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CALCULUS – Taylor polynomial

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Problem 6.1. Approximate the following values, within an error smaller than ε, by means of suitable Taylor polynomials.

- $\sin(1)\,, \quad \varepsilon = 10^{-5}\,.$
- $\sqrt[5]{\dfrac{3}{2}}\,, \quad \varepsilon = 10^{-2}\,.$

Problem 6.2. Write the Maclaurin polynomial of degree n for the given functions.

a) $f(x) = \sqrt{1+x}\,, \quad n = 3\,.$

b) $f(x) = \sin(3x^2)\,, \quad n \in \mathbb{N}$ (generic).

c) $f(x) = \tan(x)\,, \quad n = 5\,.$

d) $f(x) = e^{-x^2}\cos(x)\,, \quad n = 3\,.$

e) $f(x) = (1+e^x)^2\,, \quad n \in \mathbb{N}$ (generic).

Problem 6.3. Write the polynomial $x^4 - 5x^3 + x^2 - 3x + 4$ in terms of powers of $x-4$.

Problem 6.4. Write the Taylor formula of degree $n \in \mathbb{N}$ about $a = -1$ for the function $f(x) = 1/x$.

Problem 6.5. Write the Maclaurin polynomial of degree 5 for $f(x) = e^x \sin(x)$.

Problem 6.6. Calculate the coefficient of x^4 in the Maclaurin polynomial for the function $f(x) = \ln(\cos(x))$.

Problem 6.7. Write the Taylor polynomial of degree 3 about $a = 0$ for the given functions.

$$\begin{aligned} f(x) &= \sin(2x)\,. \\ f(x) &= e^{3x}\,. \\ f(x) &= x\,e^{-x}\,. \\ f(x) &= e^{x}\ln(1-x)\,. \\ f(x) &= \sin^2(x)\,. \\ f(x) &= \frac{\sqrt{1+x^2}\,\sin(x)}{1+\ln(1+x)}\,. \end{aligned}$$

Problem 6.8. Write the Taylor polynomial of degree $n \in \mathbb{N}$ about $a = 0$ for the following functions (here, $a \in \mathbb{R}$ is a parameter).

$$\begin{aligned} f(x) &= \cos(ax)\,. \\ f(x) &= \frac{e^{ax} - e^{-ax}}{2}\,. \\ f(x) &= e^{ax^2}\,. \\ f(x) &= \frac{1+x}{1-x}\,. \end{aligned}$$

Problem 6.9. The Taylor polynomial of degree 4 about $a = 1$ for the function $f(x)$ is given by $P_{4,1}(x) = 2(x-1)^3 - 3(x-1)^4$.

- Write an equation for the tangent line to the graph of $f(x)$ at $x = 1$.
- Calculate $\lim_{x \to 1} \dfrac{f(x)}{(x-1)^3}$.
- Compute $f^{(4)}(1)$.

Problem 6.10. Prove the following identities.

$$\begin{aligned} &\forall\, a < 1\,, \quad \sin(x) = o(x^a)\,, \quad \text{as } x \to 0\,. \\ &\ln(1+x^2) = o(x)\,, \quad \text{as } x \to 0\,. \\ &\tan(x) - \sin(x) = o(x^2)\,, \quad \text{as } x \to 0\,. \\ &\ln(x) = o(x)\,, \quad \text{as } x \to +\infty\,. \end{aligned}$$

Problem 6.11. Find the *family* of polynomials $P(x)$ such that

$$\lim_{x\to 0} \frac{\sqrt{1-x^4} - P(x)}{x^7} = 0 .$$

Problem 6.12. Approximate $f(x) = \cos(x) + e^x$ by a polynomial of degree 3 about $a = 0$ and find an *upper bound* for the involved error, when $x \in [-1/4, 1/4]$.

Problem 6.13. How many terms should you consider in the Maclaurin polynomial for $f(x) = e^x$ in order to have an approximation with three exact decimal places, when $x \in [-1, 1]$?

Problem 6.14. Using a Taylor polynomial of degree 3, approximate the value

$$\frac{1}{\sqrt{1.1}}$$

and find an *upper bound* for the involved error.

Problem 6.15. Calculate an approximation of the given values, within an error smaller than 10^{-3}, by means of suitable Taylor polynomials.

- $\cos(1)$.
- e^{-2} .
- $\ln(2)$.

Problem 6.16. How many terms should you consider in the Taylor series for the function $f(x) = \sin(x)$ about $a = 0$ in order to approximate the value $\sin(1/2)$ within an error smaller than 10^{-12} ?

Problem 6.17. Calculate the given limits by means of proper Taylor polynomials.

a) $\displaystyle\lim_{x\to 0} \frac{e^x - \sin(x) - 1}{x^2}$.

b) $\displaystyle\lim_{x\to 0} \frac{\sin(x) - x + x^3/6}{x^5}$.

c) $\displaystyle\lim_{x\to 0} \frac{\cos(x) - \sqrt{1-x}}{\sin(x)}$.

d) $\displaystyle\lim_{x\to 0} \frac{\tan(x) - \sin(x)}{x^3}$.

e) $\lim_{x\to 0} \frac{x - \sin(x)}{x\,[1-\cos(3x)]}$.

f) $\lim_{x\to 0} \frac{\cos(x) + e^{x} - x - 2}{x^3}$.

g) $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin(x)}\right)$.

h) $\lim_{x\to 0} \frac{1}{x}\left(\frac{1}{x} - \frac{\cos(x)}{\sin(x)}\right)$.

i) $\lim_{x\to +\infty} x^{3/2}\left(\sqrt{x+1} + \sqrt{x-1} - 2\sqrt{x}\right)$.

j) $\lim_{x\to +\infty} \left[x - x^2 \ln\left(1 + \frac{1}{x}\right)\right]$.

Problem 6.18. Calculate the following limits by using either Taylor polynomials or the l'Hôpital's rule, as appropriate.

a) $\lim_{x\to 0^+} x\ln(e^{x} - 1)$.

b) $\lim_{x\to +\infty} \frac{e^{x} - \arctan(x)}{\ln(1+x)}$.

c) $\lim_{x\to 0^+} x^{x}$.

d) $\lim_{x\to +\infty} \frac{\ln(x)}{x^{a}}$, $a > 0$.

e) $\lim_{x\to +\infty} \frac{x}{e^{ax}}$, $a > 0$.

f) $\lim_{x\to +\infty} x^{1/x}$ (use the change of variable $t = 1/x$) .

g) $\lim_{x\to 0} (1+x)^{1/x}$.