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CALCULUS – Sequences and series of real numbers

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Problem 2.1. Study whether the given sequences $(a_n)_{n\in\mathbb{N}}$ of real numbers are bounded, monotone, and convergent.

a) $a_n = \dfrac{1+(-1)^n}{2}$.

b) $a_n = \dfrac{(-1)^{n+1}}{n}$.

c) $a_n = \dfrac{n}{n+2}$.

d) $a_n = \dfrac{\lfloor n/2 \rfloor}{n}$.

e) $a_n = \dfrac{\lfloor nx \rfloor}{n}, \quad x \in \mathbb{R}$.

f) $a_n = \dfrac{n+\sin(\pi n/2)}{2n+1}$.

g) $a_n = \sqrt[n]{\pi^n + (\sqrt{7})^n}$.

h) $a_n = \dfrac{1}{n^2+1}\displaystyle\sum_{k=1}^{n} k$.

Problem 2.2. Calculate the limit, as $n \to \infty$, of the given sequences $(a_n)_{n\in\mathbb{N}}$ of real numbers.

a) $a_n = \dfrac{n^2}{(n-7)!} \quad (n \geq 7)$.

b) $a_n = \dfrac{n!}{n^n}$.

c) $a_n = \sqrt{n^3-1} - n$.

d) $a_n = \dfrac{\sqrt{n^3-1}-n}{5n^2 - 7\sqrt{n}}$.

e) $a_n = \dfrac{3^n + 2^{n+1}}{3^{n+1}+2^n}$.

Problem 2.3. Prove that the following *recursive* sequences $(a_n)_{n\in\mathbb{N}}$ are bounded and monotone. Then, calculate $\lim_{n\to\infty} a_n$.

a) $a_1 = \sqrt{3}, \ a_2 = \sqrt{3\sqrt{3}}, \ a_3 = \sqrt{3\sqrt{3\sqrt{3}}}, \ \ldots$.

b) $a_n = 5 + \dfrac{a_{n-1}}{4} \quad \forall n \geq 2\,, \quad a_1 = 0\,.$

c) $a_n = \dfrac{1 + 3a_{n-1}^2}{4} \quad \forall n \geq 2\,, \quad 1/3 \leq a_1 < 1\,.$

d) $a_{n+1} = \sqrt{2a_n + 3} \quad \forall n \geq 1\,, \quad a_1 = 1\,.$

Problem 2.4. Calculate the following limits.

a) $\lim\limits_{n\to\infty} n^{1/(n-1)}\,.$

b) $\lim\limits_{n\to\infty} (7n^3 - 1)^{1/n}\,.$

c) $\lim\limits_{n\to\infty} \left(\dfrac{3n^2+1}{3n^2+2}\right)^{-n^2}\,.$

Problem 2.5. Study whether the given series of real numbers, with positive terms, are convergent or not.

a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+k}\,.$

b) $\displaystyle\sum_{k=1}^{\infty} \frac{3+2\cos(k)}{k^2+k}\,.$

c) $\displaystyle\sum_{k=1}^{\infty} \frac{k+1}{k^2}\,.$

d) $\displaystyle\sum_{k=1}^{\infty} \frac{7\sqrt{k}+323}{k^2+\cos(k)}\,.$

e) $\displaystyle\sum_{k=1}^{\infty} \frac{\arctan(k)}{k^2+7}\,.$

f) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{3^k+(-1)^k}\,.$

g) $\displaystyle\sum_{k=1}^{\infty} \frac{\ln(k)}{k^4}\,.$

h) $\displaystyle\sum_{k=1}^{\infty} \frac{\ln(k)}{k}\,.$

i) $\displaystyle\sum_{k=1}^{\infty} \frac{\ln(k)}{k^2}\,.$

j) $\displaystyle\sum_{k=1}^{\infty} \frac{(k+1)^k}{k^{k+1}}\,.$

Problem 2.6. Study the convergence of the following series of real numbers.

a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2+k}\,.$

b) $\displaystyle\sum_{k=1}^{\infty} \frac{\cos(k)}{5^k}\,.$

c) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}\,.$

d) $\displaystyle\sum_{k=1}^{\infty} \frac{(-4)^k}{4+k!}\,.$

e) $\displaystyle\sum_{k=1}^{\infty} (-1)^k\,3^k\,5^{-\sqrt{k}}\,.$

f) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(\ln(k))^k}\,.$

g) $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{k^k}\,.$

h) $\displaystyle\sum_{k=1}^{\infty} \ln\left(\frac{k}{k+1}\right)\,.$

Problem 2.7. Find *all* values of the parameters $a, b, \alpha \in \mathbb{R}$ such that the given series of real numbers are convergent.

$$\text{a)} \quad \sum_{k=1}^{\infty} \frac{k^a}{b^k}\,, \quad a > 0\,,\ b \neq 0\,.$$

$$\text{b)} \quad \sum_{k=1}^{\infty} \frac{b^k}{k!}\,.$$

$$\text{c)} \quad \sum_{k=1}^{\infty} (-1)^k \frac{(2\alpha)^{3k}}{7^k \sqrt[3]{k^2+k}}\,.$$