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CALCULUS – Real functions: derivative

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Problem 4.1. Consider the function

$$f(x) \;=\; \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right), & \text{if } \; x \neq 0\,, \\ 0\,, & \text{if } \; x = 0\,. \end{cases}$$

- Is $f(x)$ continuous at $x = 0$?
- Is $f(x)$ differentiable at $x = 0$?

Problem 4.2. Study the continuity and differentiability of the function

$$f(x) = \sqrt{x+2}\, \arccos(x+2)\,.$$

Problem 4.3. Calculate the first derivative of the following functions.

1) $f(x) \;=\; \sqrt{3x^2 - 7x - 2}\,.$

2) $f(x) \;=\; x^2 \sin(x) \tan(x)\,.$

3) $f(x) \;=\; \sqrt[3]{\dfrac{x-1}{x+1}}\,.$

4) $f(x) \;=\; \sin\left(\sqrt{1+\cos(x)}\right).$

5) $f(x) \;=\; \ln\left(\dfrac{x^2 \sin(x)}{\sqrt{1+x}}\right).$

Problem 4.4. Find an equation for the *tangent line* to the graph of the function

$$g(x) \;=\; \frac{x+1}{x-1}$$

at $x = 2$.

Problem 4.5. Study the differentiability of the following functions and calculate, in each case, the first derivative (where defined).

- $f(x) = x^{1/3}$.
- $f(x) = \ln|x|$.

Problem 4.6. Consider the function

$$f(x) = \begin{cases} \cos(x), & \text{if } x \leq 0, \\ 1 - x^2, & \text{if } 0 < x < 1, \\ \arctan(x), & \text{if } x \geq 1. \end{cases}$$

Then, study whether $f(x)$ is differentiable in $\mathbb{R}$ and calculate $f'(x)$ (where defined).

Problem 4.7. Let $f(x)$ and $g(x)$ be two differentiable functions in $\mathbb{R}$. Then, in each case, calculate an expression for $h'(x)$ (where defined).

1) $h(x) = f(g(x))\, e^{f(x)}$.

2) $h(x) = \dfrac{1}{\ln\left(f(x) + g^2(x)\right)}$.

3) $h(x) = \sqrt{f^2(x) + g^2(x)}$.

4) $h(x) = \arctan\left(\dfrac{f(x)}{g(x)}\right)$.

5) $h(x) = \ln\left(g(x)\, \cos(f(x))\right)$.

Problem 4.8. Let c, c_1, $c_2 \in \mathbb{R}$. In each case, prove that the given function $f(x)$ is solution of the corresponding *differential* equation.

1) $f(x) = c/x$; $\quad xf' + f = 0$.
2) $f(x) = x\tan(x)$; $\quad xf' - f - f^2 = x^2$.
3) $f(x) = c_1 \sin(3x) + c_2 \cos(3x)$; $\quad f'' + 9f = 0$.
4) $f(x) = c_1 e^{3x} + c_2 e^{-3x}$; $\quad f'' - 9f = 0$.
5) $f(x) = c_1 e^{2x} + c_2 e^{5x}$; $\quad f'' - 7f' + 10f = 0$.
6) $f(x) = \ln\left(c_1 e^{x} + c_2 e^{-x}\right) + c_2$; $\quad f'' + (f')^2 = 1$.

Problem 4.9. Prove the following identities.

$$
\begin{aligned}
&1)\quad \arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2} && \text{(for } x > 0\text{)}.\\
&2)\quad \arctan\left(\frac{1+x}{1-x}\right) - \arctan(x) = \frac{\pi}{4} && \text{(for } x < 1\text{)}.\\
&3)\quad 2\arctan(x) + \arcsin\left(\frac{2x}{1+x^2}\right) = \pi && \text{(for } x > 1\text{)}.
\end{aligned}
$$

HINT: *calculate the first derivative of the function in the left-hand-side of each identity.*

Problem 4.10. Calculate the angle formed by the tangent lines from the right and from the left, at $x = 0$, to the graph of the function

$$
f(x) = \begin{cases} \dfrac{x}{1+e^{1/x}}, & \text{if } \quad x \neq 0, \\ 0, & \text{if } \quad x = 0. \end{cases}
$$

Problem 4.11. Given $k \in \mathbb{R}$, consider the functions

$$
f_1(x) = |x|^k, \quad f_2(x) = x\,|x|^{k-1}.
$$

- For $x \neq 0$, calculate $f_1'(x)$ and $f_2'(x)$.
- For $k > 1$, prove that both functions are differentiable at $x = 0$ and calculate $f_1'(0)$, $f_2'(0)$.
- Prove the following statement. If $f(x)$ is a function satisfying $|f(x)| \leq |x|^k$, with $k > 1$ and for all x in a neighborhood of $x_0 = 0$, then $f(x)$ is differentiable at $x_0 = 0$. Finally, calculate $f'(0)$.

Problem 4.12. Analyze the continuity and differentiability of the function

$$
f(x) = \begin{cases} (3-x^2)/2, & \text{if } \quad x < 1, \\ 1/x, & \text{if } \quad x \geq 1. \end{cases}
$$

Can we apply the *Lagrange mean-value Theorem* in the interval $[0, 2]$? In that case, find the points of the theorem statement.

Problem 4.13. The function $f(x) = 1 - x^{2/3}$ vanishes at $x = -1$ and $x = 1$. However, $f'(x) \neq 0$ for all $x \in (-1, 1)$. Does this contradict the *Rolle Theorem*?

Problem 4.14. Let $h(x)$ be a continuous function in $\mathbb{R}$ such that $h'(x)$ and $h''(x)$ are also continuous in $\mathbb{R}$. Then, consider

$$f(x) = \begin{cases} h(x)/x^2, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

Knowing that $f(x)$ is continuous, calculate $h(0)$, $h'(0)$, and $h''(0)$.

Problem 4.15. Let $f(x)$ be a continuous function in $\mathbb{R}$, with $f'(x)$ also continuous in $\mathbb{R}$, such that

$$\lim_{x\to 0} \frac{f(2x^3)}{5x^3} = 1.$$

- Prove that $f(0) = 0$ and $f'(0) = 5/2$.
- Calculate $\lim_{x\to 0} \frac{f(f(2x))}{3f^{-1}(x)}$ (supposing that $f^{-1}(x)$ exists).

Problem 4.16. Prove the following theorems.

THEOREM 1. Let $f(x)$ be a differentiable function in $[x_1, x_2]$. If $f(x)$ has $k \geq 2$ roots in $[x_1, x_2]$, then $f'(x)$ has at least $k-1$ roots in the same interval.

THEOREM 2. Let $f(x)$ be a k-times differentiable function in $[x_1, x_2]$. If $f(x)$ has $k+1 \geq 2$ roots in $[x_1, x_2]$, then $f^{(k)}(x)$ has at least one root in the same interval.

Problem 4.17. Find the *exact* number of real solutions of the following equations in the indicated intervals.

$$\begin{aligned} &1)\ \ x^7 + 4x = 3, \quad x \in \mathbb{R}. \\ &2)\ \ x^5 = 5x - 6, \quad x \in \mathbb{R}. \\ &3)\ \ x^4 - 4x^3 = 1, \quad x \in \mathbb{R}. \\ &4)\ \ \sin(x) = 2x - 1, \quad x \in \mathbb{R}. \\ &5)\ \ x^2 = \ln(1/x), \quad x \in (1, \infty). \end{aligned}$$

Problem 4.18. Calculate the following limits.

- $\lim_{x\to 0} \frac{e^x - \sin(x) - 1}{x^2}$.
- $\lim_{x\to 0^+} \frac{\ln(\sin(7x))}{\ln(\sin(x))}$.