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CALCULUS – Integration: fundamental theorems & techniques

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Problem 8.1. Calculate the area of the planar region between the graph of the function $f(x) = \sin(x)$, the horizontal axis, and the lines with equations $x = \pi/2$ and $x = \pi$.

Problem 8.2. Calculate the area of the planar region between the graph of the function $f(x) = x/(x+1)$, the horizontal axis, and the lines with equations $x = 0$ and $x = 2$.

Problem 8.3. Consider the function

$$f(x) = \begin{cases} \cos(x)\,, & \text{if } \ 0 \leq x \leq \pi/2\,, \\ -1\,, & \text{if } \ \pi/2 < x \leq \pi\,. \end{cases}$$

Calculate

$$F(x) = \int_0^x f(t)\,dt$$

for $x \in [0, \pi]$, and compare $F'(x)$ with $f(x)$ for $x \in (0, \pi)$, where $F'(x)$ exists.

Problem 8.4. Calculate an equation for the tangent line at $x = 1$ to the graph of

$$F(x) = \int_{-1}^x \frac{t^3}{t^4 - 4}\,dt\,.$$

Problem 8.5. Find the values of $x \in \mathbb{R}$ such that

$$F(x) = \int_1^x \arctan(e^t)\,dt$$

is one-to-one.

Problem 8.6. Calculate the following limits.

$$\text{a)}\ \lim_{x\to 0} \frac{1}{x}\int_0^x \frac{|\cos(t^3)|}{t^2+1}\,dt\,. \qquad \text{b)}\ \lim_{x\to \infty} \frac{1}{x}\int_0^x \frac{|\cos(t^3)|}{t^2+1}\,dt\,.$$

Problem 8.7. Calculate the first and second derivatives of the function

$$H(x) = x\int_{2x}^{3x} e^{-t^2}\,dt\,.$$

Problem 8.8. Prove that the function

$$H(x) = \int_{1-x}^{1+x} \ln(t)\,dt$$

is decreasing for $x \in [0, 1/2]$.

Problem 8.9. Find the global extrema of the function

$$H(x) = \int_{5-2x}^{1} e^{-t^4}\,dt$$

in the interval $[1, 3]$. Moreover, prove that the maximum value of $H(x)$ is larger than $2/3$.

Problem 8.10. Calculate the following limits.

$$\text{a)}\ \lim_{x\to 0^+} \frac{1}{x^{3/2}}\int_0^{x^2} \sin(t^{1/4})\,dt\,. \qquad \text{b)}\ \lim_{x\to 0} \frac{\int_0^x e^{t^2}\,dt - x}{x^3}\,.$$

Problem 8.11.

- Prove that
$$F(x) = \int_0^x \Big(1 + \sin\big(\sin(t)\big)\Big)\,dt$$
is one-to-one and $F(0) = 0$. Then, calculate $(F^{-1})'(0)$.

- Prove that

$$G(x) = \int_1^x \sin\big(\sin(t)\big)\,dt$$

is even, namely $G(x) = G(-x)$.

Problem 8.12. Write the Taylor polynomial of degree 3 about $a = 0$ for

$$F(x) = \int_0^x t^2 \cos(t^2)\,dt$$

and use it to calculate

$$\lim_{x\to 0} \frac{F(x)}{x^3}.$$

Problem 8.13. Calculate the first derivative of the following functions.

$$a)\quad H(x) = \int_3^{\left(\int_1^x \sin^3(t)\,dt\right)} \frac{dt}{1 + t^2 + \sin^6(t)}.$$

$$b)\quad K(x) = \sin\left(\int_0^x \sin\left(\int_0^t \sin^3(s)\,ds\right)\,dt\right).$$

Problem 8.14. Calculate the given indefinite integrals.

$$\int \arctan(3x)\,dx.$$

$$\int e^x \sin(x)\,dx.$$

$$\int \cos(\ln(x))\,dx.$$

$$\int \cos^2(\ln(x))\,dx.$$

$$\int \frac{dx}{\sqrt{e^x - 4}}.$$

$$\int \frac{dx}{x\sqrt{x^2 - 1}}.$$

Problem 8.15. Calculate the given integrals using suitable changes of variable.

$$\int_1^2 \frac{\sqrt{t^2-1}}{t}\,dt\,.$$
$$\int_0^{\ln(2)} \sqrt{e^t-1}\,dt\,.$$
$$\int \frac{dx}{(x+2)\sqrt{1+x}}\,.$$
$$\int \frac{dx}{1+\sqrt[3]{1-x}}\,.$$

Problem 8.16. Calculate the following integrals of rational functions.

$$\int \frac{dx}{3x^2+4x+2}\,.$$
$$\int \frac{x^5-2x^3}{x^4-2x^2+1}\,dx\,.$$
$$\int \frac{x^2+1}{x^4-x^2}\,dx\,.$$
$$\int \frac{x^3+1}{x^2+4x+13}\,dx\,.$$
$$\int \frac{x^2+6x-1}{x^3-7x^2+15x-9}\,dx\,.$$

Problem 8.17. Calculate the given integrals using the indicated hints.

a) $\displaystyle\int \cos^3(x)\,dx$ (change of variable $u = \sin(x)$).

b) $\displaystyle\int \sin^4(x)\,dx$ (identities $\cos(2\alpha) = 1-2\sin^2(\alpha) = 2\cos^2(\alpha)-1$).

c) $\displaystyle\int \frac{e^{4x}}{e^{2x}+2e^x+2}\,dx$ (change of variable $u = e^x$).

d) $\displaystyle\int \frac{\sin^3(x)}{1+\cos^2(x)}\,dx$ (change of variable $u = \cos(x)$).

e) $\displaystyle\int \sqrt{a^2-x^2}\,dx\,,\ a \in \mathbb{R}$ (change of variable $x = a\sin(u)$).