots < \log(3/2) < 0.5.$$

The exact vale is $0.405465108\ldots$

Problem 7.14

(i) We can obtain the limit

$$\ell = \lim_{x\to 0} \frac{e^x - \sin x - 1}{x^2}$$

by applying l'Hôpital's rule twice, as

$$\ell = \lim_{x\to 0} \frac{e^x + \sin x}{2} = \frac{1}{2}.$$

(ii) We can obtain the limit

$$\ell = \lim_{x\to 0} \frac{\log|\sin 7x|}{\log|x|}$$

by applying l'Hôpital's rule as

$$\ell = \lim_{x\to 0} \frac{7\cos 7x \sin x}{\sin 7x \cos x} = \lim_{x\to 0} \frac{7\cos 7x}{\cos x} \lim_{x\to 0} \frac{\sin x}{\sin 7x} = 7 \lim_{x\to 0} \frac{\sin x}{\sin 7x},$$

and then again,

$$\ell = 7 \lim_{x\to 0} \frac{\cos x}{7\cos 7x} = 1.$$

(iii) Writing the limit as

$$\ell = \lim_{x\to 1^+} \frac{\log(x-1)}{\frac{1}{\log x}}$$

it becomes a ∞/∞ indeterminacy, which we can sort out using l'Hôpital's rule. Thus,

$$\ell = \lim_{x\to 1^+} \frac{\frac{1}{x-1}}{-\frac{1}{x(\log x)^2}} = \lim_{x\to 1^+} \frac{-x(\log x)^2}{x-1} = -\lim_{x\to 1^+} \frac{(\log x)^2}{x-1}.$$

And we can solve this $0/0$ indeterminacy by applying l'Hôpital's rule once more to obtain

$$-\lim_{x\to 1^+} \frac{2\log x}{x} = 0.$$

Therefore $\ell = 0$.

(iv) This limit can be written as

$$\ell = \lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\log x/x} = \exp\left\{\lim_{x\to\infty} \frac{\log x}{x}\right\}.$$

This new limit can be obtain by applying l'Hôpital's rule as

$$\lim_{x\to\infty} \frac{1}{x} = 0,$$

therefore $\ell = 1$.

(v) The limit

$$\ell = \lim_{x\to 0} \frac{(1+x)^{1+x} - 1 - x - x^2}{x^3}$$

is a $0/0$ indeterminacy, which can be solved by applying l'Hôpital's rule three times. The denominator becomes then 6. As for the numerator, $(1-x-x^2)''' = 0$, so we have to take three derivatives of $g(x) = (1+x)^{1+x} = e^{(1+x)\log(1+x)}$. Thus,

$$\begin{aligned}
g'(x) &= g(x)\big[\log(1+x)+1\big], \\
g''(x) &= g(x)\big[\log(1+x)+1\big]^2 + \frac{g(x)}{1+x}, \\
g'''(x) &= g(x)\big[\log(1+x)+1\big]^3 + 3g(x)\frac{\log(1+x)+1}{1+x} - \frac{g(x)}{(1+x)^2}.
\end{aligned}$$

Therefore

$$\ell = \frac{1}{6}\lim_{x\to 0}\left\{g(x)\big[\log(1+x)+1\big]^3 + 3g(x)\frac{\log(1+x)+1}{1+x} - \frac{g(x)}{(1+x)^2}\right\} = \frac{1}{2}.$$

(vi) We can change the variable x to $t = 1/x$. Then

$$\ell = \lim_{x\to\infty} x\left(\tan\frac{2}{x} - \tan\frac{1}{x}\right) = \lim_{t\to 0^+} \frac{\tan 2t - \tan t}{t}.$$

We can solve this $0/0$ indeterminacy by applying l'Hôpital's rule to obtain

$$\ell = \lim_{t\to 0^+}\left(\frac{2}{\cos^2 2t} - \frac{1}{\cos^2 t}\right) = 1.$$

Problem 7.15 First of all, $h(0)=0$ because if $h(0)=c\neq 0$, then

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{h(x)}{x^2}=\pm\infty,$$

so f would not be continuous at $x=0$.

Now, since the limit above is a $0/0$ indeterminacy we can try to apply l'Hôpital's rule and calculate

$$\lim_{x\to 0}\frac{h'(x)}{2x}.$$

As h is twice differentiable $h'(x)\to h'(0)$ as $x\to 0$. For the same reason as above $h'(0)=0$, otherwise it would be $\pm\infty$ and

$$\lim_{x\to 0}\frac{h(x)}{x^2}=\pm\infty,$$

again in contradiction with the fact that f is continuous at $x=0$.

Finally, once stablished that $h'(0)=0$ we can rewrite

$$\lim_{x\to 0}\frac{h'(x)}{2x}=\frac{1}{2}\lim_{x\to 0}\frac{h'(x)-h'(0)}{x}=\frac{h''(0)}{2}.$$

This limit has to be 1 if f is to be continuous at $x=0$, thus $h''(0)=2$.

Problem 7.16

(i) We can change the variable x to $t=1/x$ to tranform the limit

$$\ell=\lim_{x\to\infty} x\left[\left(1+\frac{1}{x}\right)^x-e\right]=\lim_{t\to 0^+}\frac{(1+t)^{1/t}-e}{t}.$$

Since $(1+t)^{1/t}\to e$ as $t\to 0^+$ we face a $0/0$ indeterminacy. Let us apply l'Hôpital's rule and calculate

$$\begin{aligned}\ell&=\lim_{t\to 0^+}(1+t)^{1/t}\left(\frac{1}{t(1+t)}-\frac{\log(1+t)}{t^2}\right)=e\lim_{t\to 0^+}\frac{t-(1+t)\log(1+t)}{t^2(1+t)}\\&=e\lim_{t\to 0^+}\frac{t-(1+t)\log(1+t)}{t^2},\end{aligned}$$

another $0/0$ indeterminacy that can be solved by applying l'Hôpital's rule again twice. Doing it once we get

$$\ell=-\frac{e}{2}\lim_{t\to 0^+}\frac{\log(1+t)}{t},$$

and the second time we obtain

$$\ell=-\frac{e}{2}\lim_{t\to 0^+}\frac{1}{1+t}=-\frac{e}{2}.$$

(ii) Taking logarithms in the limit we can calculate it as

$$\log\ell=\lim_{x\to\infty}\left[x^2\log\left(1+\frac{1}{x}\right)-x\right].$$

Now we change the variable x to $t=1/x$ and write

$$\log\ell=\lim_{t\to 0^+}\frac{\log(1+t)-t}{t^2},$$

a $0/0$ indeterminacy that can be solved by applying l'Hôpital's. Thus,

$$\log \ell = \lim_{t\to 0^+} \frac{\frac{1}{1+t}-1}{2t} = -\frac{1}{2}\lim_{t\to 0^+}\frac{t}{t(1+t)} = -\frac{1}{2}\lim_{t\to 0^+}\frac{1}{1+t} = -\frac{1}{2}.$$

Therefore $\ell = 1/\sqrt{e}$.

(iii) This is an indeterminacy 1^∞ which can be calculated as

$$\ell = \lim_{x\to\infty}\left(\frac{2^{1/x}+18^{1/x}}{2}\right)^x = e^c, \qquad c = \lim_{x\to\infty} x\left(\frac{2^{1/x}+18^{1/x}}{2}-1\right).$$

Now we change the variable x to $t = 1/x$ and write

$$c = \frac{1}{2}\lim_{t\to 0^+}\frac{2^t+18^t-2}{t} = \frac{1}{2}\left(\lim_{t\to 0^+}\frac{2^t-1}{t}+\lim_{t\to 0^+}\frac{18^t-1}{t}\right) = \frac{1}{2}\left(\frac{d}{dt}2^t\bigg|_{t=0}+\frac{d}{dt}18^t\bigg|_{t=0}\right)$$
$$= \frac{1}{2}(\log 2+\log 18) = \log\sqrt{36} = \log 6.$$

Therefore $\ell = 6$.

(iv) This limit generalises the previous one. Again,

$$\ell = \lim_{x\to\infty}\left(\frac{1}{p}\sum_{k=1}^{p} a_k^{1/x}\right)^x = e^c, \qquad c = \lim_{x\to\infty} x\left(\frac{1}{p}\sum_{k=1}^{p} a_k^{1/x}-1\right).$$

Now we change the variable x to $t = 1/x$ and write

$$c = \frac{1}{p}\lim_{t\to 0^+}\frac{\sum_k a_k^t - p}{t} = \frac{1}{p}\sum_{k=1}^{p}\lim_{t\to 0^+}\frac{a_k^t-1}{t} = \frac{1}{p}\sum_{k=1}^{p}\frac{d}{dt}a_k^t\bigg|_{t=0} = \frac{1}{p}\sum_{k=1}^{p}\log a_k$$
$$= \log\left[(a_1a_2\cdots a_p)^{1/p}\right].$$

Therefore $\ell = (a_1a_2\cdots a_p)^{1/p}$.

Problem 7.17

(a) Suppose $f(0) = c \neq 0$. Then

$$\ell = \lim_{x\to 0}\frac{f(2x^3)}{5x^3} = \pm\infty,$$

in contradiction with the hypothesis. Thus $f(0) = 0$.

(b) Introduce the variable $t = 2x^3$. Then $t \to 0$ as $x \to 0$. Thus,

$$1 = \lim_{t\to 0}\frac{f(t)}{5t/2} = \frac{2}{5}\lim_{t\to 0}\frac{f(t)-f(0)}{t} = \frac{2}{5}f'(0),$$

hence $f'(0) = 5/2$.

(c) Applying l'Hôpital's rule, the limit

$$\ell = \lim_{x\to 0}\frac{(f\circ f)(2x)}{f^{-1}(3x)}$$

can be obtained through the derivatives of the functions at the numerator and denominator. But

$$\frac{d}{dx}(f\circ f)(2x) = \frac{d}{dx}f\big(f(2x)\big) = f'\big(f(2x)\big)f'(2x)2, \qquad \frac{d}{dx}f^{-1}(3x) = \frac{3}{f'\big(f^{-1}(3x)\big)},$$

and since $f(0)=0$ also $f^{-1}(0)=0$. Then

$$\begin{aligned}\ell &= \lim_{x\to 0}\frac{2f'(2x)f'\big(f(2x)\big)f'\big(f^{-1}(3x)\big)}{3}=\frac{2f'(0)f'\big(f(0)\big)f'\big(f^{-1}(0)\big)}{3}=\frac{2f'(0)^3}{3}\\ &=\frac{2}{3}\cdot\frac{5^3}{2^3}=\frac{125}{12}.\end{aligned}$$

Problem 7.18 Since $g(x)\to f^{-1}(1)=0$ as $x\to 0$, this limit is a $0/0$ indeterminacy. Thus we can apply l'Hôpital's rule to calculate it as

$$\ell=\lim_{x\to 0}\left(e^x+\cos x\,e^{-\sin x}\right)f'\left(f^{-1}(x+1)\right)=2f'(0).$$

All that remains is to compute $f'(0)$. We can do that evaluating the equation defining $f(x)$ at $x=0$. This yields,

$$e^{-f(0)}f'(0)=2 \quad\Rightarrow\quad f'(0)=2e,$$

since $f(0)=1$. Thus $\ell=4e$.

Problem 7.19

(a) f is continuous in $\mathbb{R}$ because so are polynomials and the absolute value function. As for differentiability, we can express f in a piecewise description as

$$f(x)=\begin{cases}4x^3-x^4-1, & 0<x<4,\\ x^4-4x^3-1, & \text{otherwise},\end{cases}$$

separating out the cases where $x^3(x-4)<0$ from those where $x^3(x-4)\geqslant 0$. Both pieces are differentiable (they are polynomials), so we must check the joints. Since

$$f'(x)=\begin{cases}12x^2-4x^3, & 0<x<4,\\ 4x^3-12x^2, & x<0 \text{ or } x>4,\end{cases}$$

we have $f'(0-)=f'(0^+)=0$, so f is differentiable at $x=0$, but $f'(4^-)=-64$, and $f'(4^+)=64$, so f is not differentiable at $x=4$.
Summarising, f is continuous in $\mathbb{R}$ and differentiable in $\mathbb{R}-\{4\}$.

(b) First of all we need to look where $f'(x)=0$. This means

$$4x^2(3-x)=0 \quad\Rightarrow\quad x=0,\ x=3.$$

If $x<0$ but close to $x=0$ then $f'(x)=4x^2(x-3)<0$; if $x>0$ but close to $x=0$ then $f'(x)=4x^2(3-x)>0$. Therefore f has a local minimum at $x=0$. On the other hand, if $x<3$ then $f'(x)=4x^2(3-x)>0$ and if $x>3$ then $f'(x)=4x^2(3-x)<0$, so f has a local maximum at $x=3$.
But this is not the whole story because f is not differentiable at $x=4$ —hence $x=4$ cannot be a solution to $f'(x)=0$. We need to check this point separately. Now, $f(4)=-1$, but for any $x\neq 4$ near $x=4$ we have $f(x)=|x^3(x-4)|-1>-1$, so $x=4$ is a local minimum.
Finally, -1 is the smallest value that $f(x)$ can take, and $f(0)=f(4)=-1$, so both, at $x=0$ and at $x=4$, function $f(x)$ reaches its absolute minimum. There is no absolute maximum though, because the function grows indefinitely as $x\to\pm\infty$.

(c) $f(0)=-1$ and $f(1)=2$, so Bolzano's theorem guarantees that there is at least one solution to $f(x)=0$ in $(0,1)$. On the other hand, in $(0,1)$ we have $f'(x)=4x^2(3-x)>0$ so the function is monotonically increasing. Therefore the solution is unique.

Problem 7.20

(a) The amount of material is proportional to the surface of the can, which is given by the formula $S = 2\pi r^2 + 2\pi r h$. But cans have all the same volume $V = \pi r^2 h$, so $h = V/\pi r^2$ and thefore

$$S = 2\pi \left(r^2 + \frac{V}{\pi r} \right).$$

Minimising the surface amounts to minimising the function

$$f(r) = r^2 + \frac{V}{\pi r}.$$

This is a differentiable function for all $r > 0$, so the minimum is reached at a solution of

$$f'(r) = 2r - \frac{V}{\pi r^2} = 0 \quad \Rightarrow \quad r^3 = \frac{V}{2\pi} \quad \Rightarrow \quad r = \left(\frac{V}{2\pi} \right)^{1/3}$$

and

$$h = \frac{V}{\pi r^2} = \left(\frac{4V}{\pi} \right)^{1/3}.$$

(b) Lead is proportional to the surface. If the side of the square base is a and the height h, then the surface will be $S = a^2 + 4ah$. The volume constraint, $32 = a^2 h$, implies $h = 32/a^2$, so

$$S = a^2 + \frac{128}{a} = f(a).$$

Now,

$$f'(a) = 2a - \frac{128}{a^2} \quad \Rightarrow \quad a^3 = 64 \quad \Rightarrow \quad a = 4, \quad h = 2.$$

(c) We can eliminate $y = 20 - x$, so the function to maximise is

$$f(x) = x^2(20-x)^3.$$

Now,

$$f'(x) = 2x(20-x)^3 - 3x^2(20-x)^2 = x(20-x)^2(40-2x-3x) = 5x(20-x)^2(8-x) = 0.$$

The two solutions $x = 0$, $x = 20$ clearly minimise the function. The maximum is then $x = 8$ and $y = 12$.

(d) If x is half the horizontal side of the rectangle, then

$$y = b\sqrt{1 - \frac{x^2}{a^2}}$$

is half the vertical side. Then the area of the rectangle is

$$A = 4xy = 4bx\sqrt{1 - \frac{x^2}{a^2}}.$$

Maximising this area is tantamount to maximising

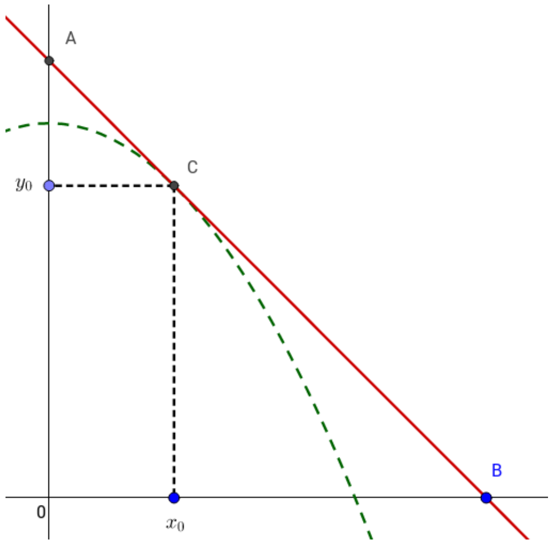
$$f(x) = \frac{A^2}{16b^2} = x^2 - \frac{x^4}{a^2},$$

which means solving the equation

$$f'(x) = 2x - \frac{4x^3}{a^2} = 2x\left(1 - \frac{2x^2}{a^2}\right) = 0.$$

One solution is $x = 0$ —which is obviously not the right one— and the other two solutions are $x = \pm a/\sqrt{2}$. Clearly the one that maximises the area has to be $x = a/\sqrt{2}$.

(e) The picture illustrates how to construct the described triangle:



We can select an arbitrary point on the parabola, $(x_0, 6-x_0^2)$. The slope of the tangent at that point will be $m = -2x_0$ (obtained differentiating $6-x^2$), so the equation of the tangent straight line will be

$$y = 6 - x_0^2 - 2x_0(x - x_0) = 6 + x_0^2 - 2x_0 x.$$

Now, this straight line meets the Y axis at $A(0, 6+x_0^2)$, and the X axis at $B\big((6+x_0^2)/2x_0, 0\big)$, so the area of the triangle will be

$$A = \frac{(6+x_0^2)^2}{4x_0} = \frac{9}{x_0} + 3x_0 + \frac{x_0^3}{4} = f(x_0).$$

Minimising the area means solving

$$f'(x_0) = -\frac{9}{x_0^2} + 3 + \frac{3x_0^2}{4} = \frac{3(x_0^4 + 4x_0^2 - 12)}{4x_0^2} = \frac{3(x_0^2+6)(x_0^2-2)}{4x_0^2} = 0.$$

The only meaningful solution to this equation is $x_0 = \sqrt{2}$.

(f) The area of the triangle at the base is $a^2\sqrt{3}/4$, and that of the lateral rectangles $3ah$, so the total cost will be

$$C = 0.20 \times a^2 \frac{\sqrt{3}}{4} + 0.10 \times 3ah = 0.10 \times \sqrt{3}\left(\frac{a^2}{2} + \sqrt{3}ah\right).$$

Since $128 = ha^2\sqrt{3}/4$ we get $\sqrt{3}ah = 512/a$, so $C = 0.10 \times \sqrt{3} f(a)$, where

$$f(a) = \frac{a^2}{2} + \frac{512}{a}.$$

The value of a minimising cost will be a solution of

$$f'(a) = a - \frac{512}{a^2} = 0 \quad \Rightarrow \quad a^3 = 512 \quad \Rightarrow \quad a = 8.$$

(g) For a given $0 \leqslant x \leqslant 2$ the corresponding y on the circumference is given by

$$y = \sqrt{1-(x-1)^2} = \sqrt{x(2-x)}.$$

Thus, the three points of the triangle are $A(0,0)$, $B\left(x, \sqrt{x(2-x)}\right)$, $C(x,0)$. The area of the triangle will then be $S = x\sqrt{x(2-x)}/2 = x^{3/2}(2-x)^{1/2}/2$. So maximising this area is tantamount to maximising

$$f(x) = 4S^2 = x^3(2-x) = 2x^3 - x^4.$$

The corresponding x will be a solution of

$$f'(x) = 6x^2 - 4x^3 = 2x^2(3-2x) = 0.$$

The only meaningful solution is $x = 3/2$.

(h) Triangle similarity implies

$$\frac{y_0+\beta}{x_0+\alpha} = \frac{\beta}{x_0} \quad \Rightarrow \quad x_0y_0 + \cancel{\beta x_0} = \cancel{\beta x_0} + \beta\alpha \quad \Rightarrow \quad \beta = \frac{x_0y_0}{\alpha}.$$

(i) The length of segment AB is

$$\ell = \sqrt{(x_0+\alpha)^2+(y_0+\beta)^2} = \sqrt{(x_0+\alpha)^2+\left(y_0+\frac{x_0y_0}{\alpha}\right)^2} = \sqrt{(x_0+\alpha)^2+\frac{y_0^2}{\alpha^2}(x_0+\alpha)^2}$$
$$= (x_0+\alpha)\sqrt{1+\frac{y_0^2}{\alpha^2}}.$$

So minimising ℓ is tantamount to minimising

$$f(\alpha) = \ell^2 = (x_0+\alpha)^2\left(1+\frac{y_0^2}{\alpha^2}\right).$$

Differentiating

$$f'(\alpha) = 2(x_0+\alpha)\left(1+\frac{y_0^2}{\alpha^2}\right) - 2(x_0+\alpha)^2\frac{y_0^2}{\alpha^3} = 2(x_0+\alpha)\left(1+\cancel{\frac{y_0^2}{\alpha^2}} - \frac{x_0y_0^2}{\alpha^3} - \cancel{\frac{y_0^2}{\alpha^2}}\right)$$
$$= 2(x_0+\alpha)\left(1-\frac{x_0y_0^2}{\alpha^3}\right) = 0.$$

This equation has the solution

$$\alpha = \left(x_0y_0^2\right)^{1/3}, \qquad \beta = \frac{x_0y_0}{\alpha} = \left(x_0^2y_0\right)^{1/3}.$$

(ii) The sum of segments OA and OB is

$$f(\alpha) = x_0+\alpha+y_0+\beta = x_0+y_0+\alpha+\frac{x_0y_0}{\alpha}.$$

Differentiating

$$f'(\alpha) = 1-\frac{x_0y_0}{\alpha^2} = 0 \quad \Rightarrow \quad \alpha = (x_0y_0)^{1/2}, \quad \beta = \frac{x_0y_0}{\alpha} = (x_0y_0)^{1/2}.$$

(iii) The area of the triangle is

$$A = \frac{1}{2}(x_0+\alpha)(y_0+\beta) = \frac{1}{2}(x_0+\alpha)\left(y_0+\frac{x_0y_0}{\alpha}\right) = \frac{y_0}{2}\frac{(x_0+\alpha)^2}{\alpha} = \frac{y_0}{2}\left(\frac{x_0^2}{\alpha}+2x_0+\alpha\right).$$

Minimising the area implies minimising

$$f(\alpha) = \frac{2A}{y_0} = \frac{x_0^2}{\alpha}+2x_0+\alpha.$$

Differentiating

$$f'(\alpha) = -\frac{x_0^2}{\alpha^2}+1 = 0 \quad \Rightarrow \quad \alpha = x_0, \quad \beta = \frac{x_0y_0}{\alpha} = y_0.$$

Problem 7.21

(a) For $a=1$ the inequality becomes a trivial equality. For $a>1$ take the function

$$f(x) = (1+x)^a - 1 - ax.$$

Differentiating,

$$f'(x) = a(1+x)^{a-1} - a = 0 \quad \Rightarrow \quad (1+x)^{a-1} = 1 \quad \Rightarrow \quad x = 0,$$

so $x=0$ is a local extremum. From the second derivative,

$$f''(x) = a(a-1)(1+x)^{a-2} \quad \Rightarrow \quad f''(0) = a(a-1) > 0$$

we conclude that $x=0$ is a minimum —the absolute minimum if $x>-1$—, therefore $f(x) \geqslant f(0) = 0$ for every $x>-1$. This means

$$(1+x)^a \geqslant 1+ax.$$

(b) Take the function

$$f(x) = e^x - 1 - x.$$

Differentiating,

$$f'(x) = e^x - 1 = 0 \quad \Rightarrow \quad x = 0,$$

so $x=0$ is a local extremum. From the second derivative,

$$f''(x) = e^x \quad \Rightarrow \quad f''(0) = 1 > 0,$$

we conclude that $x=0$ is a minimum —which is absolute in this case because there is no other one in $\mathbb{R}$. Therefore $f(x) \geqslant f(0) = 0$ for every $x \in \mathbb{R}$, i.e.,

$$e^x \geqslant 1+x.$$

(c) Take the function

$$f(x) = \log(1+x) - \frac{x}{1+x}.$$

Differentiating,

$$f'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{x}{(1+x)^2} = 0 \quad \Rightarrow \quad x = 0,$$

so $x = 0$ is a local extremum. From the second derivative,

$$f''(x) = -\frac{1}{(1+x)^2} + \frac{2}{(1+x)^3} = \frac{1-x}{(1+x)^3} \quad \Rightarrow \quad f''(0) = 1 > 0,$$

we conclude that $x = 0$ is a minimum —which is absolute in this case because there is no other one when $x > -1$. Therefore $f(x) \geqslant f(0) = 0$ for every $x > -1$. This proves the first inequality. As for the second, take

$$g(x) = x - \log(1+x)$$

and differentiate:

$$g'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} = 0 \quad \Rightarrow \quad x = 0,$$

so $x = 0$ is a local extremum. From the second derivative,

$$f''(x) = \frac{1}{(1+x)^2} \quad \Rightarrow \quad f''(0) = 1 > 0,$$

we conclude that $x = 0$ is a minimum —again absolute—, so $f(x) \geqslant f(0) = 0$ for every $x > -1$. This proves the second inequality.

Problem 7.22

(a) Take the function

$$f(x) = \frac{\log x}{x}.$$

Differentiating,

$$f'(x) = \frac{1-\log x}{x^2} = 0 \quad \Rightarrow \quad x = e.$$

From the second derivative,

$$f''(x) = \frac{2\log x - 3}{x^3} \quad \Rightarrow \quad f''(e) = -\frac{1}{e^3} < 0,$$

so $x = e$ is the absolute maximum for $x > 0$. Thus $f(x) < f(e)$ for all $x > 0$, $x \neq e$, which means

$$\frac{\log x}{x} < \frac{1}{e}.$$

(b) Multiplying the inequality by ex it becomes $e\log x < x$, and taking exponentials

$$x^e < e^x.$$

Problem 7.23

(i) The polynomial $f(x) = x^7 + 4x - 3 \sim x^7$ as $x \to \pm\infty$, so $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$. Thus $f(x) = 0$ at at least one point. What we need to know is to figure out how many times $f(x)$ bends up and down and from that determining the number of times it crosses the X axis. Now,

$$f'(x) = 7x^6 + 4 > 0$$

for all $x \in \mathbb{R}$, therefore $f(x)$ increases monotonically. The conclusion is that there is only *one* solution.

(ii) Similarly to the previous exercise, $f(x) = x^5 - 5x + 6 \sim x^5$ as $x \to \pm\infty$, so $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$. Thus $f(x) = 0$ at at least one point. Now,

$$f'(x) = 5x^4 - 5 = 0 \quad \Rightarrow \quad x = \pm 1,$$

and from the second derivative

$$f''(x) = 20x^3 \quad \Rightarrow \quad f''(1) = 20 > 0, \quad f''(-1) = -20 < 0,$$

so we conclude that $x = -1$ is a local minimum and $x = 1$ a local maximum. But $f(1) = 2 > 0$ and $f(-1) = 10 > 0$, so the local minimum is above the X axis. In conclusion, there is only *one* solution.

(iii) $f(x) = x^4 - 4x^3 - 1 \sim x^4$ as $x \to \pm\infty$, so $f(x) \to \infty$ when $x \to \pm\infty$. It is not guaranteed that there is even a single solution. From the derivative,

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) = 0$$

we conclude that $x = 0$ and $x = 3$ may be extrema. $f'(x) < 0$ around $x = 0$ (at both sides), so it is an inflection point. However, close to $x = 3$ we have $f'(x) < 0$ for $x < 3$ and $f'(x) > 0$ for $x > 3$, so at $x = 3$ the polynomial reaches its absolute minimum $f(3) = -28$. Since this value is below the X axis, $f(x)$ has to cross it twice. Therefore there are *two* solutions to the equation.

(iv) The function $f(x) = 2x - 1 - \sin x \sim 2x$ as $x \to \pm\infty$, so $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$. Thus $f(x) = 0$ at at least one point. Now,

$$f'(x) = 2 - \cos x > 0 \quad \text{for all } x \in \mathbb{R},$$

so $f(x)$ monotonically increases. Therefore there is only *one* solution.

(v) Let us first rewite the equation. Taking logarithms the equation becomes

$$f(x) = x \log x - \log 2 = 0.$$

$f(1) = -\log 2 < 0$ and $f(x) \to \infty$ as $x \to \infty$, so $f(x)$ vanishes at one point at least. Now,

$$f'(x) = \log x + 1,$$

which is $f'(x) < 0$ for $x < 1/e$ and $f'(x) > 0$ for $x > 1/e$. In other words, $f'(x) > 0$ in the interval $[1, \infty)$, so $f(x)$ monotonically increases in that interval. Therefore there is only *one* solution.

(vi) Writing the equation

$$f(x) = x^2 + \log x = 0$$

we have $f(1) = 1 > 0$, and $f(x) \sim x^2$ as $x \to \pm\infty$, so $f(x) \to \infty$ as $x \to \pm\infty$. There is no guarantee that the equation has even a single solution in that interval. From the derivative,

$$f'(x) = 2x + \frac{1}{x} = \frac{2x^2 + 1}{x}$$

we conclude that $f'(x) > 0$ in $(1, \infty)$, so $f(x)$ increases monotonically. Therefore the equation has *no* solution in that interval.

D.8 Taylor Expansions

Problem 8.1

(i) There are two ways to solve these exercises. The first one amounts to applying Taylor's formula for $P_{n,a}(x)$. For the case of $f(x) = e^x \sin x$ we have

$$\begin{aligned}
f(x) &= e^x \sin x, & f(0) &= 0,\\
f'(x) &= e^x(\sin x + \cos x), & f'(0) &= 1,\\
f''(x) &= 2e^x \cos x, & f''(0) &= 2,\\
f'''(x) &= 2e^x(\cos x - \sin x), & f'''(0) &= 2,\\
f^{(4)}(x) &= -4e^x \sin x, & f^{(4)}(0) &= 0,\\
f^{(5)}(x) &= -4e^x(\sin x + \cos x), & f^{(5)}(0) &= -4,
\end{aligned}$$

thus

$$P_{5,0}(x) = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30}.$$

The alternative way —the one we will follow here— amounts to relying upon known Taylor expansions and operate with them. For instance in this case we know that when $x \to 0$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + o(x^5), \qquad \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5),$$

therefore, multiplying the two expressions —and collecting any power higher than x^5 as $o(x^5)$— we obtain

$$\begin{aligned}
e^x \sin x &= \left[1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + o(x^5)\right]\left[x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)\right]\\
&= \left[x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)\right] + \left[x^2 - \cancel{\frac{x^4}{6}} + o(x^5)\right] + \left[\frac{x^3}{2} - \frac{x^5}{12} + o(x^5)\right]\\
&\quad + \left[\cancel{\frac{x^4}{6}} + o(x^5)\right] + \left[\frac{x^5}{24} + o(x^5)\right]\\
&= x + x^2 + \left(\frac{1}{2} - \frac{1}{6}\right)x^3 + \left(\frac{1}{120} + \frac{1}{24} - \frac{1}{12}\right)x^5 + o(x^5)\\
&= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + o(x^5),
\end{aligned}$$

and we get to the same result.

(ii) Now

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} + o(x^5), \qquad \cos 2x = 1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{24} + o(x^5) = 1 - 2x^2 + \frac{2}{3}x^4 + o(x^5),$$

so multiplying and collecting equal powers,

$$\begin{aligned}
e^{-x^2}\cos 2x &= \left[1 - x^2 + \frac{x^4}{2} + o(x^5)\right]\left[1 - 2x^2 + \frac{2}{3}x^4 + o(x^5)\right]\\
&= 1 - (1+2)x^2 + \left(\frac{1}{2} + 2 + \frac{2}{3}\right)x^4 + o(x^5)\\
&= 1 - 3x^2 + \frac{19}{6}x^4 + o(x^5).
\end{aligned}$$

Thus

$$P_{5,0}(x) = 1 - 3x^2 + \frac{19}{6}x^4.$$

(iii) Using the trigonometric identity

$$\sin\theta\cos\phi = \frac{1}{2}\left[\sin(\theta+\phi) + \sin(\theta-\phi)\right]$$

we can write

$$\sin x \cos 2x = \frac{1}{2}\left(\sin 3x - \sin x\right).$$

Now, since for $z \to 0$

$$\sin z = z - \frac{z^3}{6} + \frac{z^5}{120} + o(z^5),$$

then

$$\sin x \cos 2x = \frac{1}{2}\left(3x - \frac{9}{2}x^3 + \frac{81}{40}x^5 - x + \frac{x^3}{6} - \frac{x^5}{120}\right) + o(x^5)$$

(iv) In this case

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + o(x^5), \qquad \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} + o(x^5),$$

so

$$\begin{aligned}
e^x \log(1-x) &= -x\left[1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + o(x^4)\right]\left[1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + o(x^4)\right] \\
&= -x\left[1 + \left(1 + \frac{1}{2}\right)x + \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{3}\right)x^2 + \left(\frac{1}{6} + \frac{1}{4} + \frac{1}{3} + \frac{1}{4}\right)x^3\right. \\
&\quad \left. + \left(\frac{1}{24} + \frac{1}{12} + \frac{1}{6} + \frac{1}{4} + \frac{1}{5}\right)x^4 + o(x^4)\right] \\
&= -x - \frac{3}{2}x^2 - \frac{4}{3}x^3 - x^4 - \frac{89}{120}x^5 + o(x^5).
\end{aligned}$$

Therefore

$$P_{5,0}(x) = -x - \frac{3}{2}x^2 - \frac{4}{3}x^3 - x^4 - \frac{89}{120}x^5.$$

(v) Since $\sin^2 x = (1 - \cos 2x)/2$,

$$\sin^2 x = \frac{1}{2}\left[\cancel{1} - \cancel{1} + \frac{(2x)^2}{2} - \frac{(2x)^4}{24} + o(x^5)\right] = x^2 - \frac{x^4}{3} + o(x^5),$$

hence

$$P_{5,0}(x) = x^2 - \frac{x^4}{3}.$$

(vi) We know that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots,$$

therefore

$$\frac{1}{1-x^3} = 1 + x^3 + o(x^5),$$

which implies $P_{5,0}(x) = 1 + x^3$.

Problem 8.2 The Taylor polynomial $P_{4,4}(x)$ of $P(x) = x^4 - 5x^3 + x^2 - 3x + 4$ is obtained through

$$\begin{aligned}
&P(x) = x^4 - 5x^3 + x^2 - 3x + 4, && P(4) = -56,\\
&P'(x) = 4x^3 - 15x^2 + 2x - 3, && P'(4) = 21,\\
&P''(x) = 12x^2 - 30x + 2, && P''(4) = 74,\\
&P'''(x) = 24x - 30, && P'''(4) = 66,\\
&P^{(4)}(x) = 24, && P^{(4)}(4) = 24.
\end{aligned}$$

Hence

$$P(x) = -56 + 21(x-4) + 37(x-4)^2 + 11(x-4)^3 + (x-4)^4.$$

Problem 8.3

(i) The polynomial must be expressed in powers of $t = x+1$, so if we write

$$\frac{1}{x} = \frac{1}{t-1} = -\frac{1}{1-t} = -1 - t - t^2 - \cdots - t^n + \cdots$$

we immediately obtain $P_{n,-1}(x) = -1 - (x+1) - (x+1)^2 - \cdots - (x+1)^n$.

(ii) Since

$$\begin{aligned}
e^{-2x} &= 1 + (-2x) + \frac{(-2x)^2}{2} + \cdots + \frac{(-2x)^{n-1}}{(n-1)!} + o(x^{n-1})\\
&= 1 - 2x + 2x^2 + \cdots + (-1)^{n-1}\frac{2^{n-1}}{(n-1)!}x^{n-1} + o(x^{n-1})
\end{aligned}$$

then

$$xe^{-2x} = x - 2x^2 + 2x^3 + \cdots + (-1)^{n-1}\frac{2^{n-1}}{(n-1)!}x^n + o(x^n).$$

Thus

$$P_{n,0}(x) = x - 2x^2 + 2x^3 + \cdots + (-1)^{n-1}\frac{2^{n-1}}{(n-1)!}x^n.$$

(iii) We can expand $(1+e^x)^2 = 1 + 2e^x + e^{2x}$, so

$$\begin{aligned}
(1+e^x)^2 =& 1 + 2\left[1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + o(x^n)\right] + \left[1 + 2x + \frac{(2x)^2}{2} + \cdots + \frac{(2x)^n}{n!} + o(x^n)\right]\\
=& 4 + 4x + 3x^2 + \cdots + \frac{2+2^n}{n!}x^n + o(x^n),
\end{aligned}$$

from which

$$P_{n,0}(x) = 4 + 4x + 3x^2 + \cdots + \frac{2+2^n}{n!}x^n.$$

(iv) We must express the polynomial in powers of $t = x - \pi$, therefore $\sin x = \sin(\pi + t) = -\sin t$, and

$$\sin x = -t + \frac{t^3}{6} - \frac{t^5}{120} + \cdots + (-1)^n \frac{t^{2n-1}}{(2n-1)!} + o(t^{2n-1}).$$

Thus

$$P_{2n,\pi}(x) = P_{2n-1,\pi}(x) = -(x-\pi) + \frac{(x-\pi)^3}{6} - \frac{(x-\pi)^5}{120} + \cdots + (-1)^n \frac{(x-\pi)^{2n-1}}{(2n-1)!}.$$

Problem 8.4

(i) For $x \neq 0$,

$$f'(x) = \frac{2}{x^3} e^{-1/x^2}.$$

so $Q_1(t) = 2t^3$. Suppose now that $f^{(n)}(x) = Q_n(1/x)e^{-1/x^2}$ for $x \neq 0$ and some $n \in \mathbb{N}$. Differentiating once more,

$$f^{(n+1)} = \left[-\frac{1}{x^2} Q_n'(1/x) + \frac{2}{x^3} Q_n(1/x) \right] e^{-1/x^2} = Q_{n+1}(1/x) e^{-1/x^2},$$

where $Q_{n+1}(t) = -t^2 Q_n'(t) + 2t^3 Q_n(t)$ is a polynomial if so is $Q_n(t)$. This proves the result for all $n \in \mathbb{N}$.

(ii) First of all,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{1}{x} e^{-1/x^2} = \lim_{t\to\pm\infty} te^{-t^2} = 0.$$

Suppose now that $f^{(n)}(0) = 0$ for some $n \in \mathbb{N}$. From (i),

$$f^{(n+1)}(0) = \lim_{x\to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x} = \lim_{x\to 0} \frac{1}{x} Q_n(1/x) e^{-1/x^2} = \lim_{t\to\pm\infty} tQ_n(t) e^{-t^2} = 0.$$

This proves that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

(iii) Since we have proven that $f^{(n)}(0) = 0$ for every $n \in \mathbb{N}$, this means that $P_{n,0}(x) = 0$ for every $n \in \mathbb{N}$. In other words, the best polynomial approximation to $f(x)$ at $x = 0$ is just 0. The conclusion we get from this fact is that the reminder of this function must be $R_{n,0}(x) = f(x)$ for any $n \in \mathbb{N}$.
This is one example of a function that does not have a Taylor series which converges to it.

Problem 8.5

(i) Since $\sin x \sim x$ when $x \to 0$,

$$\lim_{x\to 0} \frac{\sin x}{x^\alpha} = \lim_{x\to 0} \frac{x}{x^\alpha} = \lim_{x\to 0} x^{1-\alpha} = 0$$

because $1 - \alpha > 0$.

(ii) Since $\log(1 + x^2) \sim x^2$ when $x \to 0$,

$$\lim_{x\to 0} \frac{\log(1+x^2)}{x} = \lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0.$$

(iii) We need to calculate the limit

$$\lim_{x\to\infty}\frac{\log x}{x}.$$

Since this is an indeterminacy $\frac{\infty}{\infty}$ we can apply l'Hôpital and calculate instead

$$\lim_{x\to\infty}\frac{1}{x}=0.$$

(iv) Since

$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^2}$$

is a $\frac{0}{0}$ indeterminacy we can apply l'Hôpital and calculate instead

$$\lim_{x\to 0}\frac{1+\tan^2 x-\cos x}{2x}.$$

And we apply l'Hôpital again because this is still a $\frac{0}{0}$ indeterminacy:

$$\lim_{x\to 0}\frac{2\tan x(1+\tan^2 x)+\sin x}{2}=0.$$

Problem 8.6

(i) When $x\to 0$ e have

$$e^x=1+x+\frac{x^2}{2}+o(x^2),\qquad \sin x=x+o(x^2),$$

thus

$$\begin{aligned}\lim_{x\to 0}\frac{e^x-\sin x-1}{x^2}&=\lim_{x\to 0}\frac{1+x+\frac{x^2}{2}+o(x^2)-x-1}{x^2}=\lim_{x\to 0}\frac{\frac{x^2}{2}+o(x^2)}{x^2}\\&=\lim_{x\to 0}\left(\frac{1}{2}+o(1)\right)=\frac{1}{2}.\end{aligned}$$

(ii) When $x\to 0$

$$\sin x=x-\frac{x^3}{6}+\frac{x^5}{120}+o(x^5),$$

so

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x-x+\frac{x^3}{6}}{x^5}&=\lim_{x\to 0}\frac{x-\frac{x^3}{6}+\frac{x^5}{120}+o(x^5)-x+\frac{x^3}{6}}{x^5}=\lim_{x\to 0}\frac{\frac{x^5}{120}+o(x^5)}{x^5}\\&=\lim_{x\to 0}\left(\frac{1}{120}+o(1)\right)=\frac{1}{120}.\end{aligned}$$

(iii) When $x\to 0$ the denominator $\sin x=x+o(x)$. On the other hand, $\cos x=1+o(x)$ and $\sqrt{1-x}=1-\frac{x}{2}+o(x)$, so

$$\lim_{x\to 0}\frac{\cos x-\sqrt{1-x}}{\sin x}=\lim_{x\to 0}\frac{1+o(x)-1+\frac{x}{2}}{x+o(x)}=\lim_{x\to 0}\frac{\frac{x}{2}+o(x)}{x+o(x)}=\lim_{x\to 0}\frac{\frac{1}{2}+o(1)}{1+o(1)}=\frac{1}{2}.$$

(iv) When $x \to 0$

$$\tan x = x + \frac{x^3}{3} + o(x^3), \qquad \sin x = x - \frac{x^3}{6} + o(x^3),$$

so

$$\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x\to 0} \frac{x + \frac{x^3}{3} + o(x^3) - x + \frac{x^3}{6}}{x^3} = \lim_{x\to 0} \frac{\frac{x^3}{2} + o(x^3)}{x^3} = \lim_{x\to 0} \left(\frac{1}{2} + o(1)\right) = \frac{1}{2}.$$

(v) When $x \to 0$

$$\cos 3x = 1 - \frac{9}{2}x^2 + o(x^2), \qquad \sin x = x - \frac{x^3}{6} + o(x^3),$$

so

$$\lim_{x\to 0} \frac{x - \sin x}{x(1-\cos 3x)} = \lim_{x\to 0} \frac{\frac{x^3}{6} + o(x^3)}{x\left(\frac{9}{2}x^2 + o(x^2)\right)} = \lim_{x\to 0} \frac{\frac{1}{6} + o(1)}{\frac{9}{2} + o(1)} = \frac{1}{27}.$$

(vi) When $x \to 0$

$$\cos x = 1 - \frac{x^2}{2} + o(x^3), \qquad e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3),$$

so

$$\lim_{x\to 0} \frac{\cos x + e^x - x - 2}{x^3} = \lim_{x\to 0} \frac{1 - \frac{x^2}{2} + o(x^3) + 1 + x + \frac{x^2}{2} + \frac{x^3}{6} - x - 2}{x^3} = \lim_{x\to 0} \frac{\frac{x^3}{6} + o(x^3)}{x^3}$$
$$= \lim_{x\to 0} \left(\frac{1}{6} + o(1)\right) = \frac{1}{6}.$$

(vii) First of all

$$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0} \frac{\sin x - x}{x \sin x}.$$

Now, $\sin x = x + o(x^2)$ when $x \to 0$, so

$$\lim_{x\to 0} \frac{\sin x - x}{x\sin x} = \lim_{x\to 0} \frac{o(x^2)}{x^2 + o(x^3)} = \lim_{x\to 0} \frac{o(1)}{1 + o(x)} = \lim_{x\to 0} \frac{o(1)}{1+o(1)} = 0.$$

(Remember that $o(x^n)$ can be replaced by $o(x^m)$ when $x \to 0$ if $n > m$.)

(viii) To begin with, since $\cot x = \cos x / \sin x$,

$$\lim_{x\to 0} \frac{1}{x}\left(\frac{1}{x} - \cot x\right) = \lim_{x\to 0} \frac{\sin x - x\cos x}{x^2 \sin x}.$$

Now,

$$\sin x = x - \frac{x^3}{6} + o(x^3) = x + o(x), \qquad x\cos x = x\left[1 - \frac{x^2}{2} + o(x^2)\right] = x - \frac{x^3}{2} + o(x^3),$$

hence

$$\lim_{x\to 0} \frac{\sin x - x\cos x}{x^2 \sin x} = \lim_{x\to 0} \frac{x - \frac{x^3}{6} + o(x^3) - x + \frac{x^3}{2}}{x^3 + o(x^3)} = \lim_{x\to 0} \frac{\frac{x^3}{3} + o(x^3)}{x^3 + o(x^3)} = \lim_{x\to 0} \frac{\frac{1}{3} + o(1)}{1 + o(1)} = \frac{1}{3}.$$

(ix) We cannot apply Taylor when $x \to \infty$, but we can rewrite the limit by changing the variable to $t = 1/x$. Then

$$\ell = \lim_{x\to\infty} x^{3/2}\left(\sqrt{x+1}+\sqrt{x-1}-2\sqrt{x}\right) = \lim_{t\to 0^+} \frac{\sqrt{\frac{1}{t}+1}+\sqrt{\frac{1}{t}-1}-\frac{2}{\sqrt{t}}}{t^{3/2}},$$

and multiplying numerator and denominator by $\sqrt{t}$,

$$\ell = \lim_{t\to 0^+} \frac{\sqrt{1+t}+\sqrt{1-t}-2}{t^2}.$$

Now we know that when $t \to 0$

$$(1+t)^\alpha = 1+\alpha t + \frac{\alpha(\alpha-1)}{2}t^2 + o(t^2),$$

so setting $\alpha = 1/2$,

$$\sqrt{1+t} = 1+\frac{t}{2}-\frac{t^2}{8}+o(t^2), \qquad \sqrt{1-t} = 1-\frac{t}{2}-\frac{t^2}{8}+o(t^2).$$

Thus,

$$\ell = \lim_{t\to 0^+} \frac{1+\frac{t}{2}-\frac{t^2}{8}+o(t^2)+1-\frac{t}{2}-\frac{t^2}{8}-2}{t^2} = \lim_{t\to 0^+} \frac{-\frac{t^2}{4}+o(t^2)}{t^2} = \lim_{t\to 0^+}\left(-\frac{1}{4}+o(1)\right) = -\frac{1}{4}.$$

(x) Changing from x to $t = 1/x$,

$$\ell = \lim_{x\to\infty}\left[x - x^2\log\left(1+\frac{1}{x}\right)\right] = \lim_{t\to 0^+}\left[\frac{1}{t}-\frac{\log(1+t)}{t^2}\right] = \lim_{t\to 0^+}\frac{t-\log(1+t)}{t^2}.$$

If we now write

$$\log(1+t) = t-\frac{t^2}{2}+o(t^2) \quad (t\to 0),$$

then

$$\ell = \lim_{t\to 0^+}\frac{t-t+\frac{t^2}{2}+o(t^2)}{t^2} = \lim_{t\to 0^+}\frac{\frac{t^2}{2}+o(t^2)}{t^2} = \lim_{t\to 0^+}\left(\frac{1}{2}+o(1)\right) = \frac{1}{2}.$$

Problem 8.7 To begin with, when $y \to 0$

$$\log(1+y) = y-\frac{y^2}{2}+o(y^2).$$

In our case

$$y = f(x) = -\frac{x}{2}-\frac{x^2}{4}+o(x^2),$$

which clearly goes to 0 when $x \to 0$. Then

$$y^2 = f(x)^2 = \left[-\frac{x}{2}-\frac{x^2}{4}+o(x^2)\right]\left[-\frac{x}{2}-\frac{x^2}{4}+o(x^2)\right] = \frac{x^2}{4}+o(x^2),$$

and $o(y^2) = o(x^2)$ because $y = -\frac{x}{2} + o(x)$. Therefore

$$\log[1+f(x)] = \left(-\frac{x}{2} - \frac{x^2}{4} + o(x^2)\right) - \frac{1}{2}\left(\frac{x^2}{4} + o(x^2)\right) + o(x^2) = -\frac{x}{2} - \frac{3}{8}x^2 + o(x^2).$$

If we now substitute

$$\lim_{x\to 0} \frac{\log[1+f(x)] + \frac{x}{2}}{x^2} = \lim_{x\to 0} \frac{-\frac{x}{2} - \frac{3}{8}x^2 + o(x^2) + \frac{x}{2}}{x^2} = \lim_{x\to 0} \frac{-\frac{3}{8}x^2 + o(x^2)}{x^2}$$
$$= \lim_{x\to 0}\left(-\frac{3}{8} + o(1)\right) = -\frac{3}{8}.$$

Problem 8.8 From the definition,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{\frac{1}{x} - \frac{1}{e^x - 1} - \frac{1}{2}}{x} = \lim_{x\to 0} \frac{(2-x)(e^x-1) - 2x}{2x^2(e^x-1)}.$$

Now, $e^x - 1 = x + o(x)$ when $x \to 0$. This means that the denominator is $2x^2(e^x-1) = 2x^3 + o(x^3)$ when $x \to 0$, and so we need to expand the numerator up to order $o(x^3)$. We need more terms of the exponential:

$$e^x - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3).$$

Substituting in the numerator we have

$$(2-x)(e^x-1) - 2x = 2x + x^2 + \frac{x^3}{3} + o(x^3) - x^2 - \frac{x^3}{2} + o(x^3) - 2x = -\frac{x^3}{6} + o(x^3).$$

Then

$$f'(0) = \lim_{x\to 0} \frac{-\frac{x^3}{6} + o(x^3)}{2x^3 + o(x^3)} = \lim_{x\to 0} \frac{-\frac{1}{6} + o(1)}{2 + o(1)} = -\frac{1}{12}.$$

Problem 8.9

(i) The difficulty of this problem is that we don't know beforehand to which order we need to do the Taylor expansions of the functions involved in order to get the first nonzero term. It turns out that the first order is the seventh. Thus we need the expansions for $x \to 0$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + o(x^7), \qquad \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + o(x^7).$$

Since $\sin x$ is the argument of $\tan(\sin x)$ we will need to calculate the expansions of $\sin^3 x$, $\sin^5 x$, and $\sin^7 x$. So,

$$\sin^2 x = \sin x \cdot \sin x = x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 + o(x^7),$$
$$\sin^3 x = \sin^2 x \cdot \sin x = x^3 - \frac{x^5}{2} + \frac{13}{120}x^7 + o(x^7),$$
$$\sin^5 x = \sin^2 x \cdot \sin^3 x = x^5 - \frac{5}{6}x^7 + o(x^7),$$
$$\sin^7 x = \sin^2 x \cdot \sin^5 x = x^7 + o(x^7).$$

Besides $o(\sin^7 x) = o(x^7)$. Likewise, $\tan x$ is the argument of $\sin(\tan x)$, therefore

$$\begin{aligned}
\tan^2 x &= \tan x \cdot \tan x = x^2 + \frac{2}{3}x^4 + \frac{17}{45}x^6 + o(x^7),\\
\tan^3 x &= \tan^2 x \cdot \tan x = x^3 + x^5 + \frac{11}{15}x^7 + o(x^7),\\
\tan^5 x &= \tan^2 x \cdot \tan^3 x = x^5 + \frac{5}{3}x^7 + o(x^7),\\
\tan^7 x &= \tan^2 x \cdot \tan^5 x = x^7 + o(x^7),
\end{aligned}$$

and $o(\tan^7 x) = o(x^7)$.
Then,

$$\begin{aligned}
\tan(\sin x) =& \sin x + \frac{1}{3}\sin^3 x + \frac{2}{15}\sin^5 x + \frac{17}{315}\sin^7 x + o(x^7),\\
=& \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + o(x^7)\right) + \frac{1}{3}\left(x^3 - \frac{x^5}{2} + \frac{13}{120}x^7 + o(x^7)\right)\\
&+ \frac{2}{15}\left(x^5 - \frac{5}{6}x^7 + o(x^7)\right) + \frac{17}{315}\left(x^7 + o(x^7)\right) + o(x^7)\\
=& x + \frac{x^3}{6} - \frac{x^5}{40} - \frac{107}{5040}x^7 + o(x^7).
\end{aligned}$$

Similarly

$$\begin{aligned}
\sin(\tan x) =& \tan x - \frac{1}{6}\tan^3 x + \frac{1}{120}\tan^5 x - \frac{1}{5040}\tan^7 x + o(x^7),\\
=& \left(x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + o(x^7)\right) - \frac{1}{6}\left(x^3 + x^5 + \frac{11}{15}x^7 + o(x^7)\right)\\
&+ \frac{1}{120}\left(x^5 + \frac{5}{3}x^7 + o(x^7)\right) - \frac{1}{5040}\left(x^7 + o(x^7)\right) + o(x^7)\\
=& x + \frac{x^3}{6} - \frac{x^5}{40} - \frac{55}{1008}x^7 + o(x^7).
\end{aligned}$$

Accordingly, substracting these two expansions,

$$f(x) = \tan(\sin x) - \sin(\tan x) = \frac{x^7}{30} + o(x^7) \quad (x \to 0).$$

(ii) We can write

$$f(x) = \frac{1}{R^2}\left[1 - (1+z)^{-2}\right], \quad z = \frac{x}{R},$$

and use the expansion $(1+z)^{-2} = 1 - 2z + o(z)$. Then

$$(1+z)^{-2} = 1 - 2\frac{x}{R} + o(x),$$

and

$$f(x) = \frac{1}{R^2}\left[1 - 1 + 2\frac{x}{R} + o(x)\right] = 2\frac{x}{R^3} + o(x).$$

(iii) We can rewrite the function as

$$f(x) = (1+x)^{1/3}(1-x)^{-1/3} - (1-x)^{1/3}(1+x)^{-1/3}$$

and then use, when $x \to 0$,

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + o(x^2).$$

For $\alpha = 1/3$ this leads to

$$(1+x)^{1/3} = 1 + \frac{x}{3} - \frac{x^2}{9} + o(x^2), \qquad (1-x)^{1/3} = 1 - \frac{x}{3} - \frac{x^2}{9} + o(x^2),$$

and for $\alpha = -1/3$

$$(1+x)^{-1/3} = 1 - \frac{x}{3} + \frac{2}{9}x^2 + o(x^2), \qquad (1-x)^{-1/3} = 1 + \frac{x}{3} + \frac{2}{9}x^2 + o(x^2).$$

Multiplying,

$$\begin{aligned}
(1+x)^{1/3}(1-x)^{-1/3} &= \left[1 + \frac{x}{3} - \frac{x^2}{9} + o(x^2)\right]\left[1 + \frac{x}{3} + \frac{2}{9}x^2 + o(x^2)\right] \\
&= 1 + \frac{2}{3}x + \frac{2}{9}x^2 + o(x^2), \\
(1-x)^{1/3}(1+x)^{-1/3} &= \left[1 - \frac{x}{3} - \frac{x^2}{9} + o(x^2)\right]\left[1 - \frac{x}{3} + \frac{2}{9}x^2 + o(x^2)\right] \\
&= 1 - \frac{2}{3}x + \frac{2}{9}x^2 + o(x^2),
\end{aligned}$$

and substracting,

$$f(x) = \frac{4}{3}x + o(x^2).$$

Problem 8.10 First of all, as $x \to 0$,

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + o(x^7),$$

so

$$\begin{aligned}
\frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} + o(x^7) &= \left[Ax^2 + Bx^4 + Cx^6 + o(x^7)\right]\left[2 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + o(x^7)\right] \\
&= 2Ax^2 + \left(2B - \frac{A}{2}\right)x^4 + \left(2C - \frac{B}{2} + \frac{A}{24}\right)x^6 + o(x^7).
\end{aligned}$$

The two expansions coincide if

$$\left.\begin{aligned}
2A &= \frac{1}{2}, \\
2B - \frac{A}{2} &= -\frac{1}{24}, \\
2C - \frac{B}{2} + \frac{A}{24} &= \frac{1}{720},
\end{aligned}\right\} \quad \Rightarrow \quad A = \frac{1}{4}, \quad B = \frac{1}{24}, \quad C = \frac{17}{2880}.$$

Problem 8.11

(i) Let $f(x) = x - (a + b\cos x)\sin x = x - a\sin x - b\sin x\cos x = x - a\sin x - (b/2)\sin 2x$. Up to fourth order,

$$\sin x = x - \frac{x^3}{6} + o(x^4), \qquad \sin 2x = 2x - \frac{4}{3}x^3 + o(x^4),$$

therefore

$$f(x) = x - ax + \frac{a}{6}x^3 - bx + \frac{2b}{3}x^3 + o(x^4) = (1-a-b)x + \frac{a+4b}{6}x^3 + o(x^4).$$

This function will be $f(x) = o(x^4)$ if and only if

$$\begin{cases} a+b=1, \\ a+4b=0, \end{cases} \qquad \Rightarrow \qquad a = \frac{4}{3}, \quad b = -\frac{1}{3}.$$

(ii) Both $\cot x$ and the rational function diverge when $x \to 0$, so we can multiply by x and write the equivalent equation

$$f(x) = x\cot x - \frac{1+ax^2}{1+bx^2} = o(x^5) \quad (x \to 0),$$

where neither of the two functions involved is singular at $x = 0$. Take first the rational function. This is a product of two functions, namely

$$\frac{1+ax^2}{1+bx^2} = (1+ax^2)(1+bx^2)^{-1}.$$

But $(1+z)^{-1} = 1 - z + z^2 - z^3 + o(z^3)$, $(z \to 0)$, therefore

$$(1+bx^2)^{-1} = 1 - bx^2 + b^2x^4 - b^3x^6 + o(x^6) = 1 - bx^2 + b^2x^4 + o(x^5)$$

(we don't need to keep powers higher than x^5 in the expansion). Substituting

$$\begin{aligned} \frac{1+ax^2}{1+bx^2} &= \left[1 - bx^2 + b^2x^4 + o(x^5)\right] + \left[ax^2 - abx^4 + o(x^5)\right] \\ &= 1 + (a-b)x^2 - b(a-b)x^4 + o(x^5). \end{aligned}$$

As for $x\cot x$, we can also write it as the product of two functions

$$x\cot x = x\frac{\cos x}{\sin x} = \cos x \left(\frac{\sin x}{x}\right)^{-1}.$$

Now,

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5), \qquad \frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} + o(x^5) = 1 - y,$$

where $y = \frac{x^2}{6} - \frac{x^4}{120} + o(x^5)$. Then

$$\begin{aligned} \left(\frac{\sin x}{x}\right)^{-1} &= 1 + y + y^2 + y^3 + o(y^3) = 1 + \left[\frac{x^2}{6} - \frac{x^4}{120} + o(x^5)\right] + \left[\frac{x^4}{36} + o(x^5)\right] + o(x^5) \\ &= 1 + \frac{x^2}{6} + \frac{7}{360}x^4 + o(x^5) \end{aligned}$$

(where we have taken into account that $y^3 + o(y^3) \propto x^6 + o(x^6) = o(x^5)$). Accordingly,

$$x\cot x = \left[1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)\right]\left[1 + \frac{x^2}{6} + \frac{7}{360}x^4 + o(x^5)\right] = 1 - \frac{x^2}{3} - \frac{x^4}{45} + o(x^5)$$

and

$$\begin{aligned} f(x) &= \left[1 - \frac{x^2}{3} - \frac{x^4}{45} + o(x^5)\right] - \left[1 + (a-b)x^2 - b(a-b)x^4 + o(x^5)\right] \\ &= \left(b - a - \frac{1}{3}\right)x^2 + \left(b(a-b) - \frac{1}{45}\right)x^4 + o(x^5). \end{aligned}$$

We will have $f(x) = o(x^5)$ if, and only if,

$$\begin{cases} b - a = \frac{1}{3}, \\ b(a-b) = \frac{1}{45}, \end{cases} \qquad \Rightarrow \qquad a = -\frac{2}{5}, \quad b = -\frac{1}{15}.$$

The interest of this exercise is to show that when $x \to 0$,

$$\cot x = \frac{1 - \frac{2}{5}x^2}{x - \frac{1}{15}x^3} + o(x^4),$$

which provides a reasonable approximation of $\cot x$ as a rational function near $x = 0$.

Problem 8.12 We can transform the expression into

$$e^x(1 + cx + dx^2) = 1 + ax + bx^2 + o(x^4), \quad (x \to 0),$$

because $o(x^4)(1+cx+dx^2) = o(x^4) + o(x^5) + o(x^6) = o(x^4)$. Using the expasion for the exponential

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + o(x^4), \quad (x \to 0),$$

and multiplying we get

$$e^x(1+cx+dx^2) = 1 + (1+c)x + \left(\frac{1}{2} + c + d\right)x^2 + \left(\frac{1}{6} + \frac{c}{2} + d\right)x^3 + \left(\frac{1}{24} + \frac{c}{6} + \frac{d}{2}\right)x^4 + o(x^4).$$

Comparing the two sides of the first equation we obtain

$$1 + c = a, \qquad \frac{1}{6} + \frac{c}{2} + d = 0,$$

$$\frac{1}{2} + c + d = b, \qquad \frac{1}{24} + \frac{c}{6} + \frac{d}{2} = 0.$$

The two equations on the right can be rewritten as

$$\begin{cases} c + 2d = -\dfrac{1}{3}, \\ c + 3d = -\dfrac{1}{4}, \end{cases} \quad \Rightarrow \quad d = \frac{1}{12}, \quad c = -\frac{1}{2},$$

and from these values we get $a = \dfrac{1}{2}$, $b = \dfrac{1}{12}$ from the equations on the left. Hence

$$e^x = \frac{1 + \frac{x}{2} + \frac{x^2}{12}}{1 - \frac{x}{2} + \frac{x^2}{12}} + o(x^4), \quad (x \to 0).$$

Problem 8.13

(i) First of all we can write

$$\sqrt{1+n^2} = n\sqrt{1+\frac{1}{n^2}} = n\left[1+\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)\right] = n+\frac{1}{2n}+o\left(\frac{1}{n}\right) \quad (n\to\infty).$$

Let us denote $\varepsilon_n = \frac{\pi}{2n}+o\left(\frac{1}{n}\right)$. Then

$$\sin\pi\sqrt{1+n^2} = \sin(\pi n+\varepsilon_n) = (-1)^n\sin\varepsilon_n = (-1)^n\left[\varepsilon_n+o(\varepsilon_n)\right].$$

But $o(\varepsilon_n) = o\left(\frac{1}{n}\right)$, so

$$\sin\pi\sqrt{1+n^2} = (-1)^n\frac{\pi}{2n}+o\left(\frac{1}{n}\right) = (-1)^n\frac{\pi}{2n}[1+o(1)] \quad (n\to\infty).$$

Thus

$$\lim_{n\to\infty}\sin\pi\sqrt{1+n^2} = 0.$$

(ii) From the previous result

$$\sin^2\pi\sqrt{1+n^2} = \frac{\pi^2}{4n^2}[1+o(1)] \quad (n\to\infty);$$

in other words, $\sin^2\pi\sqrt{1+n^2}\sim\frac{\pi^2}{4n^2}$ when $n\to\infty$. The convergence of the series follows from the fact that $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}<\infty$.

Problem 8.14 Since $\sin x = x+o(x)$, then $f(x) = 1+x^4+o(x^4)$, when $x\to 0$. Thus $P_{4,0}(x) = 1+x^4$. Accordingly f has a local minimum at $x=0$.

Problem 8.15

(i) Let us consider the function

$$f(x) = \frac{1}{\sqrt{1+x}}.$$

The value we want to obtain is $f(0.1)$. The Taylor expansion for this function near $a=0$ follows from

$$\begin{aligned}
f(x) &= (1+x)^{-1/2}, & f(0) &= 1,\\
f'(x) &= -\frac{1}{2}(1+x)^{-3/2}, & f'(0) &= -\frac{1}{2},\\
f''(x) &= \frac{3}{4}(1+x)^{-5/2}, & f''(0) &= \frac{3}{4},\\
f'''(x) &= -\frac{15}{8}(1+x)^{-7/2}, & f'''(0) &= -\frac{15}{8},\\
f^{(4)}(x) &= \frac{105}{16}(1+x)^{-9/2},
\end{aligned}$$

which implies

$$P_{3,0}(x) = 1-\frac{x}{2}+\frac{3}{8}x^2-\frac{5}{16}x^3, \qquad R_{3,0}(x) = \frac{35}{128}\left(\frac{1}{\sqrt{1+\theta x}}\right)^9 x^4, \quad 0<\theta<1.$$

Now $P_{3,0}(0.1) = 0.9534375$ and since $\sqrt{1+\theta x} > 1$ for every $x > 0$,

$$|R_{3,0}(x)| < \frac{35}{128}x^4 \quad \Rightarrow \quad |R_{3,0}(0.1)| < 2.7 \times 10^{-5}.$$

Hence $1/\sqrt{1.1} = 0.9534(3)$ —where the figure in brackets may be affected by the error. (The exact value is $1/\sqrt{1.1} = 0.953462589\dots$)

(ii) Consider the function $f(x) = \sqrt[3]{27+x} =$. Then $\sqrt[3]{28} = f(1)$. To ontain the second degree Taylor expansion around $a = 0$ we calculate

$$\begin{aligned}
f(x) &= (27+x)^{1/3}, & f(0) &= 3,\\
f'(x) &= \frac{1}{3}(27+x)^{-2/3}, & f'(0) &= \frac{1}{27},\\
f''(x) &= -\frac{2}{9}(27+x)^{-5/3}, & f''(0) &= -\frac{2}{2187},\\
f'''(x) &= \frac{10}{27}(27+x)^{-8/3}, & &
\end{aligned}$$

from which

$$P_{2,0}(x) = 3 + \frac{x}{27} - \frac{x^2}{2187}, \qquad R_{2,0}(x) = \frac{5}{81}\frac{x^3}{\left(\sqrt[3]{27+\theta x}\right)^8}, \qquad 0 < \theta < 1.$$

Now $P_{2,0}(1) = 3.03657979$ and since $\sqrt[3]{27+\theta x} > \sqrt[3]{27} = 3$ for every $x > 0$,

$$|R_{2,0}(x)| < \frac{5x^3}{531441} \qquad \Rightarrow \qquad |R_{2,0}(1)| < \frac{5}{531441} = 0.9408 \times 10^{-5}.$$

Hence $\sqrt[3]{28} = 3.0365(8)$. (As a matter of fact $\sqrt[3]{28} = 3.036588972\dots$)

Problem 8.16

(i) Since for $x \to 0$

$$\cos x = 1 - \frac{x^2}{2} + o(x^3), \qquad e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3),$$

then

$$P_{3,0}(x) = 2 + x + \frac{x^3}{6}.$$

(ii) First of all $(\cos x)^{(4)} = \cos x$ and $(e^x)^{(4)} = e^x$, so $f^{(4)}(x) = f(x)$. Therefore

$$R_{3,0}(x) = \frac{\cos\theta x + e^{\theta x}}{24}x^4, \quad 0 < \theta < 1.$$

Now $|\cos\theta x| \leqslant 1$ and $e^{\theta x} \leqslant \max\{e^x, 1\}$. Thus for $-1/4 \leqslant x \leqslant 1/4$

$$|R_{3,0}(x)| < \frac{1+e^{1/4}}{24}\left(\frac{1}{4}\right)^4 = 3.72 \times 10^{-4}.$$

Problem 8.17 The reminder of the Taylor expansion of $f(x) = e^x$ around $a = 0$ is

$$R_{n,0}(x) = \frac{e^{\theta x}}{(n+1)!}x^{n+1}, \quad 0 < \theta < 1,$$

so an upper bound for $-1 \leqslant x \leqslant 1$ will be

$$|R_{n,0}(x)| < \frac{e}{(n+1)!}.$$

If we want to have three exact decimal places the error should be smaller than 10^{-3}, so we must look for the smallest n for which $(n+1)! > 10^3$. Since $6! = 720$ and $7! = 5040$ then $n = 6$.

Problem 8.18

(i)

$$\frac{1}{\rho} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{2^n n^2}} = \frac{1}{2} \qquad \Rightarrow \qquad \rho = 2.$$

For $x = 2$

$$\sum_{n=1}^{\infty} \frac{2^n}{2^n n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

so the interval of absolute convergence is $[-2, 2]$.

(ii)

$$\rho = \lim_{n\to\infty} \frac{n!}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)!} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

For $x = e$

$$\frac{n! e^n}{n^n} \sim \sqrt{2\pi n}$$

so the series does not converge absolutely at $x = \pm e$. Therefore the inveral of absolute convergence is $(-e, e)$.

(iii)

$$\frac{1}{\rho} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n 10^{n-1}}} = \frac{1}{10} \qquad \Rightarrow \qquad \rho = 10.$$

For $x = 10$

$$\sum_{n=1}^{\infty} \frac{10^n}{n 10^{n-1}} = 10, \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

so the interval of absolute convergence is $(-10, 10)$.

(iv)

$$\rho = \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} = 1.$$

For $x = 1$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty,$$

so the interval of absolute convergence is $(-1, 1)$.

(v) We can rewrite the series as

$$\sum_{n=0}^{\infty}(-2)^n\left(x-\frac{3}{2}\right)^n,$$

so the radius of convergence is

$$\frac{1}{\rho}=\lim_{n\to\infty}\sqrt[n]{2^n}=2 \qquad \Rightarrow \qquad \rho=\frac{1}{2}.$$

When $x-3/2=\pm 1/2$

$$\sum_{n=0}^{\infty}2^n\left|x-\frac{3}{2}\right|^n=\sum_{n=0}^{\infty}2^n\frac{1}{2^n}=\sum_{n=0}^{\infty}1=\infty.$$

Therefore the interval of absolute convergence is given by $|x-3/2|<1/2$, i.e., $(1,2)$.

(vi)

$$\rho=\lim_{n\to\infty}\frac{\sqrt{2(n+1)}}{\sqrt{2n}}=1,$$

and for $|x-2|=1$

$$\sum_{n=1}^{\infty}\frac{|x-2|^n}{\sqrt{2n}}=\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n}}=\infty,$$

so the interval of absolute convergence is given by $|x-2|<1$, i.e., $(1,3)$.

Problem 8.19 We can rewrite the function as $f(x)=(1-x)^{-k}$, which matches the function

$$(1+t)^{\alpha}=\sum_{n=0}^{\infty}\binom{\alpha}{n}t^n, \qquad \binom{\alpha}{n}=\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!},$$

for $t=-x$ and $\alpha=-k$. Thus

$$\frac{1}{(1-x)^k}=\sum_{n=0}^{\infty}\binom{-k}{n}(-1)^n x^n.$$

Now

$$\begin{aligned}(-1)^n\binom{-k}{n}&=(-1)^n\frac{(-k)(-k-1)\cdots(-k-n+1)}{n!}=(-1)^n(-1)^n\frac{k(k+1)\cdots(k+n-1)}{n!}\\&=\frac{(k+n-1)!}{n!(k-1)!}=\binom{k+n-1}{k-1}.\end{aligned}$$

Therefore

$$\frac{1}{(1-x)^k}=\sum_{n=0}^{\infty}\binom{n+k-1}{k-1}x^n.$$

For $k=1$ the coefficients are $\binom{n}{0}=1$, so

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n,$$

as expected.

For $k=2$ the coefficients are $\binom{n+1}{1}=n+1$, so

$$\frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}(n+1)x^n.$$

Finally, for $k=3$ the coefficients are $\binom{n+2}{2}=(n+2)(n+1)/2$, so

$$\frac{1}{(1-x)^3}=\frac{1}{2}\sum_{n=0}^{\infty}(n+2)(n+1)x^n.$$

An alternative way of getting the same result goes as follows. We know the result for $k=1$ because it is the geometric series,

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n.$$

Now differentiating this equation we obtain

$$\frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}(n+1)x^n,$$

which is the result for $k=2$. And differentiating again,

$$\frac{2}{(1-x)^3}=\sum_{n=0}^{\infty}(n+2)(n+1)x^n,$$

which, divided by 2, yields the result for $k=3$.

Problem 8.20 Using the hint we write

$$\frac{1}{x^2+x+1}=\frac{1-x}{1-x^3}=\frac{1}{1-x^3}-\frac{x}{1-x^3}.$$

Now,

$$\frac{1}{1-x^3}=\sum_{n=0}^{\infty}\left(x^3\right)^n=\sum_{n=0}^{\infty}x^{3n}, \qquad \frac{x}{1-x^3}=\sum_{n=0}^{\infty}x^{3n+1},$$

therefore

$$\frac{1}{x^2+x+1}=\sum_{n=0}^{\infty}x^{3n}-\sum_{n=0}^{\infty}x^{3n+1}.$$

In other words, the coefficients a_n are such that $a_{3n}=1$, $a_{3n+1}=-1$, and $a_{3n+2}=0$. Accordingly $a_{300}=1$, $a_{301}=-1$, $a_{302}=0$.

Problem 8.21 For a function $f(x)$ that can be expanded as a Taylor series

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n.$$

Therefore the coefficient of x^n in this series is $f^{(n)}(0)/n!$. Since

$$\log(1+u)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}u^n$$

then

$$\log(1+x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^{2n}.$$

Therefore $f^{(231)}(0) = 0$ because the coefficients of the odd terms are all zero, and the coefficient of x^{100} is

$$\frac{(-1)^{51}}{50} = -\frac{1}{50} \quad \Rightarrow \quad \frac{f^{(100)}(0)}{100!} = -\frac{1}{50} \quad \Rightarrow \quad f^{(100)}(0) = -\frac{100!}{50}.$$

Problem 8.22

(i) Let us denote $f(x)$ the sum of the series. Then

$$f'(x) = \sum_{n=1} x^{n-1} = \sum_{n=0} x^n = \frac{1}{1-x}, \quad |x| < 1.$$

Therefore $f(x) = -\log(1-x) + c$. To determine the constant we just calculate $f(0)$. From the series $f(0) = 0$, and from the latter expression $f(0) = c$. Therfore $c = 0$ and

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x), \quad |x| < 1.$$

(ii) First of all we can write the series as

$$\sum_{n=0}^{\infty} (n+1)\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} (n+1)t^n,$$

where $t = x/2$. Now

$$\sum_{n=0}^{\infty} (n+1)t^n = \sum_{n=1}^{\infty} nt^{n-1} = \sum_{n=1}^{\infty} (t^n)' \underset{(*)}{=} \left(\sum_{n=0}^{\infty} t^n\right)' = \left(\frac{1}{1-t}\right)' = \frac{1}{(1-t)^2}$$

(in (*) we have added the term $n = 0$ to the sum because it is a constant, and the derivative of a constant is zero). Therefore

$$\sum_{n=0}^{\infty} (n+1)2^{-n}x^n = \frac{1}{(1-x/2)^2} = \frac{4}{(2-x)^2}, \quad |x| < 2.$$

(The radius of convergence of the geometric series is 1, i.e., converges for $|t| < 1$; since $t = x/2$, our series —which is the derivative of the geometric— converges for $|x| < 2$.)

Problem 8.23

(i) To begin with

$$f(x) = \sin^2 x = \frac{1}{2}(1 - \cos 2x),$$

and since

$$\cos t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!}, \quad t \in \mathbb{R},$$

substituting we obtain

$$f(x) = \frac{1}{2}\left[1 - \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!}\right] = \frac{1}{2}\left[-\sum_{n=1}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!}\right] = \frac{1}{2}\sum_{n=1}^{\infty} (-1)^{n+1} 2^{2n} \frac{x^{2n}}{(2n)!}$$
$$= \sum_{n=1}^{\infty} (-1)^{n+1} 2^{2n-1} \frac{x^{2n}}{(2n)!}, \quad x \in \mathbb{R}.$$

(ii) We can rewrite

$$f(x) = \log\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\log(1+x) - \frac{1}{2}\log(1-x)$$

and use

$$\log(1+t) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{t^n}{n}, \quad |t| < 1,$$

to obtain

$$f(x) = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{x^n}{n} = \sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}+1}{2}\right]\frac{x^n}{n}, \quad |x| < 1.$$

But

$$\frac{(-1)^{n+1}+1}{2} = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

Therefore

$$f(x) = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}, \quad |x| < 1.$$

(iii) We can rewrite

$$f(x) = \frac{x}{a}\cdot\frac{1}{1+bx/a}.$$

Now since

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n, \quad |t| < 1,$$

then

$$f(x) = \frac{x}{a}\sum_{n=0}^{\infty}(-1)^n\frac{b^n}{a^n}x^n = \sum_{n=0}^{\infty}(-1)^n\frac{b^n}{a^{n+1}}x^{n+1} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{b^{n-1}}{a^n}x^n, \quad |x| < \left|\frac{a}{b}\right|.$$

(iv) We can express

$$f(x) = \frac{1}{2}\frac{1}{1-x^2/2} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x^2}{2}\right)^n = \sum_{n=0}^{\infty}\frac{x^{2n}}{2^{n+1}},$$

and the converge requires $x^2/2 < 1$, i.e., $|x| < \sqrt{2}$.

(v) We can rewrite

$$f(x) = (1+x)e^{-x} - (1-x)e^x.$$

At this point we can already expand the exponential, so that

$$\begin{aligned}(1-x)e^x &= (1-x)\sum_{n=0}^{\infty}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{x^n}{n!} - \sum_{n=0}^{\infty}\frac{x^{n+1}}{n!} = \sum_{n=0}^{\infty}\frac{x^n}{n!} - \sum_{n=1}^{\infty}\frac{x^n}{(n-1)!} \\ &= 1 + \sum_{n=1}^{\infty}\left(\frac{1}{n!} - \frac{1}{(n-1)!}\right)x^n = 1 + \sum_{n=1}^{\infty}\left(\frac{1}{n!} - \frac{n}{n!}\right)x^n = 1 + \sum_{n=1}^{\infty}\frac{1-n}{n!}x^n.\end{aligned}$$

The expansion of $(1+x)e^{-x}$ is the same one but changing x by $-x$, i.e.,

$$(1+x)e^{-x} = 1 + \sum_{n=1}^{\infty} \frac{1-n}{n!}(-x)^n = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1-n}{n!} x^n.$$

Substracting both expansions

$$\begin{aligned} f(x) = (1+x)e^{-x} - (1-x)e^{x} &= \sum_{n=1}^{\infty} (-1)^n \frac{1-n}{n!} x^n - \sum_{n=1}^{\infty} \frac{1-n}{n!} x^n \\ &= \sum_{n=1}^{\infty} \frac{n-1}{n!} x^n - \sum_{n=1}^{\infty} (-1)^n \frac{n-1}{n!} x^n = \sum_{n=1}^{\infty} [1-(-1)^n] \frac{n-1}{n!} x^n. \end{aligned}$$

But $1-(-1)^n = 2$ when n is odd and $= 0$ when n is even, therefore

$$f(x) = \sum_{k=0}^{\infty} \frac{4k}{(2k+1)!} x^{2k+1}.$$

An alternative derivation arises from realising that

$$f(x) = e^{-x} + xe^{-x} - e^{x} + xe^{x} = 2x\cosh x - 2\sinh x,$$

for then

$$\begin{aligned} f(x) &= 2x \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} - 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n)!} - 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \\ &= 2\sum_{n=0}^{\infty} \left(\frac{1}{(2n)!} - \frac{1}{(2n+1)!} \right) x^{2n+1} = 2\sum_{n=0}^{\infty} \left(\frac{2n+1}{(2n+1)!} - \frac{1}{(2n+1)!} \right) x^{2n+1} \\ &= 2\sum_{n=0}^{\infty} \frac{2n}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \frac{4n}{(2n+1)!} x^{2n+1}. \end{aligned}$$

Problem 8.24

(i)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} = \sum_{n=0}^{\infty} \frac{(-1/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \bigg|_{x=-1/2} = e^{-1/2}.$$

(ii)

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n}{2^n} &= \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n = \frac{1}{2} \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^{n-1} = \frac{1}{2} \sum_{n=1}^{\infty} n x^{n-1} \bigg|_{x=1/2} = \frac{1}{2} \left(\sum_{n=0}^{\infty} x^n \right)' \bigg|_{x=1/2} \\ &= \frac{1}{2} \left(\frac{1}{1-x} \right)' \bigg|_{x=1/2} = \frac{1}{2(1-x)^2} \bigg|_{x=1/2} = \frac{1}{2(1-1/2)^2} = 2. \end{aligned}$$

(iii)

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{x^n}{n} \bigg|_{x=1/2} = -\log(1-x)\Big|_{x=1/2} = -\log(1/2) = \log 2.$$

(iv)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \bigg|_{x=1} = \arctan 1 = \frac{\pi}{4}.$$

Problem 8.25 For $x = 0$

$$f(0) = \sum_{n=1}^{\infty} \frac{1}{n!} = e - 1.$$

For $x = 1$

$$f(1) = \sum_{n=1}^{\infty} \frac{n}{n!} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = \sum_{n=0}^{\infty} \frac{1}{n!} = e.$$

For $x = 2$

$$f(2) = \sum_{n=1}^{\infty} \frac{n^2}{n!} = \sum_{n=1}^{\infty} \frac{n}{(n-1)!} = \sum_{n=0}^{\infty} \frac{n+1}{n!} = \sum_{n=1}^{\infty} \frac{n}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!} = f(1) + e = 2e.$$

Problem 8.26 Since $f(0) = 2$ the series must be

$$f(x) = 2 + \sum_{n=1}^{\infty} a_n x^n.$$

Now,

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n,$$

so $f'(x) = f(x) + x$ implies

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = x + 2 + \sum_{n=1}^{\infty} a_n x^n,$$

or equivalently

$$a_1 + 2a_2 x + \sum_{n=2}^{\infty} (n+1) a_{n+1} x^n = 2 + (1 + a_1)x + \sum_{n=2}^{\infty} a_n x^n.$$

From this equality we get $a_1 = 2$, $a_2 = (1 + a_1)/2 = 3/2$ and for $n > 1$

$$a_{n+1} = \frac{a_n}{n+1}.$$

The iteration yields

$$a_n = \frac{1}{n} a_{n-1} = \frac{1}{n(n-1)} a_{n-2} = \frac{1}{n(n-1)(n-2)} a_{n-3} = \cdots = \frac{1}{n(n-1)(n-2)\cdots 4 \cdot 3} a_2.$$

The denominator is $n!/2$, so

$$a_n = \frac{2a_2}{n!} = \frac{3}{n!}, \quad n > 1.$$

Therefore

$$f(x) = 2 + 2x + 3 \sum_{n=2}^{\infty} \frac{x^n}{n!} = 2 + 2x + 3(e^x - 1 - x) = 3e^x - 1 - x.$$

It is straightforward to check that this function satisfies both $f(0) = 2$ and $f'(x) = f(x) + x$.

Problem 8.27 Let us compute two derivatives of h:

$$h' = (f' \circ g)g', \qquad h'' = (f' \circ g)'g' + (f' \circ g)g'' = (f'' \circ g)(g')^2 + (f' \circ g)g''.$$

Since f is convex $f'' \circ g > 0$; since f is increasing $f' \circ g > 0$; since g is convex $g'' > 0$; and of course $(g')^2 \geqslant 0$. Therefore $h'' > 0$, hence h is convex.

Problem 8.28

(i) $f(x) = x^{5/3} - 2x^{2/3}$, so

$$f'(x) = \frac{5}{3}x^{2/3} - \frac{4}{3}x^{-1/3}, \qquad f''(x) = \frac{10}{9}x^{-1/3} + \frac{4}{9}x^{-4/3} = \frac{10}{9}x^{-4/3}\left(x + \frac{2}{5}\right).$$

Since $x^{-4/3} > 0$ for all $x \neq 0$, then $f(x)$ is concave for $x < -2/5$ and convex in $-2/5 < x < 0$ and $x > 0$. At $x = -2/5$ it has an inflection point, and at $x = 0$ the function has a nondifferentiable cusp.

(ii) $f(x)$ is not differentiable at $x = 0$. Now, for $x > 0$

$$f(x) = xe^x, \qquad f'(x) = (x+1)e^x, \qquad f''(x) = (x+2)e^x,$$

so the funtion is always convex. On the other hand, the function is even (because $f(-x) = f(x)$), so it is convex also for $x < 0$.

(iii) $x^2 - 6x + 8 = (x-2)(x-4)$, so the domain of this function is $(-\infty, 2) \cup (4, \infty)$. On the other hand, in its domain

$$f(x) = \log(x^2 - 6x + 8) = \log|x^2 - 6x + 8| = \log|x-2| + \log|x-4|,$$

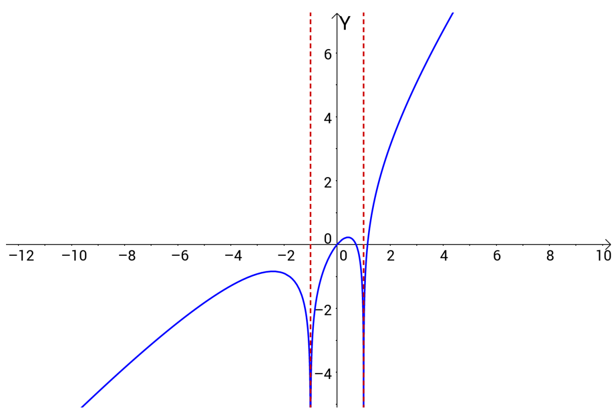
so

$$f'(x) = \frac{1}{x-2} + \frac{1}{x-4}, \qquad f''(x) = -\frac{1}{(x-2)^2} - \frac{1}{(x-4)^2},$$

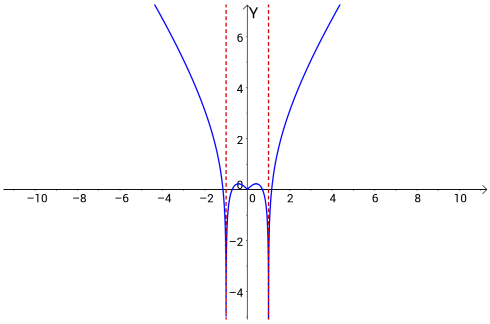
and then we have $f''(x) < 0$ in the whole domain of the function. Thus $f(x)$ is concave.

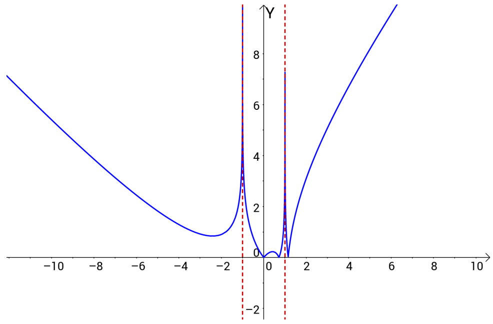
Problem 8.29

(i) $f(x) = x + \log|x^2 - 1|$:



(ii) $g(x) = f(|x|)$ $h(x) = |f(x)|$:

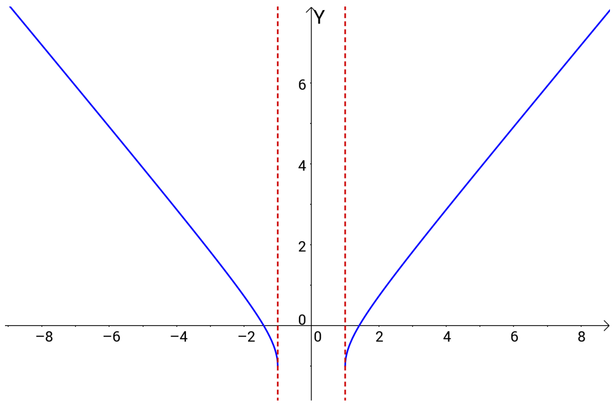




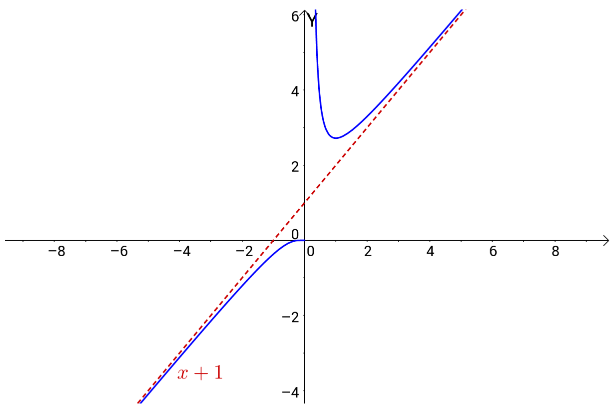
Problem 8.30

(i) $f(x) = e^x \sin x$: this function oscillates between $y = e^x$ and $y = -e^x$, crossing the X axis at $x = n\pi$, where $n \in \mathbb{Z}$.

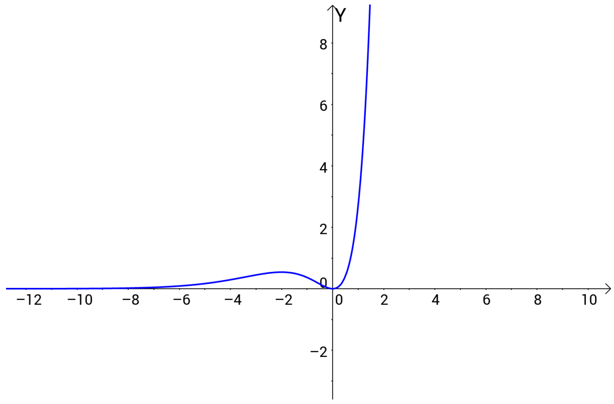
(ii) $f(x) = \sqrt{x^2 - 1} - 1$:



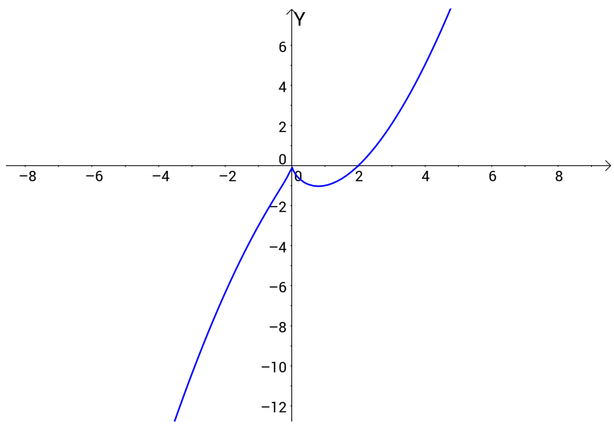
(iii) $f(x) = xe^{1/x}$:



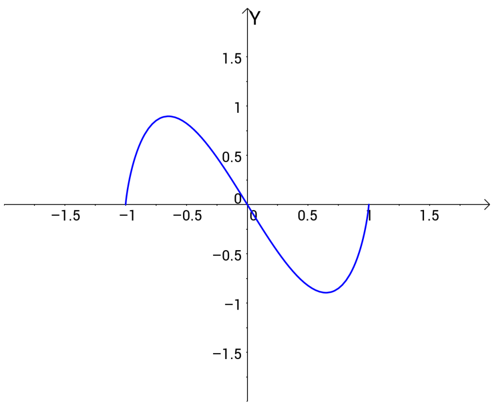
(iv) $f(x) = x^2 e^x$:



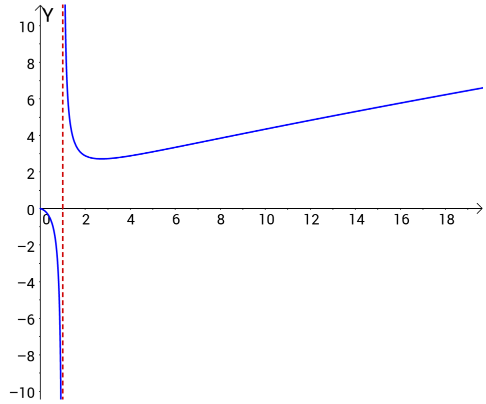
(v) $f(x) = (x-2)x^{2/3}$:



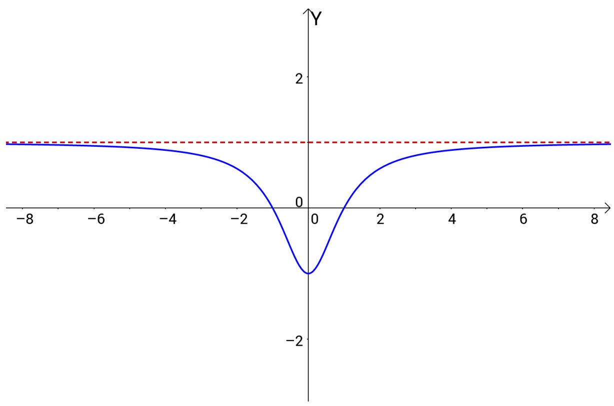
(vi) $f(x) = (x^2-1)\log\left(\dfrac{1+x}{1-x}\right)$:



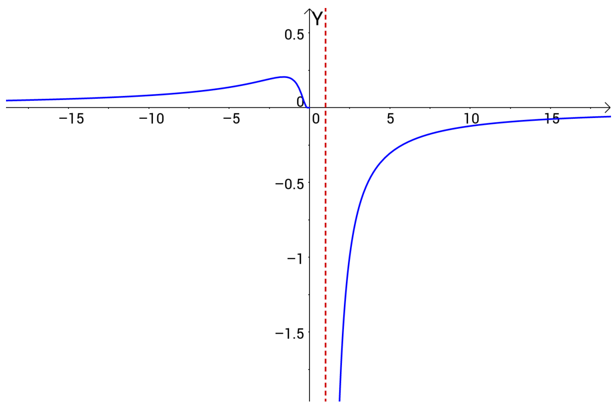
(vii) $f(x) = \dfrac{x}{\log x}$:



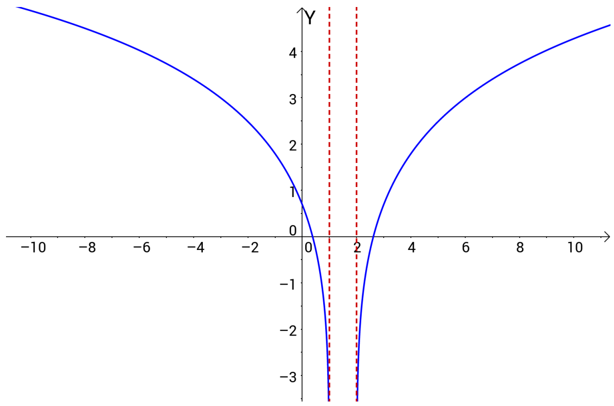
(viii) $f(x) = \dfrac{x^2-1}{x^2+1}$:



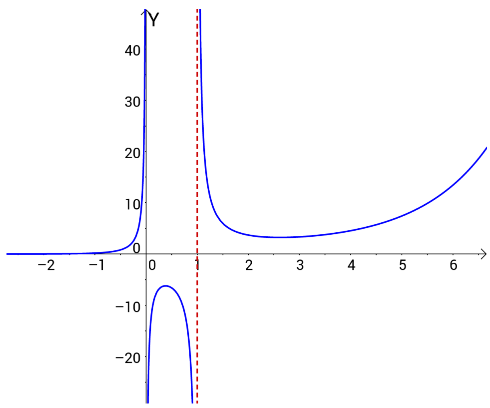
(ix) $f(x) = \dfrac{e^{1/x}}{1-x}$:



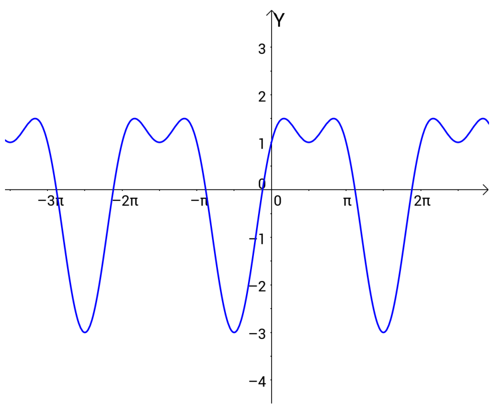
(x) $f(x) = \log\left[(x-1)(x-2)\right]$:



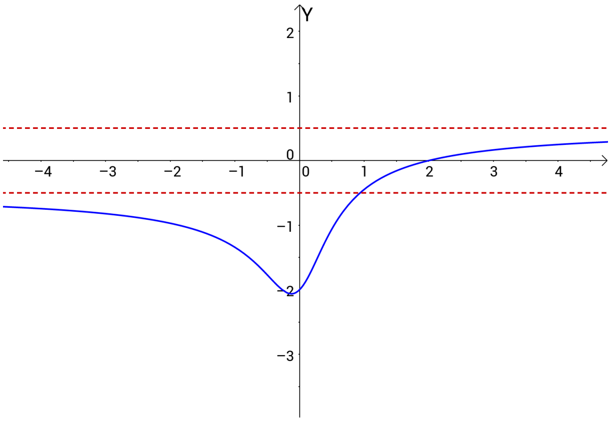
(xi) $f(x) = \dfrac{e^x}{x(x-1)}$:



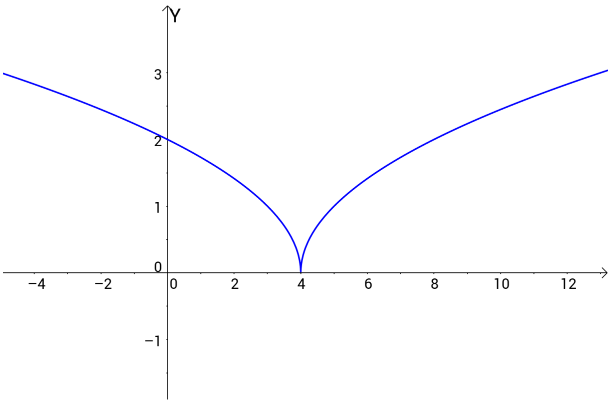
(xii) $f(x) = 2\sin x + \cos 2x$:



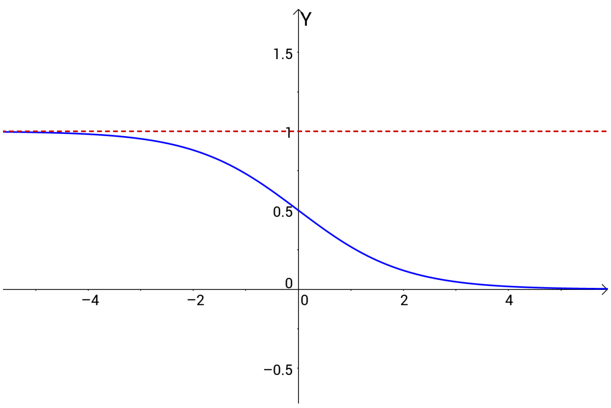
(xiii) $f(x) = \dfrac{x-2}{\sqrt{4x^2+1}}$:



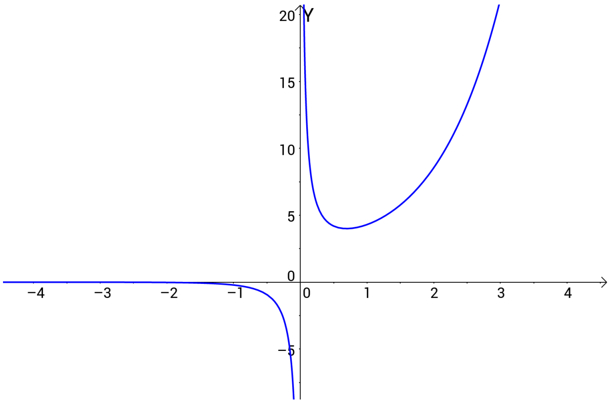
(xiv) $f(x) = \sqrt{|x-4|}$:



(xv) $f(x) = \dfrac{1}{1+e^x}$:



(xvi) $f(x) = \dfrac{e^{2x}}{e^x - 1}$:



(xvii) $f(x) = e^{-x}\sin x$: this function oscillates between $y = e^{-x}$ and $y = -e^{-x}$, crossing the X axis at $x = n\pi$, where $n \in \mathbb{Z}$.

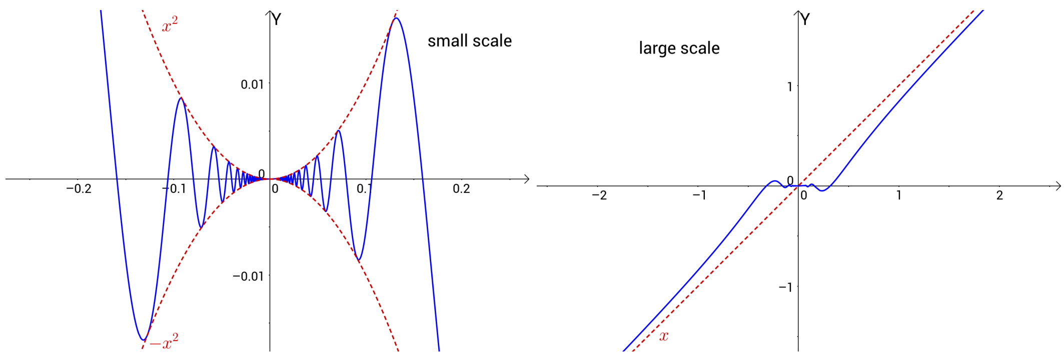
(xviii) $f(x) = x^2 \sin\dfrac{1}{x}$: this function has an oblique asymptote because

$$\sin\frac{1}{x} = \frac{1}{x} + o\left(\frac{1}{x^2}\right) \qquad (x \to \pm\infty)$$

(given that $\sin t = t + o(t^2)$ $(t \to 0)$); hence

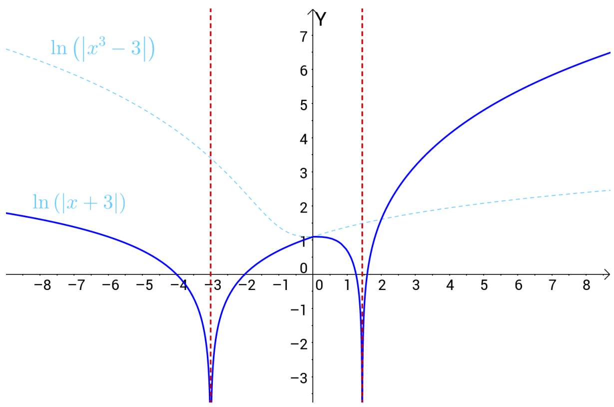
$$f(x) = x^2\left[\frac{1}{x} + o\left(\frac{1}{x^2}\right)\right] = x + o(1) \qquad (x \to \pm\infty).$$

Therefore the function looks different on a small scale and on a large scale. On a small scale it is an oscillatory function between $-x^2$ and x^2 that crosses the X axis at $x = \pm\frac{1}{n\pi}$, for all $n \in \mathbb{Z} - \{0\}$; on a large scale it asymptotes to $y = x$:



Problem 8.31

(i) $f(x) = \min\{\log|x^3 - 3|, \log|x+3|\}$:



(ii) $f(x) = \dfrac{1}{|x|-1} - \dfrac{1}{|x-1|}$: this function has a different form for $x > 1$, for $0 < x < 1$ and for $x < 0$.
For $x > 1$

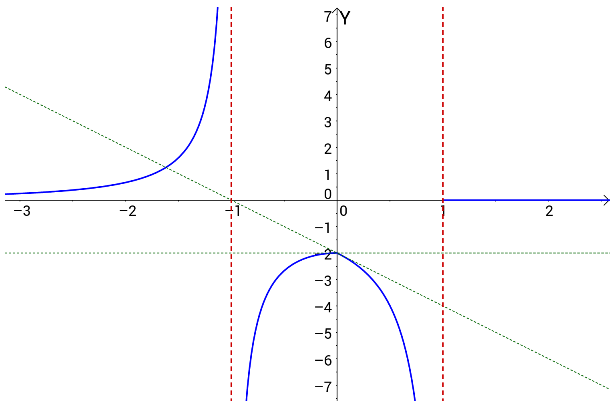
$$f(x) = \frac{1}{x-1} - \frac{1}{x-1} = 0.$$

For $0 < x < 1$ we have $|x-1| = -(x-1)$ so

$$f(x) = \frac{1}{x-1} + \frac{1}{x-1} = \frac{2}{x-1}.$$

For $x < 0$ we have $|x| - 1 = -(x+1)$ and $|x-1| = -(x-1)$, so

$$f(x) = -\frac{1}{x+1} + \frac{1}{x-1} = \frac{2}{x^2-1}.$$



(iii) $f(x) = \dfrac{1}{1+|x|} - \dfrac{1}{1+|x-a|}$ $(a > 0)$: this function also has different definitions depending on whether $x > a$, $0 < x < a$, or $x < 0$.
For $x > a$

$$f(x) = \frac{1}{1+x} - \frac{1}{1+x-a} = \frac{-a}{(x+1)(x-a+1)},$$

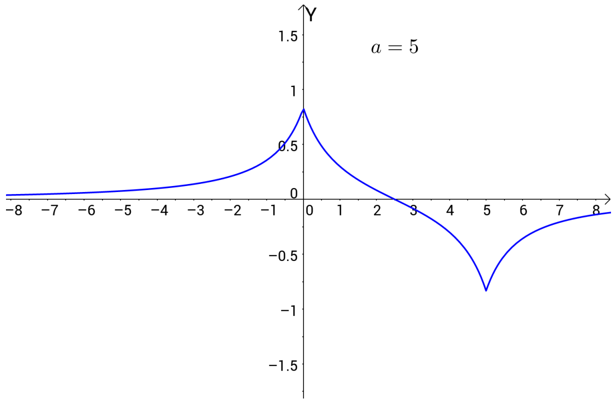
which has two vertical asymptotes, $x = -1$ and $x = a-1$, both out of the region $x > a$.
For $0 < x < a$

$$f(x) = \frac{1}{1+x} - \frac{1}{1+a-x} = \frac{2x-a}{(x+1)(x-a-1)},$$

which again has two asymptotes, $x = -1$ and $x = a+1$, both out of the region $0 < x < a$.
For $x < 0$

$$f(x) = \frac{1}{1-x} - \frac{1}{1+a-x} = \frac{a}{(x-1)(x-a-1)},$$

which also has two asymptotes, $x = 1$ and $x = a+1$, both out of the region $x < 0$.
Here is a plot for $a = 5$ (which is generic):

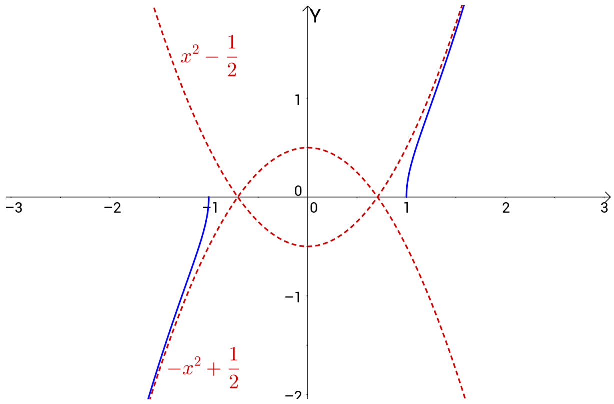


(iv) $f(x) = x\sqrt{x^2-1}$: notice that

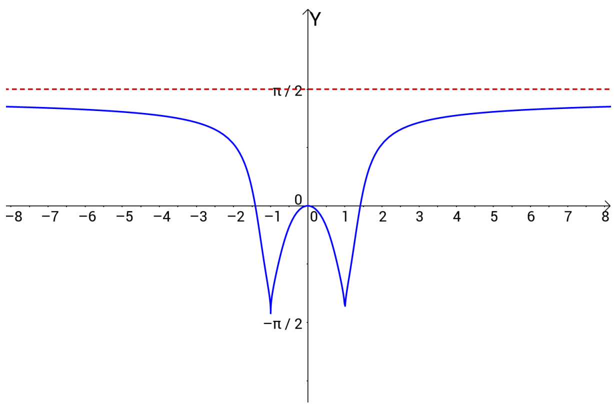
$$f(x) = x|x|\sqrt{1-\frac{1}{x^2}},$$

and since $\sqrt{1-t} = 1 - t/2 + o(t)$ $(t \to 0)$, when $x \to \pm\infty$,

$$f(x) = x|x|\left[1 - \frac{1}{2x^2} + o\left(\frac{1}{x^2}\right)\right] = x|x| - \frac{|x|}{2x} + o(1) = \begin{cases} x^2 - \frac{1}{2} + o(1) & (x \to \infty), \\ -x^2 + \frac{1}{2} + o(1) & (x \to -\infty). \end{cases}$$



(v) $f(x) = \arctan\log|x^2-1|$: when $x \to \pm 1$ the logarithm diverges to $-\infty$, so $f(x) \to -\pi/2$. In other words, even though the function is not well defined in $x = \pm 1$, at these two points it has an *avoidable* discontinuity which can be remedied by setting $f(\pm 1) = -\pi/2$. On the other hand, as $x \to \pm\infty$ the logarithm diverges to ∞ and therefore $f(x) \to \pi/2$.



(vi) $f(x) = 2\arctan x + \arcsin\left(\dfrac{2x}{1+x^2}\right)$: the domain of this function is $\mathbb{R}$ because so is the domain of $\arctan x$ and the argument of the arcsin is within $[-1,1]$. To see this

$$(x-1)^2 \geqslant 0 \quad\Leftrightarrow\quad x^2-2x+1\geqslant 0 \quad\Leftrightarrow\quad x^2+1\geqslant 2x \quad\Leftrightarrow\quad \frac{2x}{x^2+1}\leqslant 1,$$

$$(x+1)^2 \geqslant 0 \quad\Leftrightarrow\quad x^2+2x+1\geqslant 0 \quad\Leftrightarrow\quad x^2+1\geqslant -2x \quad\Leftrightarrow\quad -\frac{2x}{x^2+1}\leqslant 1$$

$$\Leftrightarrow\quad \frac{2x}{x^2+1}\geqslant -1.$$

if we calculate $f'(x)$, using the fact that

$$\left(\frac{2x}{1+x^2}\right)' = \frac{2(1+x^2)-(2x)^2}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2},$$

we obtain

$$f'(x) = \frac{2}{1+x^2} + \frac{1}{\sqrt{1-\frac{4x^2}{(1+x^2)^2}}}\frac{2(1-x^2)}{(1+x^2)^2}.$$

But

$$1-\frac{4x^2}{(1+x^2)^2} = \frac{1+2x^2+x^4-4x^2}{(1+x^2)^2} = \frac{1-2x^2+x^4}{(1+x^2)^2} = \frac{(1-x^2)^2}{(1+x^2)^2},$$

so

$$f'(x) = \frac{2}{1+x^2} + \frac{(1+x^2)}{|1-x^2|}\cdot\frac{2(1-x^2)}{(1+x^2)^2} = \frac{2}{1+x^2}\left[1+\frac{1-x^2}{|1-x^2|}\right].$$

Now

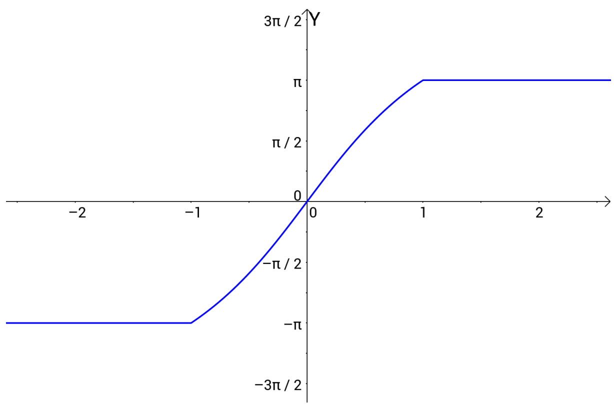
$$\frac{1-x^2}{|1-x^2|} = \begin{cases} 1, & |x|<1, \\ -1, & |x|>1, \end{cases}$$

therefore

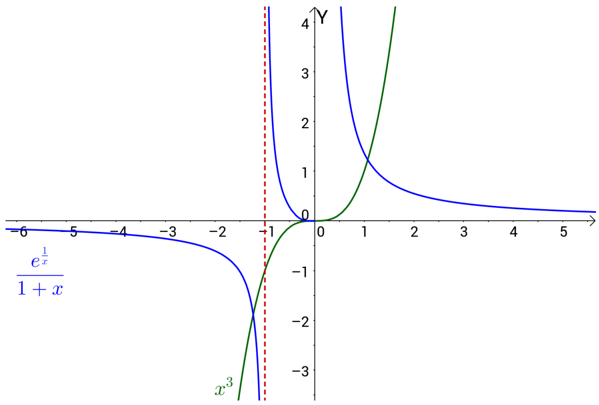
$$f'(x) = \begin{cases} \dfrac{4}{1+x^2}, & |x|<1, \\ 0, & |x|>1. \end{cases}$$

Function $f(x)$ is thus constant if $|x|>1$ and strictly increasing if $|x|<1$. Besides, $f(x)$ is obviously continuous because so are all functions involved, so the constant values it takes for $x\geqslant 1$ and $x\leqslant -1$ can be found as

$$f(\pm 1) = \pm 2\arctan 1 + \pm\arcsin 1 = \pm\left(2\cdot\frac{\pi}{4}+\frac{\pi}{2}\right) = \pm\pi.$$



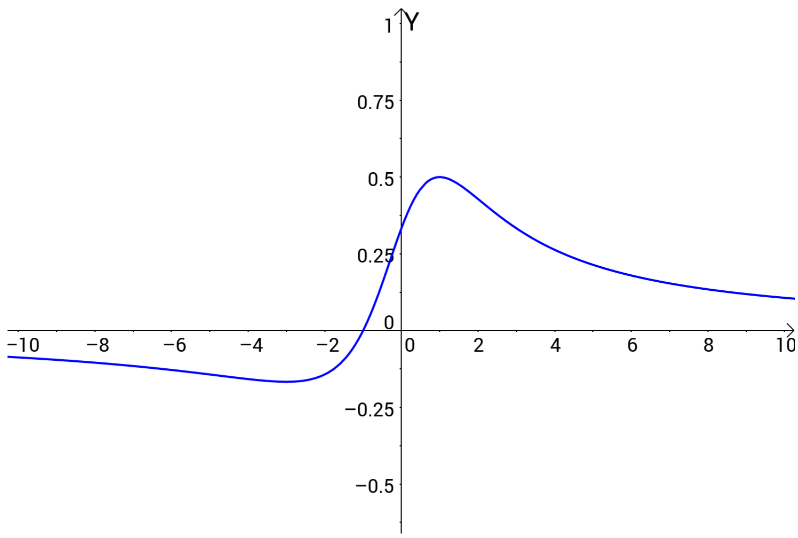
Problem 8.32



Problem 8.33 Let us calculate the derivative of $f(x)$:

$$f'(x) = \frac{3+x^2-(1+x)2x}{(3+x^2)^2} = \frac{3-2x-x^2}{(3+x^2)^2} = \frac{(3+x)(1-x)}{(3+x^2)^2},$$

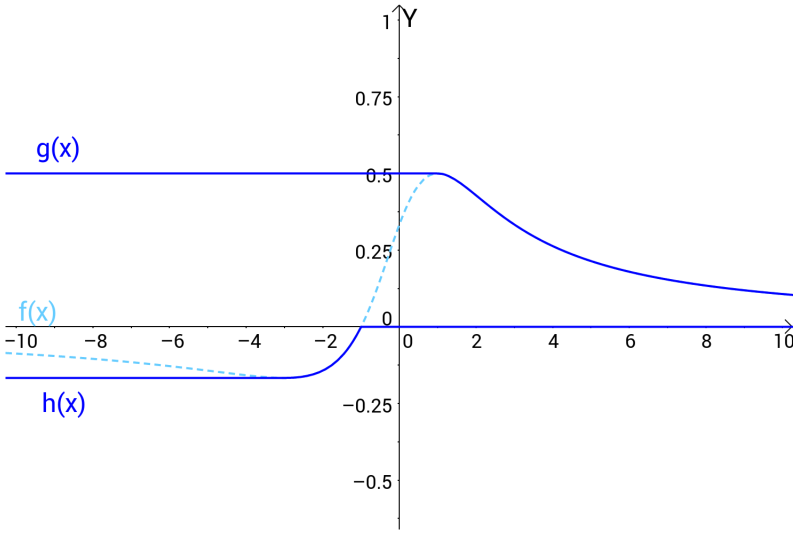
so the function increases for $-3 < x < 1$ and decreases for $x < -3$ and $x > 1$, hence it has a local maximum at $x = 1$ and a local minimum at $x = -3$. The X axis is a horizontal asymptote, and $f(1) = 1/2$, $f(-3) = -1/6$. Besides $f(x) = 0$ for $x = -1$ only. Here is a plot of the function:



Therefore

$$g(x) = \sup_{y>x} f(y) = \begin{cases} \frac{1}{2}, & x < 1, \\ f(x), & x \geqslant 1, \end{cases} \qquad h(x) = \inf_{y>x} f(y) = \begin{cases} -\frac{1}{6}, & x < -3, \\ f(x), & -3 \geqslant x \leqslant -1, \\ 0, & -1 < x. \end{cases}$$

Here is a plot of these functions:



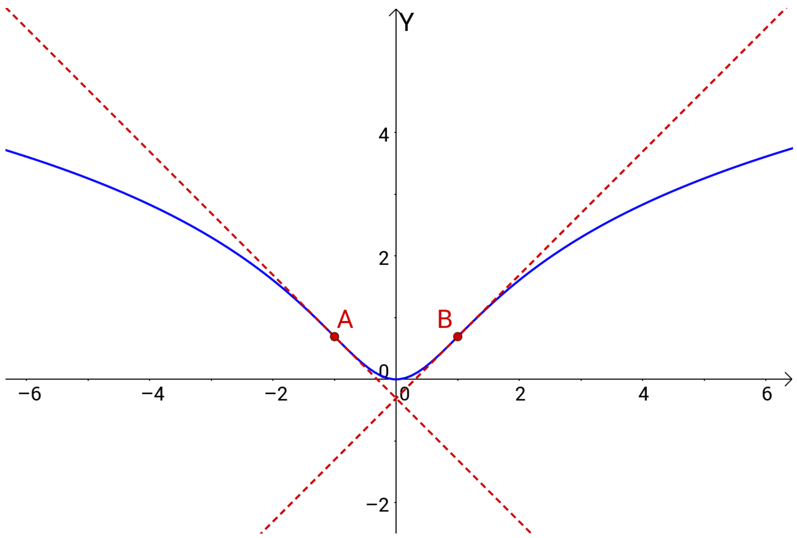
Problem 8.34 We first need to calculate two derivatives of $f(x)$:

$$f'(x) = \frac{2x}{1+x^2}, \qquad f''(x) = \frac{2(1-x^2)}{(1+x^2)^2},$$

Thus $x = \pm 1$ are inflection points of $f(x)$ (because $f''(x) < 0$ on one side of them and $f''(x) > 0$ on the other side). The slopes at these two points are $f'(1) = 1$, $f'(-1) = -1$, and the coordinates of those points are $\big(1, f(1)\big) = (1, \log 2)$, $\big(-1, f(-1)\big) = (-1, \log 2)$. Therefore the two straight tangents are

$$y = \log 2 + (x-1), \qquad y = \log 2 - (x+1).$$

The plot of $f(x)$ along with these two tangents goes as follows:



D.9 Primitives

Problem 9.1

(i) $\cos^{-2}x = \sec^2 x = 1+\tan^2 x = (\tan x)'$; hence

$$\int \frac{dx}{\cos^2 x} = \tan x + c.$$

(ii) $\sin x - \cos x = -(\cos x + \sin x)'$; hence

$$\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx = -\log|\sin x + \cos x| + c.$$

(iii) $2x = (x^2+1)'$; hence

$$\int \frac{x}{(x^2+1)^{5/2}}\,dx = -\frac{1}{3}\left(x^2+1\right)^{-3/2} + c.$$

(iv) $\dfrac{1+\sin x}{1+\cos x} = \dfrac{(1+\sin x)(1-\cos x)}{1-\cos^2 x} = \dfrac{1+\sin x - \cos x - \sin x\cos x}{\sin^2 x}$, thus

$$\begin{aligned}\int \frac{1+\sin x}{1+\cos x}\,dx &= \int \csc^2 x\,dx + \int \csc x\,dx - \int \frac{\cos x}{\sin^2 x}\,dx - \int \cot x\,dx\\ &= -\cot x - \log|\csc x + \cot x| + \csc x - \log|\sin x|\\ &= \frac{1-\cos x}{\sin x} - \log(1+\cos x) + c.\end{aligned}$$

(v) $\dfrac{1}{1-\sin x} = \dfrac{1+\sin x}{1-\sin^2 x} = \dfrac{1+\sin x}{\cos^2 x}$, thus

$$\int \frac{dx}{1-\sin x} = \int \sec^2 x\,dx + \int \frac{\sin x}{\cos^2 x}\,dx = \tan x + \sec x + c = \frac{\sin x + 1}{\cos x} + c.$$

(vi) $2x = (1+x^2)'$; hence

$$\int \frac{x}{\sqrt{1+x^2}}\,dx = \sqrt{1+x^2} + c.$$

(vii) As $\left(1-\sqrt{x}\right)' = -\dfrac{1}{2\sqrt{x}}$,

$$\begin{aligned}\int \frac{1+\sqrt{1-\sqrt{x}}}{\sqrt{x}}\,dx &= -2\int \left(1+\sqrt{1-\sqrt{x}}\right)\left(1-\sqrt{x}\right)'\,dx\\ &= -2\left(1-\sqrt{x}\right) - \frac{4}{3}\left(1-\sqrt{x}\right)^{3/2} + c = 2\sqrt{x} - \frac{4}{3}\left(1-\sqrt{x}\right)^{3/2} + c'\end{aligned}$$

(where $c' = c-2$).

(viii) $\dfrac{\cos^3 x}{\sin^4 x} = \dfrac{\cos^2 x}{\sin^4 x}(\sin x)' = \dfrac{1-\sin^2 x}{\sin^4 x}(\sin x)'$, hence

$$\int \frac{\cos^3 x}{\sin^4 x}\,dx = \int \left(\frac{1}{\sin^4 x} - \frac{1}{\sin^2 x}\right)(\sin x)'\,dx = -\frac{1}{3}\csc^3 x + \csc x + c.$$

(ix) $x^3\sqrt{1-x^2} = x(x^2-1)\sqrt{1-x^2} + x\sqrt{1-x^2} = -x(1-x^2)^{3/2} + x(1-x^2)^{1/2}$. Since $(1-x^2)' = -2x$, then

$$\int x^3\sqrt{1-x^2}\,dx = \frac{1}{5}\left(1-x^2\right)^{5/2} - \frac{1}{3}\left(1-x^2\right)^{3/2} + c.$$

Problem 9.2

(i) $-\dfrac{1}{2(x-1)^2}-\dfrac{2}{x-1}+\log|x-1|+c$;

(ii) $-\dfrac{1}{3(x-1)}-\dfrac{1}{3}\log|x-1|+\dfrac{1}{6}\log(x^2+x+1)+\dfrac{1}{3\sqrt{3}}\arctan\left(\dfrac{2x+1}{\sqrt{3}}\right)+c$;

(iii) $5\log|x-1|-3\log|x|+\dfrac{3}{x}+c$;

(iv) $2\arctan(x-1)+c$;

(v) $2x^2+7x+3\log|x-2|-4\log|x-3|+5\log|x+3|+c$;

(vi) $\dfrac{x^2}{2}+\dfrac{1}{4(x-1)}-\dfrac{1}{4(x+1)}+c$.

Problem 9.3

(i) $\dfrac{2}{7}(x-1)^{7/2}+\dfrac{4}{5}(x-1)^{5/2}+\dfrac{2}{3}(x-1)^{3/2}+c$;

(ii) $\dfrac{2}{3}\left(-x^{3/2}\cos x^{3/2}+\sin x^{3/2}\right)+c$;

(iii) $\dfrac{x}{2}\cos(\log x)+\dfrac{x}{2}\sin(\log x)+c$;

(iv) $-\dfrac{x}{2}\cos(\log x)+\dfrac{x}{2}\sin(\log x)+c$;

(v) $\dfrac{x}{2}+\dfrac{x}{10}\cos(2\log x)+\dfrac{x}{5}\sin(2\log x)+c$;

(vi) $2\sqrt{x}+\log|x+3|-2\sqrt{3}\arctan\sqrt{\dfrac{x}{3}}+c$;

(vii) $\dfrac{1}{3}\left[1-(x+1)^2\right]^{3/2}-\left[1-(x+1)^2\right]^{1/2}+c$;

(viii) $-\dfrac{1}{2(1+x^2)}+\dfrac{1}{4(1+x^2)^2}+c$;

(ix) $2\arctan\sqrt{1+x}+c$;

(x) $-\dfrac{3}{2}(1-x)^{2/3}+3(1-x)^{1/3}-3\log\left|1+(1-x)^{1/3}\right|+c$;

(xi) $\dfrac{e^{2x}}{2}-2e^x+\log\left(e^{2x}+2e^x+2\right)+2\arctan(e^x+1)+c$;

(xii) $\arctan\sqrt{e^{2x}-1}+c$;

(xiii) $2\sqrt{e^x-1}-2\arctan\sqrt{e^x-1}+c$;

(xiv) $\dfrac{1}{7}\cos^7 x+\dfrac{1}{5}\cos^5 x+\dfrac{1}{3}\cos^3 x+\cos x+\dfrac{1}{2}\log(1-\cos x)-\dfrac{1}{2}\log(1+\cos x)+c$;

(xv) $\sqrt{2x+5}-3\log\left(3+\sqrt{2x+5}\right)+c$;

(xvi) $-\dfrac{1}{t-1}+\log|t-1|-\dfrac{1}{t+1}-\log|t+1|+c$, with $t=\sqrt{(x-1)/(x+1)}$;

(xvii) $\dfrac{4}{5}\left(\sqrt{x}+1\right)^{5/2}-\dfrac{4}{3}\left(\sqrt{x}+1\right)^{3/2}+c$;

(xviii) $x-2\sqrt{x+2}+2\log\left(\sqrt{x+2}+1\right)+c$;

(xix) $2\sqrt{2+e^x}+\sqrt{2}\log\left(\sqrt{2+e^x}-\sqrt{2}\right)-\sqrt{2}\log\left(\sqrt{2+e^x}+\sqrt{2}\right)+c$;

(xx) $\frac{1}{5}\log|\tan x+2|-\frac{1}{10}\log\left(\tan^2 x+1\right)+\frac{7}{5}x+c$;

(xxi) $\log\left|\tan\frac{x}{2}\right|-2\log\left(\tan^2\frac{x}{2}+3\right)+\frac{2}{\sqrt{3}}\arctan\left(\frac{\tan(x/2)}{\sqrt{3}}\right)+c$;

(xxii) $\frac{6}{5}\left(1+x^{1/3}\right)^{5/2}-2\left(1+x^{1/3}\right)^{3/2}+c$;

(xxiii) $\log\left|(x+2)^{1/3}-1\right|-\frac{1}{2}\log\left[(x+2)^{2/3}+(x+2)^{1/3}+1\right]+\sqrt{3}\arctan\left(\frac{2(x+2)^{1/3}+1}{\sqrt{3}}\right)+c$;

(xxiv) $\frac{1}{4}\log|e^x-2|-\frac{1}{4}\log(e^x+2)+c$.

Problem 9.4

(i) $\sin^2 x=\frac{1}{2}-\frac{\cos 2x}{2}$, hence

$$\int \sin^2 x\,dx=\frac{x}{2}-\frac{1}{4}\sin 2x+c.$$

(ii) $\cos^2 x=\frac{1}{2}+\frac{\cos 2x}{2}$, hence

$$\int \cos^2 x\,dx=\frac{x}{2}+\frac{1}{4}\sin 2x+c.$$

(iii) $\sin^4 x=\frac{1}{4}(1-\cos 2x)^2=\frac{1}{4}-\frac{1}{2}\cos 2x+\frac{1}{4}\cos^2 2x=\frac{1}{4}-\frac{1}{2}\cos 2x+\frac{1}{8}+\frac{1}{8}\cos 4x=\frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$, hence

$$\int \sin^4 x\,dx=\frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+c.$$

(iv) $\cos^4 x=\frac{1}{4}(1+\cos 2x)^2=\frac{1}{4}+\frac{1}{2}\cos 2x+\frac{1}{4}\cos^2 2x=\frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$, hence

$$\int \cos^4 x\,dx=\frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+c.$$

(v) $\cos^6 x=\frac{1}{8}(1+\cos 2x)^3=\frac{1}{8}+\frac{3}{8}\cos 2x+\frac{3}{8}\cos^2 2x+\frac{1}{8}\cos^3 2x=\frac{5}{16}+\frac{3}{8}\cos 2x+\frac{3}{16}\cos 4x+\frac{1}{16}(1-\sin^2 2x)(\sin 2x)'$, hence

$$\begin{aligned}\int \cos^6 x\,dx&=\frac{5}{16}x+\frac{3}{16}\sin 2x+\frac{3}{64}\sin 4x+\frac{1}{16}\sin 2x-\frac{1}{48}\sin^3 2x+c\\&=\frac{5}{16}x+\frac{1}{4}\sin 2x+\frac{3}{64}\sin 4x-\frac{1}{48}\sin^3 2x+c.\end{aligned}$$

(vi) $\sin^2 x\cos^2 x=\frac{1}{4}-\frac{1}{4}\cos^2 2x=\frac{1}{8}-\frac{1}{8}\cos 4x$, hence

$$\int \sin^2 x\cos^2 x\,dx=\frac{x}{8}-\frac{1}{32}\sin 4x+c.$$

(vii) $\tan^2 x=(\tan x)'-1$, hence

$$\int \tan^2 x\,dx=\tan x-x+c.$$

(viii) $\tan^4 x = \tan^2 x(\tan x)' - \tan^2 x = \tan^2 x(\tan x)' - (\tan x)' + 1$, hence

$$\int \tan^4 x\,dx = \frac{1}{3}\tan^3 x - \tan x + x + c.$$

(ix) $\dfrac{1}{\cos^4 x} = (1+\tan^2 x)(\tan x)'$, hence

$$\int \frac{dx}{\cos^4 x}dx = \tan x + \frac{1}{3}\tan^3 x + c.$$

(x) $\sin^5 x = -\sin^4 x(\cos x)' = -(1-\cos^2 x)^2(\cos x)' = (-1+2\cos^2 x - \cos^4 x)(\cos x)'$, thus

$$\int \sin^5 x\,dx = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c.$$

(xi) $\cos^3 \sin^2 x = (1-\sin^2 x)\sin^2 x(\sin x)' = (\sin^2 x - \sin^4 x)(\sin x)'$, so

$$\int \cos^3 \sin^2 x\,dx = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c.$$

(xii) $\sec^6 x = (1+\tan^2 x)^2(\tan x)' = (1+2\tan^2 x + \tan^4 x)(\tan x)'$, hence

$$\int \sec^6 x\,dx = \tan x + \frac{2}{3}\tan^3 x + \frac{1}{5}\tan^5 x + c.$$

(xiii) $\sin^3 x\cos^2 x = -(1-\cos^2 x)\cos^2 x(\cos x)' = (\cos^4 x - \cos^2 x)(\cos x)'$, therefore

$$\int \sin^3 x\cos^2 x\,dx = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + c.$$

(xiv) $\tan^3 x = \tan x[(\tan x)' - 1] = \tan x(\tan x)' - \tan x$, thus

$$\int \tan^3 x\,dx = \frac{1}{2}\tan^2 x + \log|\cos x| + c.$$

(xv) $\tan^3 x\sec^4 x = \tan^3 x(1+\tan^2 x)(\tan x)' = (\tan^3 x + \tan^5 x)(\tan x)'$, thus

$$\int \tan^3 x\sec^4 x\,dx = \frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x + c.$$

Problem 9.5

(i) $\tan^2(2x) = \dfrac{1}{2}(\tan 2x)' - 1$, so

$$\int x\tan^2(2x)\,dx = \frac{1}{2}\int x(\tan 2x)'\,dx - \int x\,dx = -\frac{x^2}{2} + \frac{x}{2}\tan 2x - \frac{1}{2}\int \tan 2x\,dx$$
$$= -\frac{x^2}{2} + \frac{x}{2}\tan 2x + \frac{1}{4}\log|\cos 2x| + c.$$

(ii) Since $(e^x)' = e^x$,

$$\int e^x \sin\pi x\,dx = e^x\sin\pi x - \pi\int e^x\cos\pi x\,dx = e^x\sin\pi x - \pi e^x\cos\pi x - \pi^2\int e^x\sin\pi x\,dx.$$

Therefore

$$(1+\pi^2)\int e^x\sin\pi x\,dx = e^x(\sin\pi x - \pi\cos\pi x)$$

and finally

$$\int e^x\sin\pi x\,dx = \frac{e^x}{1+\pi^2}(\sin\pi x - \pi\cos\pi x) + c.$$

(iii) Since $(e^x)' = e^x$,

$$\int e^x \cos 2x\,dx = e^x \cos 2x + 2\int e^x \sin 2x\,dx = e^x \cos 2x + 2e^x \sin 2x - 4\int e^x \cos 2x\,dx.$$

Then

$$5\int e^x \cos 2x\,dx = e^x(\cos 2x + 2\sin 2x)$$

and

$$\int e^x \cos 2x\,dx = \frac{e^x}{5}(\cos 2x + 2\sin 2x) + c.$$

(iv) Since $\sec^2 x = 1 + \tan^2 x = (\tan x)'$ and $(\sec x)' = \sec x \tan x$,

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x \tan x - \int \tan^2 x \sec x\,dx = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx\\ &= \sec x \tan x - \int \sec^3 x\,dx + \log|\sec x + \tan x|,\end{aligned}$$

therefore

$$2\int \sec^3 x\,dx = \sec x \tan x + \log|\sec x + \tan x|$$

and finally

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\log|\sec x + \tan x| + c.$$

(v) First of all $\tan^2(3x)\sec^3(3x) = \frac{1}{3}\tan^2(3x)(\tan 3x)'\sec 3x = \frac{1}{9}\left[\tan^3(3x)\right]'\sec 3x$. Thus,

$$\begin{aligned}I(x) = \int \tan^2(3x)\sec^3(3x)\,dx &= \frac{1}{9}\tan^3(3x)\sec 3x - \frac{1}{9}\int \tan^3(3x)(\sec 3x)'\,dx\\ &= \frac{1}{9}\tan^3(3x)\sec 3x - \frac{1}{3}\int \tan^4(3x)\sec 3x\,dx.\end{aligned}$$

But $\tan^2(3x) = \sec^2(3x) - 1$, so

$$\begin{aligned}\int \tan^4(3x)\sec 3x\,dx &= \int \tan^2(3x)\sec^3(3x)\,dx - \int \tan^2(3x)\sec 3x\,dx\\ &= I(x) - \int \sec^3(3x)\,dx + \int \sec 3x\,dx.\end{aligned}$$

And from the previous exercise,

$$\begin{aligned}&\int \sec^3(3x)\,dx = \frac{1}{6}\sec 3x \tan 3x + \frac{1}{6}\log|\sec 3x + \tan 3x|,\\ &\int \sec 3x\,dx = \frac{1}{3}\log|\sec 3x + \tan 3x|.\end{aligned}$$

Therefore,

$$\int \tan^4(3x)\sec 3x\,dx = I(x) - \frac{1}{6}\sec 3x \tan 3x + \frac{1}{6}\log|\sec 3x + \tan 3x|.$$

This yields

$$I(x)=\frac{1}{9}\tan^3(3x)\sec 3x-\frac{1}{3}I(x)+\frac{1}{18}\sec 3x\tan 3x-\frac{1}{18}\log|\sec 3x+\tan 3x|,$$

from which

$$\frac{4}{3}I(x)=\frac{1}{9}\tan^3(3x)\sec 3x+\frac{1}{18}\sec 3x\tan 3x-\frac{1}{18}\log|\sec 3x+\tan 3x|,$$

which leads to

$$\begin{aligned}I(x)&=\frac{1}{12}\tan^3(3x)\sec 3x+\frac{1}{24}\sec 3x\tan 3x-\frac{1}{24}\log|\sec 3x+\tan 3x|+c\\&=\frac{1}{12}\tan 3x\sec^3(3x)-\frac{1}{24}\sec 3x\tan 3x-\frac{1}{24}\log|\sec 3x+\tan 3x|+c.\end{aligned}$$

(The last line is obtain by replacing $\tan^2(3x)=\sec^2(3x)-1$.)

(vi) Since $e^{\sin x}\cos x=\left(e^{\sin x}\right)'$,

$$\begin{aligned}\int e^{\sin x}\cos^3 x\,dx&=e^{\sin x}\cos^2 x+2\int e^{\sin x}\cos x\sin x\,dx\\&=e^{\sin x}\cos^2 x+2e^{\sin x}\sin x-2\int e^{\sin x}\cos x\,dx\\&=e^{\sin x}\cos^2 x+2e^{\sin x}\sin x-2e^{\sin x}+c=e^{\sin x}\left(\cos^2 x+2\sin x-2\right)+c\\&=e^{\sin x}\left(-\sin^2 x+2\sin x-1\right)+c=-e^{\sin x}(1-\sin x)^2+c.\end{aligned}$$

(vii) $\displaystyle\int x^2\log x\,dx=\frac{x^3}{3}\log x-\frac{1}{3}\int x^3\frac{1}{x}\,dx=\frac{x^3}{3}\log x-\frac{x^3}{9}+c.$

(viii) If $m\neq -1$,

$$\int x^m\log x\,dx=\frac{x^{m+1}}{m+1}\log x-\frac{1}{m+1}\int x^{m+1}\frac{1}{x}\,dx=\frac{x^{m+1}}{m+1}\log x-\frac{x^{m+1}}{(m+1)^2}+c.$$

If $m=-1$,

$$\int\frac{1}{x}\log x\,dx=\int\log x(\log x)'\,dx=\frac{1}{2}(\log x)^2+c.$$

(ix) Taking $1=(x)'$,

$$\int(\log x)^3\,dx=x(\log x)^3-3\int\cancel{x}(\log x)^2\frac{1}{\cancel{x}}\,dx.$$

Similarly,

$$\int(\log x)^2\,dx=x(\log x)^2-2\int\cancel{x}\log x\frac{1}{\cancel{x}}\,dx=x(\log x)^2-2x\log x+2x.$$

Thus,

$$\int(\log x)^3\,dx=x(\log x)^3-3x(\log x)^2+6x\log x-6x+c.$$

(x) $\displaystyle\int x(\log x)^2\,dx=\frac{x^2}{2}(\log x)^2-\int x^2\log x\frac{1}{x}\,dx.$ Also,

$$\int x\log x\,dx=\frac{x^2}{2}\log x-\frac{1}{2}\int x^2\frac{1}{x}\,dx=\frac{x^2}{2}\log x-\frac{x^2}{4}.$$

Therefore

$$\int x(\log x)^2\,dx=\frac{x^2}{2}(\log x)^2-\frac{x^2}{2}\log x+\frac{x^2}{4}+c.$$

(xi) On the one hand, $\dfrac{x}{(1+x^2)^2} = -\dfrac{1}{2}\left(\dfrac{1}{1+x^2}\right)'$, so

$$\int \frac{x\log x}{(1+x^2)^2}\,dx = -\frac{1}{2}\frac{\log x}{1+x^2} + \frac{1}{2}\int \frac{dx}{x(1+x^2)}\,dx.$$

On the other hand,

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2} \quad \Rightarrow \quad 1 = A(1+x^2) + Bx^2 + Cx.$$

Setting $x = 0$ we obtain $A = 1$. Thus,

$$1 = 1 + x^2 + Bx^2 + Cx \quad \Rightarrow \quad -x^2 = Bx^2 + CX \quad \Rightarrow \quad B = -1, \quad C = 0.$$

Hence,

$$\int \frac{dx}{x(1+x^2)}\,dx = \int \frac{dx}{x} - \int \frac{x}{1+x^2}\,dx = \log x - \frac{1}{2}\log(1+x^2).$$

(Notice: we do not need to write $\log|x|$ because the $\log x$ of the integrand forces $x > 0$.) Therefore,

$$\int \frac{x\log x}{(1+x^2)^2}\,dx = -\frac{\log x}{2(1+x^2)} + \frac{1}{2}\log x - \frac{1}{4}\log(1+x^2) + c.$$

(xii) Using $1 = (x)'$ we get

$$\int \arctan\sqrt[3]{x}\,dx = x\arctan\sqrt[3]{x} - \frac{1}{3}\int x\frac{x^{-2/3}}{1+x^{2/3}}\,dx = x\arctan\sqrt[3]{x} - \frac{1}{3}\int \frac{x^{1/3}}{1+x^{2/3}}\,dx.$$

Now,

$$\frac{x^{1/3}}{1+x^{2/3}} = x^{-1/3}\frac{x^{2/3}}{1+x^{2/3}} = x^{-1/3}\left(1 - \frac{1}{1+x^{2/3}}\right) = \frac{3}{2}\left(1 - \frac{1}{1+x^{2/3}}\right)\left(x^{2/3}\right)',$$

thus

$$\frac{1}{3}\int \frac{x^{1/3}}{1+x^{2/3}}\,dx = \frac{x^{2/3}}{2} - \frac{1}{2}\log\left(1+x^{2/3}\right).$$

Accordingly,

$$\int \arctan\sqrt[3]{x}\,dx = x\arctan\sqrt[3]{x} - \frac{\sqrt[3]{x^2}}{2} + \frac{1}{2}\log\left(1+\sqrt[3]{x^2}\right) + c.$$

Problem 9.6

(i) Using the change of variable $x = \sec t$, with $dx = \sec t\tan t\,dt$,

$$\int \frac{x^2+1}{\sqrt{x^2-1}}\,dx = \int \frac{\sec^2 t + 1}{\cancel{\tan t}}\sec t\cancel{\tan t}\,dt = \int \sec^3 t\,dt + \int \sec t\,dt.$$

Using

$$\int \sec^3 t\,dt = \frac{1}{2}\sec t\tan t + \frac{1}{2}\log|\sec t + \tan t|,$$

$$\int \sec t\,dt = \log|\sec t + \tan t|,$$

$$\int \frac{x^2+1}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\sec t\tan t + \frac{3}{2}\log|\sec t + \tan t| + c = \frac{1}{2}x\sqrt{x^2-1} + \frac{3}{2}\log\left|x + \sqrt{x^2-1}\right| + c.$$

(ii) Using the change of variable $x=\tan t$, with $dx=\sec^2 t\,dt$, and the identity $\sec^2 t = 1+\tan^2 t$,

$$\int \frac{x^2}{(x^2+1)^{5/2}}\,dx = \int \frac{\tan^2 t}{\sec^5 t}\sec^2 t\,dt = \int \tan^2 t\cos^3 t\,dt = \int \sin^2 t\cos t\,dt = \frac{1}{3}\sin^3 t.$$

Since $\sin t = \dfrac{\tan t}{\sqrt{1+\tan^2 t}}$,

$$\int \frac{x^2}{(x^2+1)^{5/2}}\,dx = \frac{x^3}{3(x^2+1)^{3/2}} + c.$$

(iii) Using the change of variable $x=\sin t$, with $dx=\cos t\,dt$,

$$\int \frac{x^2}{(1-x^2)^{3/2}}\,dx = \int \frac{\sin^2 t}{\cos^3 t}\cos t\,dt = \int \tan^2 t\,dt = \tan t - t + c = \frac{x}{\sqrt{1-x^2}} - \arcsin x + c.$$

(iv) Using the change of variable $x=\sin t$, with $dx=\cos t\,dt$,

$$\int \frac{dx}{x^2\sqrt{1-x^2}} = \int \frac{\cancel{\cos t}}{\sin^2 t\,\cancel{\cos t}}\,dt = \int \csc^2 t\,dt = -\cot t + c = -\frac{\sqrt{1-x^2}}{x} + c.$$

(v) Using the change of variable $x=3y$, with $dx=3\,dy$,

$$\int \frac{dx}{x^2\sqrt{9-x^2}} = \frac{3}{27}\int \frac{dy}{y^2\sqrt{1-y^2}} = -\frac{\sqrt{1-y^2}}{9y} + c = -\frac{\sqrt{9-x^2}}{9x} + c.$$

Problem 9.7

(i) First of all we write

$$\begin{aligned} I_m &= \int \sin^m x\,dx = \int \sin^2 x\sin^{m-2} x\,dx = \int (1-\cos^2 x)\sin^{m-2} x\,dx \\ &= I_{m-2} - \int \cos^2 x\sin^{m-2} x\,dx. \end{aligned}$$

Now, since

$$\frac{d}{dx}\sin^{m-1} x = (m-1)\sin^{m-2} x\cos x,$$

we can integrate by parts

$$\begin{aligned} \int \cos^2 x\sin^{m-2} x\,dx &= \frac{1}{m-1}\sin^{m-1} x\cos x - \frac{1}{m-1}\int \sin^{m-1} x(-\sin x)\,dx \\ &= \frac{1}{m-1}\sin^{m-1} x\cos x + \frac{1}{m-1} I_m. \end{aligned}$$

Therefore

$$I_m = I_{m-2} - \frac{1}{m-1} I_m - \frac{1}{m-1}\sin^{m-1} x\cos x,$$

which can be rewritten as

$$\left(1+\frac{1}{m-1}\right) I_m = I_{m-2} - \frac{1}{m-1}\sin^{m-1} x\cos x \quad\Rightarrow\quad I_m = \frac{m-1}{m} I_{m-2} - \frac{1}{m}\sin^{m-1} x\cos x.$$

(ii) Integrating by parts,

$$I_m = \int (\log x)^m \, dx = x(\log x)^m - \int \cancel{x} m(\log x)^{m-1} \frac{1}{\cancel{x}} \, dx = x(\log x)^m - mI_{m-1}.$$

(iii) Integrating by parts,

$$I_m = \int x^m e^{-x} \, dx = -x^m e^{-x} + \int m x^{m-1} e^{-x} \, dx = -x^m e^{-x} + mI_{m-1}.$$

(iv) First of all,

$$\tan^m x = \tan^{m-2} x \tan^2 x = \tan^{m-2} x(\tan^2 x + 1 - 1) = \tan^{m-2} x(\tan x)' - \tan^{m-2} x.$$

Thus,

$$I_m = \int \tan^m x \, dx = -I_{m-2} + \int \tan^{m-2} x(\tan x)' \, dx.$$

Now, integrating by parts,

$$\begin{aligned}\int \tan^{m-2} x(\tan x)' \, dx &= \tan^{m-1} x - \int (m-2) \tan^{m-3} x(1+\tan^2 x) \tan x \, dx \\ &= \tan^{m-1} x - (m-2)(I_{m-2} + I_m).\end{aligned}$$

Therefore,

$$I_m = -I_{m-2} + \tan^{m-1} x - (m-2)(I_{m-2} + I_m) \quad \Rightarrow \quad (m-1)I_m = -(m-1)I_{m-2} + \tan^{m-1} x,$$

so

$$I_m = -I_{m-2} + \frac{1}{m-1} \tan^{m-1} x.$$

(v) First of all

$$\sec^m x = \sec^{m-2} x \sec^2 x = \sec^{m-2} x(\tan x)',$$

so integrating by parts,

$$I_m = \int \sec^m x \, dx = \tan x \sec^{m-2} x - \int (m-2) \sec^{m-3} x(\sec x \tan x) \tan x \, dx.$$

But

$$\sec^{m-3} x(\sec x \tan x) \tan x = \sec^{m-2} x \tan^2 x = \sec^{m-2} x(\sec^2 x - 1),$$

therefore

$$I_m = \tan x \sec^{m-2} x - (m-2)(I_m - I_{m-2}) \quad \Rightarrow \quad (m-1)I_m = \tan x \sec^{m-2} x + (m-2)I_{m-2},$$

and finally

$$I_m = \frac{1}{m-1} \tan x \sec^{m-2} x + \frac{m-2}{m-1} I_{m-2}.$$

(vi) First of all

$$\sin^m x \cos^n x = \sin^{m-1} x(-\cos x)' \cos^n x, = -\sin^{m-1} x \frac{1}{n+1} \left(\cos^{n+1} x\right)',$$

so integrating by parts,

$$\begin{aligned} I_{m,n} &= \int \sin^m x \cos^n x \, dx \\ &= -\frac{1}{n+1} \sin^{m-1} x \cos^{n+1} x + \frac{1}{n+1} \int (m-1) \sin^{m-2} x \cos x \cos^{n+1} x \, dx \\ &= -\frac{1}{n+1} \sin^{m-1} x \cos^{n+1} x + \frac{m-1}{n+1} \int \sin^{m-2} x \cos^{n+2} x \, dx. \end{aligned}$$

But

$$\begin{aligned} \sin^{m-2} x \cos^{n+2} x &= \sin^{m-2} x \cos^n x \cos^2 x = \sin^{m-2} x \cos^n x (1 - \sin^2 x) \\ &= \sin^{m-2} x \cos^n x - \sin^m x \cos^n x, \end{aligned}$$

therefore

$$I_{m,n} = -\frac{1}{n+1} \sin^{m-1} x \cos^{n+1} x + \frac{m-1}{n+1} (I_{m-2,n} - I_{m,n}),$$

from which

$$\frac{m+n}{n+1} I_{m,n} = -\frac{1}{n+1} \sin^{m-1} x \cos^{n+1} x + \frac{m-1}{n+1} I_{m-2,n},$$

and finally

$$I_{m,n} = -\frac{1}{m+n} \sin^{m-1} x \cos^{n+1} x + \frac{m-1}{m+n} I_{m-2,n}.$$

Problem 9.8 There is no need to calculate the integral. If the right-hand side is the primitive of the integrand then

$$\left(Ax + B \log |c \cos x + d \sin x|\right)' = \frac{a \cos x + b \sin x}{c \cos x + d \sin x},$$

in other words,

$$A + B \frac{-c \sin x + d \cos x}{c \cos x + d \cos x} = \frac{a \cos x + b \sin x}{c \cos x + d \sin x},$$

which when reduced to a single fraction becomes

$$\frac{(Ac + Bd) \cos x + (Ad - Bc) \sin x}{c \cos x + d \cos x} = \frac{a \cos x + b \sin x}{c \cos x + d \sin x}.$$

The two functions are the same if, and only if,

$$\left.\begin{aligned} Ac + Bd &= a, \\ Ad - Bc &= b, \end{aligned}\right\} \quad \Rightarrow \quad A = \frac{ac + bd}{c^2 + d^2}, \quad B = \frac{ad - bc}{c^2 + d^2}.$$

D.10 Fundamental Theorem of Calculus

Problem 10.1 For $x<0$,

$$\frac{4-x^2}{(4+x^2)^2}=\frac{4+x^2-2x^2}{(4+x^2)^2}=\frac{1}{4+x^2}-\frac{2x^2}{(4+x^2)^2}.$$

Now,

$$\int\frac{dx}{4+x^2}=\frac{1}{2}\arctan\frac{x}{2}$$

and

$$\int\frac{2x^2}{(4+x^2)^2}\,dx=\int x\frac{2x}{(4+x^2)^2}\,dx=-\frac{x}{4+x^2}+\int\frac{dx}{4+x^2}=-\frac{x}{4+x^2}+\frac{1}{2}\arctan\frac{x}{2}.$$

Thus,

$$f(x)=\frac{x}{4+x^2}+a.$$

For $x>0$, with the change $t=\sqrt{x}$ $(dx=2t\,dt)$,

$$f(x)=\int e^{\sqrt{x}}\,dx=2\int te^t\,dt=2(t-1)e^t+b=2\left(\sqrt{x}-1\right)e^{\sqrt{x}}+b.$$

Continuity and $f(0)=0$ requires $f(0^-)=a=0$ and $f(0^+)=-2+b=0$, thus

$$f(x)=\begin{cases}\dfrac{x}{4+x^2}, & x<0,\\ 2\left(\sqrt{x}-1\right)e^{\sqrt{x}}+2, & x\geqslant 0.\end{cases}$$

Problem 10.2

(a) Changing $x=-t$,

$$I=\int_{-a}^{a}f(x)\,dx=\int_{-a}^{a}f(-t)\,dt=-\int_{-a}^{a}f(t)\,dt=-I\quad\Rightarrow\quad 2I=0\quad\Rightarrow\quad I=0.$$

(b) Using the same change,

$$\int_{-a}^{a}f(x)\,dx=\int_0^a f(x)\,dx+\int_{-a}^{0}f(x)\,dx=\int_0^a f(x)\,dx+\int_0^a\underbrace{f(-t)}_{=f(t)}\,dt=2\int_0^a f(x)\,dx.$$

(c) Changing $t=x-8$,

$$\int_6^{10}\sin\left(\sin\left((x-8)^3\right)\right)dx=\int_{-2}^{2}\sin\left(\sin\left(t^3\right)\right)dt=0$$

because the integrand is an odd function.

Problem 10.3 These are all Riemann's sums:

(i)

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\cdot\frac{1}{1+(k/n)^2}=\int_0^1\frac{dx}{1+x^2}=\arctan 1=\frac{\pi}{4}.$$

(ii)

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sqrt[n]{e^{2k}}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\cdot e^{2k/n}=\int_0^1 e^{2x}\,dx=\frac{e^2-1}{2}.$$

(iii)

$$\lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{1}{\sqrt{n^2-k^2}}=\lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{1}{n}\cdot\frac{1}{\sqrt{1-(k/n)^2}}=\int_0^1\frac{dx}{\sqrt{1-x^2}}=\arcsin 1=\frac{\pi}{2}.$$

Problem 10.4

(i) For $x<0$,

$$F(x)=\int_{-1}^{x}(-t)e^t\,dt=(1-x)e^x-\frac{2}{e}.$$

For $x\geqslant 0$,

$$F(x)=\int_{-1}^{0}(-t)e^t\,dt+\int_0^x te^{-t}\,dt=1-\frac{2}{e}+1-(1+x)e^{-x}=2-\frac{2}{e}-(1+x)e^{-x}.$$

(ii) For $x<1/2$,

$$F(x)=\int_{-1}^{x}\left(\frac{1}{2}-t\right)\,dt=\frac{2+x-x^2}{2}=\frac{(2-x)(1+x)}{2}$$

For $x\geqslant 1/2$,

$$F(x)=\int_{-1}^{1/2}\left(\frac{1}{2}-t\right)\,dt+\int_{1/2}^{x}\left(t-\frac{1}{2}\right)\,dt=\frac{9}{4}+\frac{(x-2)(1+x)}{2}.$$

(iii) For $x<0$,

$$F(x)=\int_{-1}^{x}(-1)\,dt=-1-x.$$

For $x\geqslant 0$,

$$F(x)=\int_{-1}^{0}(-1)\,dt+\int_0^x dt=-1+x.$$

Thus, $F(x)=|x|-1$.

(iv) For $x<0$,

$$F(x)=\int_{-1}^{x}t^2\,dt=\frac{x^3+1}{3}.$$

For $x\geqslant 0$,

$$F(x)=\int_{-1}^{0}t^2\,dt+\int_0^x(t^2-1)dt=\int_{-1}^{x}t^2\,dt-\int_0^x dt=\frac{x^3+1}{3}-x=\frac{x^3-3x+1}{3}.$$

(v) For $x\leqslant 0$,

$$F(x)=\int_{-1}^{x}dt=x+1.$$

For $x>0$,

$$F(x)=\int_{-1}^{0}dt+\int_0^x(t+1)dt=\int_{-1}^{x}dt+\int_0^x t\,dt=\frac{x^2}{2}+x+1.$$

(vi) For $x \leqslant -1/2$,

$$F(x) = \int_{-1}^{x} (1+t)\,dt = \frac{(1+x)^2}{2}.$$

For $-1/2 < x < 1/2$,

$$F(x) = \int_{-1}^{-1/2} (1+t)\,dt + \frac{1}{2}\int_{-1/2}^{x} dt = \frac{1}{8} + \frac{2x+1}{4} = \frac{4x+3}{8}.$$

For $x \geqslant 1/2$,

$$F(x) = \int_{-1}^{-1/2} (1+t)\,dt + \frac{1}{2}\int_{-1/2}^{1/2} dt + \int_{1/2}^{x} (1-t)\,dt = \frac{3}{4} - \frac{(1-x)^2}{2}.$$

(vii) For $-1 \leqslant x < 1/2$ we have $\cos(\pi x/2) > \sin(\pi x/2)$, hence

$$F(x) = \int_{-1}^{x} \cos(\pi t/2)\,dt = \frac{2}{\pi}\big[1+\sin(\pi x/2)\big].$$

For $1/2 < x \leqslant 1$ we have $\sin(\pi x/2) > \cos(\pi x/2)$, hence

$$F(x) = \int_{-1}^{1/2} \cos(\pi t/2)\,dt + \int_{1/2}^{x} \sin(\pi t/2)\,dt = \frac{2}{\pi}\left[1+\sqrt{2}-\cos(\pi x/2)\right].$$

Problem 10.5

(i) With the change $t = \sqrt{e^x - 1}$, i.e., $x = \log\left(1+t^2\right)$ (hence $dx = 2t\,dt/(1+t^2)$), we get

$$\int_0^{\log 2} \sqrt{e^x - 1}\,dx = \int_0^1 \frac{2t^2}{1+t^2}\,dt = 2 - 2\arctan 1 = \frac{4-\pi}{2}.$$

(ii) With the change $x = \sec t$ (hence $x^2 - 1 = \tan^2 t$ and $dx = \sec t \tan t\,dt$) we obtain

$$\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx = \int_0^{\pi/3} \tan^2 t\,dt = (\tan t - t)\Big|_0^{\pi/3} - \frac{\pi}{3} = \sqrt{3} - \frac{\pi}{3}.$$

Problem 10.6

(i) $F'(x) = \dfrac{3e^{x^3} - 2e^{x^2}}{x}.$

(ii) $F'(x) = \dfrac{6x^2}{1+\sin^2(x^3)}.$

(iii) $F'(x) = \dfrac{\sin^3 x}{1+\sin^6\left(\int_1^x \sin^3 t\,dt\right) + \left(\int_1^x \sin^3 t\,dt\right)^2}.$

(iv) $F'(x) = \dfrac{2x\tan x}{\displaystyle\int_1^{x^2} \tan\sqrt{t}\,dt} \exp\left\{\int_1^{x^2} \tan\sqrt{t}\,dt\right\}.$

(v) $F'(x) = 2x\displaystyle\int_0^x f(t)\,dt + x^2 f(x).$

(vi) $F'(x) = \cos\left(\displaystyle\int_0^x \sin\left(\int_0^y \sin^3 t\,dt\right)dy\right)\sin\left(\int_0^x \sin^3 t\,dt\right).$

Problem 10.7 $f'(x) = e^{-(x-1)^2} - e^{-2(x-1)}$, so $f'(x) = 0$ when $(x-1)^2 = 2(x-1)$, i.e., when $x = 1$ or $x = 3$. Between those two values $(x-1)^2 < 2(x-1)$, and for $x > 3$ the opposite holds. Therefore $f'(x) > 0$ for $1 < x < 3$ and $f'(x) < 0$ for $x > 3$. Thus there is a local maximum at $x = 3$ —which is the absolute maximum. To obtain the absolute minimum we need to obtain

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(\int_0^{x-1} e^{-t^2}\,dt - \int_0^{x-1} e^{-2t}\,dt \right) = \frac{\sqrt{\pi}}{2} - \lim_{x\to\infty} \frac{1}{2}\left(1 - e^{-2(x-1)}\right) = \frac{\sqrt{\pi}-1}{2} > 0.$$

Since $f(1) = 0$, the absolute minimum is reached at $x = 1$.

Problem 10.8 Function $f(x) = \displaystyle\int_0^x e^{t^2}\,dt - 1$ is an increasing function because $f'(x) = e^{x^2} > 0$. Further $f(0) = -1$. On the other hand, $e^{t^2} > 1$ for all $t > 0$, so

$$f(1) = \int_0^1 e^{t^2}\,dt - 1 > \int_0^1 dt - 1 = 0.$$

Therefore $f(x) = 0$ has a unique solution in $(0,1)$.

Problem 10.9 $F(x)$ is a continuous function (is the difference of two integrals) in $[0,1]$. On the other hand,

$$F(0) = 2\cancel{\int_0^0 f(t)\,dt} - \int_0^1 f(t)\,dt = -\int_0^1 f(t)\,dt < 0$$

(it is negative because $f(x) > 0$ in $[0,1]$, therefore the integral is positive), and

$$F(1) = 2\int_0^1 f(t)\,dt - \cancel{\int_1^1 f(t)\,dt} = 2\int_0^1 f(t)\,dt > 0$$

(it is positive for the same reason). Since $F(x)$ has opposite signs at the extremes of the interval it must be zero somewhere in between. Thus, the equation $F(x) = 0$ has at least one solution. To see that there are no more solutions we differentiate

$$F'(x) = 2f(x) - f(x)(-1) = 3f(x) > 0.$$

Therefore $F(x)$ increases monotonically in $[0,1]$, hence can be zero only once in the interval.

Problem 10.10 If $x > 0$ the equation $G'(x) = 2x\sin\left(x^2\right)e^{\sin\left(x^2\right)} = 0$ has solutions $x = \sqrt{n\pi}$, with $n \in \mathbb{N}$. Since the exponential is always positive, the sign of $G'(x)$ is determined by $\sin\left(x^2\right)$. So it starts being positive and alternates sign every other solution. So $\sqrt{(2k-1)\pi}$ are maxima and $\sqrt{2k\pi}$ are minima ($k \in \mathbb{N}$).

Problem 10.11 For $x = \sqrt[4]{\pi/4}$ we get $y = 0$. On the other hand, since $y' = -2x\tan\left(x^4\right)$, the slope at $x = \sqrt[4]{\pi/4}$ will be $-2\sqrt[4]{\pi/4} = -\sqrt[4]{4\pi}$. This yields for the tangent straight line the equation

$$y = -\sqrt[4]{4\pi}\left(x - \sqrt[4]{\pi/4}\right) = \sqrt{\pi} - \sqrt[4]{4\pi}x.$$

Problem 10.12 If the function must be continuous at 0 then $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x)$. But

$$\begin{aligned}
\lim_{x\to 0^-} f(x) &= \lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{1 + x + x^2/2 + o(x^2) - 1 - x}{x^2} = \lim_{x\to 0} \frac{x^2/2 + o(x^2)}{x^2} \\
&= \lim_{x\to 0}\left[\frac{1}{2} + o(1)\right] = \frac{1}{2}, \\
\lim_{x\to 0^+} f(x) &= \lim_{x\to 0}\left(a + b\int_0^x e^{-t^4}\,dt\right) = a + b\int_0^0 e^{-t^4}\,dt = a.
\end{aligned}$$

Hence $a = 1/2$. Now, for the function to be differentiable at $x = 0$ it must hold

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x}.$$

Since $f(0) = 1/2$,

$$\begin{aligned}
\lim_{x\to 0^-} \frac{f(x)-f(0)}{x} &= \lim_{x\to 0} \frac{\dfrac{e^x-1-x}{x^2}-\dfrac{1}{2}}{x} = \lim_{x\to 0} \frac{e^x-1-x-x^2/2}{x^3} \\
&= \lim_{x\to 0} \frac{1+x+x^2/2+x^3/6+o(x^3)-1-x-x^2/2}{x^3} = \lim_{x\to 0} \frac{x^3/6+o(x^3)}{x^3} \\
&= \lim_{x\to 0} \left[\frac{1}{6}+o(1)\right] = \frac{1}{6}, \\
\lim_{x\to 0^+} \frac{f(x)-f(0)}{x} &= \lim_{x\to 0} \frac{\dfrac{1}{2}+b\displaystyle\int_0^x e^{-t^4}\,dt-\dfrac{1}{2}}{x} = \lim_{x\to 0} \frac{b}{x}\int_0^x e^{-t^4}\,dt = b\frac{d}{dx}\left(\int_0^x e^{-t^4}\,dt\right)\bigg|_{x=0} \\
&= be^{-x^4}\Big|_{x=0} = b.
\end{aligned}$$

Therefore $b = 1/6$.

Here is a shorter alternative. We can Taylor expand both functions up to first order. On the one hand

$$e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+0(x^3),$$

therefore

$$\frac{e^x-1-x}{x^2} = \frac{1}{2}+\frac{x}{6}+o(x).$$

On the other hand if

$$g(x) = \int_0^x e^{-t^4}\,dt$$

then $g(0) = 0$, $g'(x) = e^{-x^4}$ and $g'(0) = 1$, so

$$g(x) = x+o(x),$$

therefore

$$a+b\int_0^x e^{-t^4}\,dt = a+bx+o(x).$$

If $f(x)$ has to be continuous and differentiable at $x = 0$ both expansions must coincide up to first order, hence we obtain the same values for a and b.

Problem 10.13

(i) Using l'Hôpital's rule once we get

$$\lim_{x\to 0} \frac{e^{x^2}-1}{3x^2} = \frac{1}{3}.$$

(ii) Since $\lim_{x\to 0}\cos x = 1$,

$$\lim_{x\to 0}\frac{\cos x}{x^4}\int_0^x \sin\left(t^3\right)\,dt = \lim_{x\to 0}\frac{1}{x^4}\int_0^x \sin\left(t^3\right)\,dt.$$

Applying l'Hôpital's rule once we get

$$\lim_{x\to 0}\frac{\sin\left(x^3\right)}{4x^3} = \frac{1}{4}.$$

Problem 10.14 Using l'Hôpital's rule once we get

$$\lim_{x\to 0^\pm}\frac{2x\tan|x|}{6x^2} = \lim_{x\to 0^\pm}\frac{\tan|x|}{3x} = \pm\frac{1}{3}.$$

Problem 10.15

(a) Since

$$\frac{\sin t}{t} = \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n+1)!}, \qquad \int_0^{x^2} t^{2n}\,dt = \frac{x^{4n+2}}{2n+1},$$

we obtain

$$f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{x^{2n+2}}{2n+1}.$$

(b) $1-\cos x = \frac{x^2}{2} + o\left(x^2\right)$ $(x\to 0)$ and $f(x) = x^2 + o\left(x^2\right)$ $(x\to 0)$, so

$$\lim_{x\to 0}\frac{f(x)}{1-\cos x} = 2.$$

(c) The series converges because

$$f(1/n) = \frac{1}{n^2} + o\left(\frac{1}{n^2}\right) \quad\text{and}\quad \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty.$$

Problem 10.16

$$f'(x) = \frac{1}{a^2+x^2} - \frac{1}{x^2}\frac{1}{a^2+1/x^2} = \frac{1}{a^2+x^2} - \frac{1}{a^2x^2+1},$$

so in order to have $f'(x) = 0$ for any x we need $a = \pm 1$.

Problem 10.17

(a) $f(x) = \sum_{n=0}^{\infty}\frac{x^{2n}}{n!} - x^2 - 1 = \sum_{n=2}^{\infty}\frac{x^{2n}}{n!}$. Then

$$g(x) = \sum_{n=2}^{\infty}\frac{x^{2n+1}}{n!(2n+1)}.$$

(b) Since $g(x) = x^5/10 + o\left(x^5\right)$ $(x\to 0)$ —i.e., the first nonzero derivative at $x=0$ is the fifth—, the point $x=0$ is an inflection point.

Problem 10.18

(a) $dt = 2\sin\theta\cos\theta\,d\theta = \sin 2\theta\,d\theta$, therefore

$$I = \int_0^1 \arcsin\sqrt{t}\,dt = \int_0^{\pi/2} \arcsin(\sin\theta)\,\sin 2\theta\,d\theta = \int_0^{\pi/2} \theta\sin 2\theta\,d\theta.$$

We can now integrate by parts, where $u = \theta$ and $v' = \sin 2\theta$, and then

$$I = -\frac{\theta}{2}\cos 2\theta\Big|_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2}\cos 2\theta\,d\theta = \frac{\pi}{4} + \frac{1}{4}\sin 2\theta\Big|_0^{\pi/2} = \frac{\pi}{4} + 0.$$

Thus

$$\int_0^1 \arcsin\sqrt{t}\,dt = \frac{\pi}{4}.$$

(b) Differentiating,

$$f'(x) = 2\sin x\cos x\arcsin(\sin x) - 2\cos x\sin x\arccos(\cos x) = x\sin 2x - x\sin 2x = 0.$$

Therefore $f(x)$ is constant.

(c) We can calculate c by substituting any value of x, for instance $x = \pi/2$. Then

$$c = f(\pi/2) = \int_0^1 \arcsin\sqrt{t}\,dt + \int_0^0 \arccos\sqrt{t}\,dt = \int_0^1 \arcsin\sqrt{t}\,dt.$$

But this is precisely the integral we have obtained in (a), so $c = \pi/4$.

Problem 10.19

(a) Setting $x = 0$ in the equation

$$\int_0^{g(0)} \left(e^{t^2} + e^{-t^2}\right)\,dt = 0.$$

Since the integrand is a strictly positive function, the only possibility for this equation to hold is that $g(0) = 0$.

Differentiating,

$$g'(x)\left(e^{g(x)^2} + e^{-g(x)^2}\right) = 3x^2 + \frac{3}{1+x^2},$$

thus, using $g(0) = 0$, we obtain $g'(0) = 3/2$.

Finally, we know that $g(0) = 0$ so $g^{-1}(0) = 0$. Then

$$\left(g^{-1}\right)'(0) = \frac{1}{g'\left(g^{-1}(0)\right)} = \frac{1}{g'(0)} = \frac{2}{3}.$$

(b) Since it is an indeterminacy $\frac{0}{0}$ we can use l'Hôpital's rule and calculate

$$\lim_{x\to 0}\frac{\left(g^{-1}\right)'(x)}{g'(x)} = \frac{\left(g^{-1}\right)'(0)}{g'(0)} = \frac{2/3}{3/2} = \frac{4}{9}.$$

Problem 10.20

(a) With this change of variables the limits remain the same, so

$$I=\int_0^\pi xf(\sin x)\,dx=\int_0^\pi(\pi-y)f\big(\sin(\pi-y)\big)\,dy.$$

But since $\sin(\pi-y)=\sin y$, we have

$$I=\int_0^\pi(\pi-y)f(\sin y)\,dy=\pi\int_0^\pi f(\sin y)\,dy-I.$$

Thus

$$I=\frac{\pi}{2}\int_0^\pi f(\sin x)\,dx.$$

(b) Since

$$\frac{\sin x}{1+\cos^2 x}=\frac{\sin x}{2-\sin^2 x}=f(\sin x),$$

we are in the situation described in the previous item. Hence

$$\begin{aligned}I&=\int_0^\pi\frac{x\sin x}{1+\cos^2 x}\,dx=\frac{\pi}{2}\int_0^\pi\frac{\sin x}{1+\cos^2 x}\,dx=-\frac{\pi}{2}\int_0^\pi\frac{(\cos x)'}{1+\cos^2 x}\,dx\\&=-\frac{\pi}{2}\arctan(\cos x)\Big|_0^\pi=-\frac{\pi}{2}(-2\arctan 1)=\frac{\pi^2}{4}.\end{aligned}$$

Problem 10.21 Differentiating the equation,

$$f(x)=-x^2f(x)+2x^{15}+2x^{17}\quad\Rightarrow\quad(1+x^2)f(x)=2x^{15}(1+x^2)\quad\Rightarrow\quad f(x)=2x^{15}.$$

Now substituting back into the equation and setting $x=1$,

$$\int_0^1 f(t)\,dt=\frac{1}{8}+\frac{1}{9}+c\quad\Rightarrow\quad\frac{t^{16}}{8}\Big|_0^1=\frac{1}{8}+\frac{1}{9}+c\quad\Rightarrow\quad\frac{1}{8}=\frac{1}{8}+\frac{1}{9}+c\quad\Rightarrow\quad c=-\frac{1}{9}.$$

Problem 10.22 By definition $f(x)\sim g(x)$ $(x\to a)$ if

$$\lim_{x\to a}\frac{f(x)}{g(x)}=1.$$

In our case we have to calculate the limit

$$\ell=\lim_{x\to\infty}\frac{\int_0^x e^{t^2}\,dt}{e^{x^2}/2x}.$$

Since this is a $\frac{\infty}{\infty}$ indeterminacy, we can apply l'Hôpital and obtain

$$\ell=\lim_{x\to\infty}\frac{e^{x^2}}{(4x^2e^{x^2}-2e^{x^2})/4x^2}=\lim_{x\to\infty}\frac{4x^2}{4x^2-2}=1.$$

This proves the equivalence.

Problem 10.23

(a) $R_0(x)=\displaystyle\int_a^x f'(t)\,dt=f(x)-f(a).$

(b) $R_n(x)=\displaystyle\frac{1}{n!}\left[(x-t)^nf^{(n)}(t)\Big|_a^x+n\int_a^x(x-t)^{n-1}f^{(n)}(t)\,dt\right]=-\frac{f^{(n)}(a)}{n!}(x-a)^n+R_{n-1}(x).$

(c) Using the recurrence iteratively we obtain

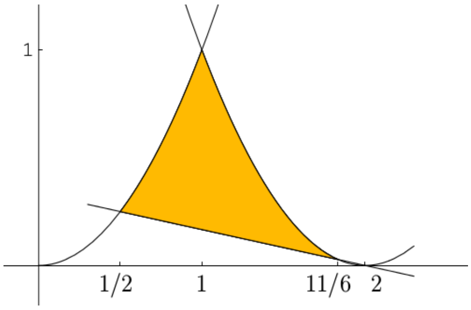
$$
\begin{aligned}
R_0(x) &= f(x) - f(a), \\
R_1(x) &= f(x) - f(a) - f'(a)(x-a), \\
R_2(x) &= f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2}(x-a)^2, \\
&\vdots \\
R_n(x) &= f(x) - f(a) - f'(a)(x-a) - \cdots - \frac{f^{(n)}(a)}{n!}(x-a)^n.
\end{aligned}
$$

In other words, $R_n(x) = f(x) - P_{n,a}(x)$, where $P_{n,a}(x)$ is Taylor's polynomial of $f(x)$ at the point a. Function $R_n(x)$ is therefore the remainder of order n of Taylor's approximation.

D.11 Geometric Applications of Integrals

Problem 11.1

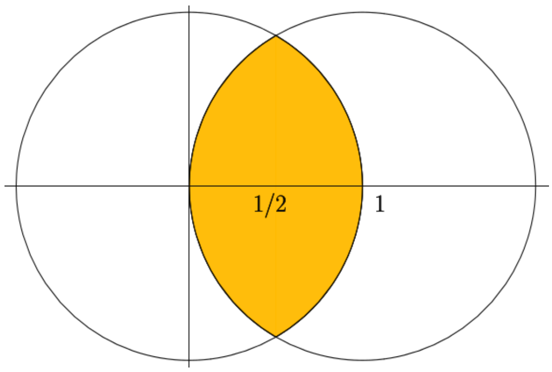
(i) Here is the figure delimited by the three curves:



$y = x^2$ and $y = (x-2)^2$ meet at $x = 1$; $y = x^2$ and $y = (2-x)/6$ meet at $x = 1/2$; and $y = (x-2)^2$ and $y = (2-x)/6$ meet at $x = 11/6$ and $x = 2$. The area enclosed by this three curves is calculated as

$$A = \int_{1/2}^{1} \left(x^2 - \frac{2-x}{6} \right) dx + \int_{1}^{11/6} \left((x-2)^2 - \frac{2-x}{6} \right) dx = \frac{71}{162}.$$

(ii) Here is the figure delimited by the two circumferences:

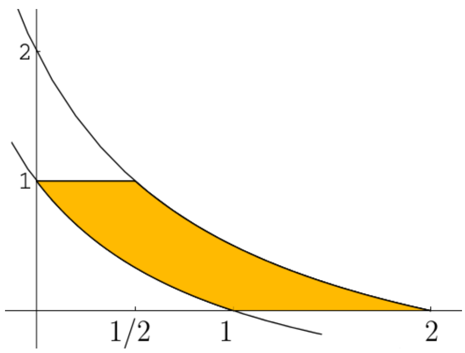


They meet at $x = 1/2$. By symmetry,

$$A = 4 \int_{1/2}^{1} \sqrt{1-x^2}\, dx = 4 \int_{\pi/6}^{\pi/2} \cos^2 t\, dt = 2 \int_{\pi/6}^{\pi/2} (1 + \cos 2t)\, dt = \frac{2\pi}{3} - \frac{\sqrt{3}}{2},$$

using the change $x = \sin t$ and the identity $2\cos^2 t = 1 + \cos 2t$.

(iii) Here is the figure delimited by the four curves:



The area is then

$$A = \int_0^{1/2} \left(1 - \frac{1-x}{1+x}\right) dx + \int_{1/2}^{1} \left(\frac{2-x}{1+x} - \frac{1-x}{1+x}\right) dx + \int_1^2 \frac{2-x}{1+x} dx$$
$$= \int_0^{1/2} \frac{2x}{1+x} dx + \int_{1/2}^{1} \frac{dx}{1+x} + \int_1^2 \frac{2-x}{1+x} dx = 1 - 2\log(1+x)\Big|_0^{1/2} + \log(1+x)\Big|_{1/2}^{1}$$
$$+ 3\log(1+x)\Big|_1^2 - 1 = -2\log(3/2) + \log 2 - \log(3/2) + 3\log 3 - 3\log 2 = \log 2.$$

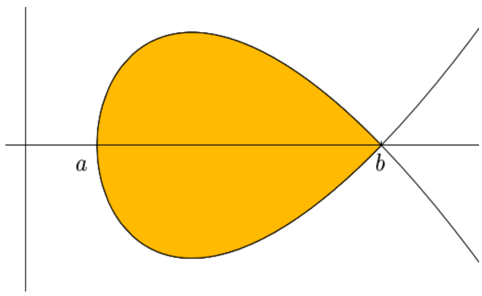
The result is easier to obtain if we express the curves as

$$x = \frac{1-y}{1+y}, \qquad x = \frac{2-y}{1+y},$$

for then

$$A = \int_0^1 \left(\frac{2-y}{1+y} - \frac{1-y}{1+y}\right) dy = \int_0^1 \frac{dy}{1+y} dy = \log 2.$$

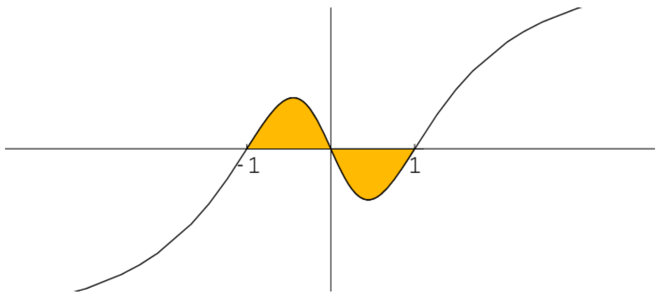
(iv) Here is the figure delimited by the curve:



By symmetry, the area is

$$A = 2\int_a^b (b-x)\sqrt{x-a}\, dx = 2\int_a^b \left[(b-a) - (x-a)\right]\sqrt{x-a}\, dx = \frac{8}{15}(b-a)^{5/2}.$$

Problem 11.2 Given the figure



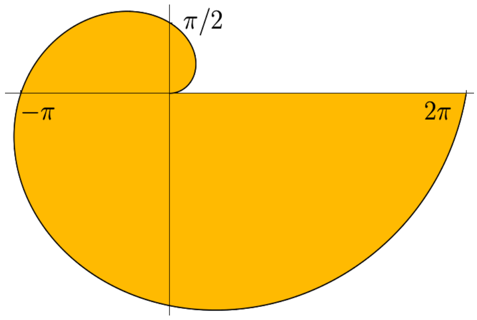
the area is obtained, by symmetry, as

$$A = 2\int_0^1 \frac{x(1-x^2)}{(x^2+1)^{3/2}} dx = 2\int_0^{\pi/4} \tan t \underbrace{(1-\tan^2 t)}_{=2-\frac{1}{\cos^2 t}} \cos^3 t \frac{dt}{\cos^2 t} = 2\int_0^{\pi/4} \left(2\sin t - \frac{\sin t}{\cos^2 t}\right) dt$$
$$= 4(-\cos t)\Big|_0^{\pi/4} - \frac{2}{\cos t}\Big|_0^{\pi/4} = 6 - 4\sqrt{2},$$

where we have made the change of variable $x = \tan t$ and used the identity $1 + \tan^2 x = 1/\cos^2 x$.

Problem 11.3

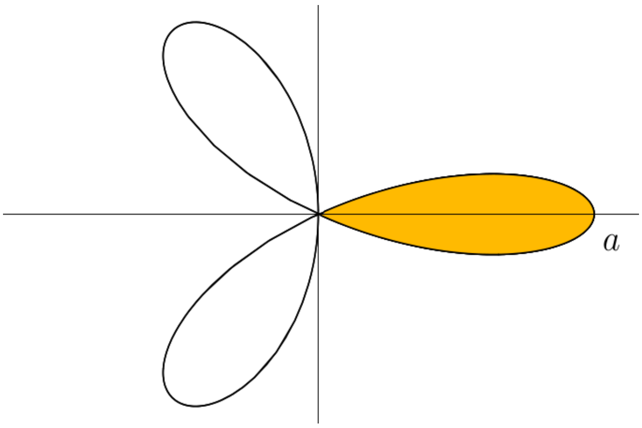
(i) Here is the figure:



The area is obtained as

$$A = \frac{1}{2}\int_0^{2\pi} a^2\theta^2\,d\theta = \frac{4}{3}\pi^3 a^2.$$

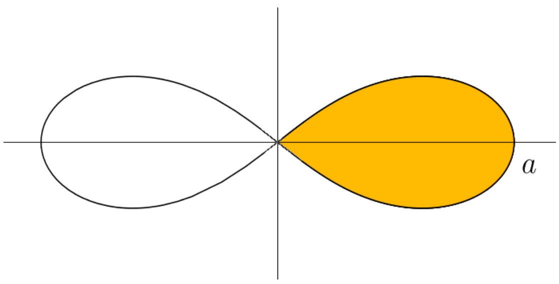
(ii) Here is the figure:



The area is obtained, by symmetry, as

$$A = \int_0^{\pi/6} a^2\cos^2 3\theta\,d\theta = \frac{a^2}{2}\int_0^{\pi/6}(1+\cos 6\theta)\,d\theta = \frac{\pi}{12}a^2.$$

(iii) Here is the figure:



The area is obtained, by symmetry, as

$$A = \int_0^{\pi/4} a^2\cos 2\theta\,d\theta = \frac{a^2}{2}.$$

Problem 11.4

(a) Both curves meet at $x = 0$ and $x = 1$, and within $[0,1]$ we have $\sqrt{x} \geqslant x^2$. Then

$$A = \int_0^1 \left(\sqrt{x} - x^2\right) dx = \frac{1}{3}.$$

(b)

$$V = \pi \int_0^1 x\,dx - \pi \int_0^1 x^4\,dx = \frac{3\pi}{10}.$$

Problem 11.5

(i)

$$\begin{aligned} V =& \pi \int_0^{2\pi} (1+\sin x)^2\,dx = \pi \int_0^{2\pi} \left(1 + 2\sin x + \sin^2 x\right) dx \\ &= \pi \int_0^{2\pi} \left(\frac{3}{2} + 2\sin x - \frac{1}{2}\sin 2x\right) dx = 3\pi^2. \end{aligned}$$

(ii)

$$V = \frac{4}{3}\pi(2R)^3 - \frac{4}{3}\pi R^3 = \frac{28}{3}\pi R^3.$$

(iii) Since $x \geqslant \sin x$ within $[0,\pi]$,

$$V = \pi \int_0^{\pi} \left(x^2 - \sin^2 x\right) dx = \pi \int_0^{\pi} \left(x^2 - \frac{1}{2} + \frac{1}{2}\sin 2x\right) dx = \frac{\pi^4}{3} - \frac{\pi^2}{2}.$$

Problem 11.6

(i)

$$V = \pi \int_{-a}^{a} b^2 \left(1 - \frac{x^2}{a^2}\right) dx = \frac{4}{3}\pi b^2 a.$$

(ii)

$$V = 4\pi \int_0^a xb\sqrt{1 - \frac{x^2}{a^2}}\,dx = -\frac{4}{3}\pi b a^2 \left(1 - \frac{x^2}{a^2}\right)^{3/2} \Bigg|_0^a = \frac{4}{3}\pi b a^2.$$

(iii) The area of the triangular section at x will be

$$a(x) = 2b\sqrt{1 - \frac{x^2}{a^2}},$$

hence

$$V = 2b \int_{-a}^{a} \sqrt{1 - \frac{x^2}{a^2}}\,dx = 2ab \int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta = ab \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta)\,d\theta = \pi ab,$$

using the change of variable $x = a\sin\theta$.

Problem 11.7

(a) By symmetry

$$A = 2\int_{-a}^{a} b\sqrt{1 - \frac{x^2}{a^2}}\,dx = \pi ab$$

(see Problem 11.6(iii)).

(b) Rewrite the equation of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{z^2}{c^2} \quad\Rightarrow\quad \frac{x^2}{a(z)^2}+\frac{y^2}{b(z)^2}=1,$$

where

$$a(z)=a\sqrt{1-\frac{z^2}{c^2}}, \qquad b(z)=b\sqrt{1-\frac{z^2}{c^2}}.$$

This means that the sections of the ellipsoid perpendicular to the Z axis are ellipses, with axis $a(z)$ and $b(z)$ $(-c\leqslant z\leqslant c)$. Their area is $A(z)=\pi a(z)b(z)$ (see (a)), therefore

$$V=\pi ab\int_{-c}^{c}\left(1-\frac{z^2}{c^2}\right)dz=\frac{4}{3}\pi abc.$$

(c) If the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ rotates around the X axis, it generates an ellipsoid with $c=b$, hence $V=4\pi ab^2/3$. If it does around the Y axis, the ellipsoid will have $c=a$, hence $V=4\pi a^2b/3$.

Problem 11.8

(i) $\mathbf{r}(x)=\big(x,e^{x/2}+e^{-x/2}\big)$, thus $\mathbf{r}'(x)=\big(1,(e^{x/2}-e^{-x/2})/2\big)$ and $\|\mathbf{r}'(x)\|=\sqrt{1+\sinh^2(x/2)}=\cosh(x/2)$. Accordingly

$$L=\int_0^2\cosh\frac{x}{2}\,dx=2\sinh 1=e-e^{-1}.$$

(ii) $\mathbf{r}'(t)=\big(a(1-\cos t),a\sin t\big)$,

$$\|\mathbf{r}'(t)\|=a\sqrt{(1-\cos t)^2+\sin^2 t}=a\sqrt{2(1-\cos t)}=2a\left|\sin\frac{t}{2}\right|$$

Therefore

$$L=2a\int_0^{2\pi}\sin\frac{t}{2}\,dt=4a\int_0^{\pi}|\sin u|\,du=8a.$$

(iii) One arc of the curve can be parametrised $\mathbf{r}(x)=\big(x,(4-x^{2/3})^{3/2}\big)$, where $0\leqslant x\leqslant 8$ (the other three have identical length). Thus $\mathbf{r}'(x)=\big(1,-x^{-1/3}(4-x^{2/3})^{1/2}\big)$ and $\|\mathbf{r}'(x)\|=\sqrt{1+x^{-2/3}(4-x^{2/3})}=2x^{-1/3}$. Accordingly

$$L=8\int_0^8 x^{-1/3}\,dx=48.$$

(iv) Taking $\mathbf{r}(x)=\big(x,y(x)\big)$ we get, after a lengthy calculation,

$$\mathbf{r}'(x)=\left(1,-\frac{\sqrt{a^2-x^2}}{x}\right) \quad\Rightarrow\quad \|\mathbf{r}'(x)\|=\sqrt{1+\frac{a^2-x^2}{x^2}}=\frac{a}{x}.$$

Therefore

$$L=a\int_{a/2}^{a}\frac{dx}{x}=a\log 2.$$

(v) The parametrisation is $\mathbf{r}(\theta) = \big(r(\theta)\cos\theta, r(\theta)\sin\theta\big)$, so

$$\mathbf{r}'(\theta) = \big(r'(\theta)\cos\theta - r(\theta)\sin\theta, r'(\theta)\sin\theta + r(\theta)\cos\theta\big)$$

and therefore, using $\cos^2\theta + \sin^2\theta = 1$,

$$\|\mathbf{r}'(\theta)\| = \sqrt{r'(\theta)^2 + r(\theta)^2} = \sqrt{\sin^2\theta + (1+\cos\theta)^2} = \sqrt{2(1+\cos\theta)} = 2\left|\cos\frac{\theta}{2}\right|.$$

Accordingly,

$$L = 2\int_0^{2\pi} \left|\cos\frac{\theta}{2}\right| d\theta = 4\int_0^{\pi} |\cos t|\, dt = 8\int_0^{\pi/2} \cos t\, dt = 8.$$