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CALCULUS – Solutions to proposed problems

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1. Sets of numbers

Problem 1.1.

1) $x \in (-\infty, -2) \cup (0, 1) \cup (1, +\infty)$.

2) $x \in [0, 25]$.

3) $\forall\, x \in \mathbb{R}$.

4) $x \in [-5, 11]$.

5) $x \in \left(\dfrac{3}{2}, 2\right) \cup \left(2, \dfrac{5}{2}\right)$.

6) $x \in (-\infty, 2] \cup [3, +\infty)$.

7) $x \in (-3, 0) \cup (5, +\infty)$.

8) $x \in (-7, -4) \cup (-1, +\infty)$.

9) $x \in (-\infty, 1) \cup (2, +\infty)$.

10) $x = \dfrac{-1 \pm \sqrt{21}}{2}$.

Problem 1.2.

1) $\sup(A_1) = 1$, $\inf(A_1) = 0$, $\max(A_1) = 1$, $\nexists \ \min(A_1)$.

2) $\sup(A_2) = 1$, $\inf(A_2) = -1$, $\max(A_2) = 1$, $\min(A_2) = -1$.

3) $\sup(A_3) = \sqrt{2}$, $\inf(A_3) = 0$, $\nexists \ \max(A_3)$, $\min(A_3) = 0$.

4) $\nexists\ \sup(A_4)\,,\ \nexists\ \inf(A_4)\,,\ \nexists\ \max(A_4)\,,\ \nexists\ \min(A_4)\,.$

5) $\sup(A_5) = \dfrac{\sqrt{5}-1}{2}\,,\ \inf(A_5) = \dfrac{-\sqrt{5}-1}{2}\,,\ \nexists\ \max(A_5)\,,\ \nexists\ \min(A_5)\,.$

6) $\sup(A_6) = 0\,,\ \inf(A_6) = \dfrac{-\sqrt{5}-1}{2}\,,\ \nexists\ \max(A_6)\,,\ \nexists\ \min(A_6)\,.$

7) $\sup(A_7) = \dfrac{3}{2}\,,\ \inf(A_7) = -1\,,\ \max(A_7) = \dfrac{3}{2}\,,\ \nexists\ \min(A_7)\,.$

8) $\sup(A_8) = 3\,,\ \inf(A_8) = \dfrac{1}{3}\,,\ \nexists\ \max(A_8)\,,\ \nexists\ \min(A_8)\,.$

9) $\sup(A_9) = d\,,\ \inf(A_9) = a\,,\ \nexists\ \max(A_9)\,,\ \nexists\ \min(A_9)\,.$

10) $\sup(A_{10}) = \dfrac{7}{10}\,,\ \inf(A_{10}) = 0\,,\ \max(A_{10}) = \dfrac{7}{10}\,,\ \nexists\ \min(A_{10})\,.$

Problem 1.3.

1) Proof by contradiction.

2) Principle of induction.

3) Principle of induction.

4) Prove each inequality in the right-hand-side separately.

5) Find the values of x and y satisfying the assumption $0 < x < y$ such that the inequality in the right-hand-side holds.

6) In order to prove $\Longrightarrow$, calculate the square of both sides of the identity in the left-hand-side; in order to prove $\Longleftarrow$, consider three different cases, namely (i) $x = 0$ or $y = 0$, (ii) $x > 0$ and $y > 0$, (iii) $x < 0$ and $y < 0$.

2. Sequences and series of real numbers

Problem 2.1.

a) Bounded; not monotone; divergent.

b) Bounded; not monotone; convergent to 0 (for instance, use the properties of limits or apply the sandwich theorem).

c) Bounded; monotone; convergent to 1.

d) Bounded; not monotone; convergent to $1/2$.

e) Bounded; not monotone for any $x \in \mathbb{R}$; convergent to x (for instance, apply the sandwich theorem).

f) Bounded; not monotone; convergent to $1/2$ (for instance, apply the sandwich theorem).

g) Bounded; monotone; convergent to π.

h) Bounded; monotone for $n \geq 2$; convergent to $1/2$ (for instance, consider the formula for the sum of the first n natural numbers).

Problem 2.2.

a) The sequence converges to 0.

b) The sequence converges to 0 (for instance, apply the sandwich theorem).

c) The sequence diverges.

d) The sequence converges to 0.

e) The sequence converges to $1/3$.

Problem 2.3.

a) The sequence can be written as $a_n = \sqrt{3\, a_{n-1}}\,, \ \forall n \geq 2\,,$ with $a_1 = \sqrt{3}\,$. Use the principle of induction to prove that $(a_n)_{n\in\mathbb{N}}$ is bounded. Moreover, the sequence is increasing and converges to 3.

b) Use the principle of induction to prove that $(a_n)_{n\in\mathbb{N}}$ is bounded. In addition, the sequence is increasing and converges to 20/3.

c) Use the principle of induction to prove that $(a_n)_{n\in\mathbb{N}}$ is bounded. Moreover, the sequence is decreasing and converges to 1/3.

d) Use the principle of induction to prove that $(a_n)_{n\in\mathbb{N}}$ is bounded. In addition, the sequence is increasing and converges to 3.

Problem 2.4.

a) The limit is 1 (note that $\sqrt[n]{n} \longrightarrow 1$ as $n \to \infty$).

b) The limit is 1 (note that $\sqrt[n]{n} \longrightarrow 1$ as $n \to \infty$).

c) The limit is $e^{1/3}$ (note that $(1 + a_n)^{1/a_n} \longrightarrow e$ as $n \to \infty$, if $a_n \longrightarrow 0$).

Problem 2.5.

a) Convergent *telescoping* series (its sum is equal to 1).

b) Convergent (by the comparison test with $\sum_{k=1}^{\infty} b_k,\ b_k = 5/k^2$).

c) Divergent (by the comparison test with $\sum_{k=1}^{\infty} a_k,\ a_k = 1/k$).

d) Convergent (by the limit comparison test with $\sum_{k=1}^{\infty} b_k,\ b_k = 1/k^{3/2}$).

e) Convergent (by the limit comparison test with $\sum_{k=1}^{\infty} b_k,\ b_k = 1/k^2$).

f) Convergent (by the limit comparison test with $\sum_{k=1}^{\infty} b_k,\ b_k = (2/3)^k$).

g) Convergent (by the limit comparison test with $\sum_{k=1}^{\infty} b_k,\ b_k = 1/k^3$).

h) Divergent (by the comparison test with $\sum_{k=1}^{\infty} a_k,\ a_k = 1/k$).

i) Convergent (by the limit comparison test with $\sum_{k=1}^{\infty} b_k,\ b_k = 1/k^{3/2}$).

j) Divergent (by the comparison test with $\sum_{k=1}^{\infty} a_k,\ a_k = 1/k$).

Problem 2.6.

a) *Alternating* series: convergent by the *Leibniz test*.

b) Convergent (consider the series with general term given by $|\cos(k)|/5^k$, then use the comparison test with $\sum_{k=1}^{\infty} b_k,\ b_k = (1/5)^k$).

c) *Alternating* series: convergent by the *Leibniz test*.

d) Convergent (by the *ratio test*).

e) Divergent (by the *root test*).

f) Convergent (by the *root test*).

g) Convergent (by the *ratio test*).

h) Divergent *telescoping* series.

Problem 2.7.

a) Convergent for $|b| > 1$ and $a > 0$, divergent for $|b| < 1$ and $a > 0$ (use the *ratio test*). In addition, the series is divergent for $b = \pm 1$ $(a > 0)$ since the general term does not tend to zero, as $k \to \infty$.

b) Convergent for all $b \in \mathbb{R}$ (by the *ratio test*).

c) Convergent for $|\alpha| < \sqrt[3]{7}/2$ and divergent for $|\alpha| > \sqrt[3]{7}/2$ (use the *ratio test*). If $\alpha = \sqrt[3]{7}/2$, the series is convergent by the *Leibniz test*; if $\alpha = -\sqrt[3]{7}/2$, the series is divergent by the limit comparison test with $\sum_{k=1}^{\infty} b_k,\ b_k = 1/k^{2/3}$.

3. Real functions: limits and continuity

Problem 3.2.

1) The domain is $\mathbb{R}$, the image is $\mathbb{Z}$, and $f(x)$ is continuous in $\mathbb{R} \setminus \mathbb{Z}$.

2) The domain is $\mathbb{R}$, the image is $[0, 1)$, and $f(x)$ is continuous in $\mathbb{R} \setminus \mathbb{Z}$.

3) The domain is $\mathbb{R}$, the image is $[0, 1)$, and $f(x)$ is continuous in $\mathbb{R} \setminus \mathbb{Z}$.

4) The domain is $\mathbb{R}$, the image is $\mathbb{R}$, and $f(x)$ is continuous in $\mathbb{R}$.

5) The domain is $\mathbb{R}\setminus\{0\}$, the image is $\mathbb{Z}$, and $f(x)$ is continuous in $\mathbb{R}\setminus\left\{0, \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \ldots\right\}$.

Problem 3.3.

1) The function $f(x)$ is continuous in $\mathbb{R}$.

2) The function $f(x)$ is continuous in $\mathbb{R}$.

3) The function $f(x)$ is continuous in $\mathbb{R} \setminus \{0\}$.

4) The function $f(x)$ is continuous in $[-1, 1]$.

5) The function $f(x)$ is continuous in $(4/9, +\infty)$.

6) The function $f(x)$ is continuous in $(4/9, 1]$.

Problem 3.4.

- The function $f(x)$ is continuous in $\mathbb{R} \setminus \{0\}$ since it is the product of continuous functions ($\cos(1/x)$ is a composition of continuous functions). At $x = 0$, the function $f(x)$ is also continuous as $\lim_{x\to 0} f(x) = f(0) = 0$.
- The function $g(x)$ is continuous in $(-\infty, 0) \cup (0, 1) \cup (1, +\infty)$ since defined in terms of continuous functions. At $x = 1$, the function $g(x)$ is also continuous as $\lim_{x\to 1^+} g(x) = \lim_{x\to 1^-} g(x) = g(1) = 0$. At $x = 0$, the function $g(x)$ is not continuous as $\lim_{x\to 0^+} g(x) = 0$ and $\lim_{x\to 0^-} g(x) = 1$, namely the limit of $g(x)$ when $x \to 0$ does not exist.
- The function $h(x)$ is continuous in $\mathbb{R} \setminus \{0\}$ since it is the sum of continuous functions. At $x = 0$, the function $h(x)$ is not continuous as $\lim_{x\to 0^+} h(x) = 1$ and $\lim_{x\to 0^-} h(x) = 5$, namely the limit of $h(x)$ when $x \to 0$ does not exist.

Problem 3.5. In order to prove that the function $f(x)$ is bounded in the interval $[-7, 5]$ (closed and bounded), we can prove that it is continuous. Thus, $f(x)$ is continuous in $[-7, 0) \cup (0, 5]$ since it is the composition of continuous functions. At $x = 0$, $f(x)$ is also continuous as $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0) = 0$. Finally, we can conclude that $f(x)$ is continuous, hence bounded in $[-7, 5]$.

Problem 3.6. Apply *Bolzano Theorem* to the function $f(x) = \cos(x) - x$ in a proper interval, for instance in $[0, 1]$.

Problem 3.7.

1) Apply *Bolzano Theorem* to the function $F(x) = f(x) - x$ in the interval $[0, 1]$.
2) Apply *Bolzano Theorem* to the function $F(x) = f(x) - g(x)$ in the interval $[x_1, x_2]$.

4. Real functions: derivative

Problem 4.1. The function $f(x)$ is continuous at $x = 0$ as $\lim_{x\to 0} f(x) = f(0) = 0$ (this limit can be calculated by taking into account that $\sin(1/x)$ is bounded). In addition, $f(x)$ is differentiable at $x = 0$ as $\lim_{h\to 0} [f(0 + h) - f(0)]/h$ exists and is equal to 0.

Problem 4.2. The function $f(x)$ is continuous in $[-2, -1]$ and differentiable in $(-2, -1)$.

Problem 4.3.

1) $f'(x) = \dfrac{6x-7}{2\sqrt{3x^2-7x-2}}\,.$

2) $f'(x) = x\sin(x)\left(2\tan(x) + x + \dfrac{x}{\cos^2(x)}\right).$

3) $f'(x) = \dfrac{2}{3(x-1)^{2/3}(x+1)^{4/3}}\,.$

4) $f'(x) = \dfrac{-\sin(x)\,\cos\left(\sqrt{1+\cos(x)}\right)}{2\sqrt{1+\cos(x)}}\,.$

5) $f'(x) = \dfrac{2}{x} + \dfrac{1}{\tan(x)} - \dfrac{1}{2x+2}\,.$

Problem 4.4. An equation for the desired *tangent line* is $y = -2x + 7$.

Problem 4.5.

- For $x \neq 0$, $f(x)$ is differentiable and $f'(x) = 1/(3x^{2/3})$. However, at $x = 0$, $f(x)$ is not differentiable.
- For $x \neq 0$, $f(x)$ is differentiable and $f'(x) = 1/x$. However, at $x = 0$, $f(x)$ is not differentiable.

Problem 4.6. The function $f(x)$ is differentiable in $(-\infty, 0) \cup (0, 1) \cup (1, +\infty)$ since it is defined in terms of differentiable elementary functions (in addition, $f'(x)$ is obtained by deriving the corresponding function in each interval). At $x = 0$, we can write $\lim_{h\to 0^+} [f(0+h) - f(0)]/h = \lim_{h\to 0^-} [f(0+h) - f(0)]/h = 0$, hence $f(x)$ is differentiable and $f'(0) = 0$. At $x = 1$, the function $f(x)$ is not continuous (hence not differentiable) as $\lim_{x\to 1^+} f(x) = \pi/4$ and $\lim_{x\to 1^-} f(x) = 0$, namely the limit of $f(x)$ as $x \to 1$ does not exist.

Problem 4.7. The following expressions are obtained by means of the *chain rule*.

1) $h'(x) = f'(g(x))\,g'(x)\,e^{f(x)} + f(g(x))\,f'(x)\,e^{f(x)}\,.$

2) $h'(x) = \dfrac{-f'(x) - 2g(x)g'(x)}{(f(x) + g^2(x))\ln^2(f(x) + g^2(x))}\,.$

3) $h'(x) = \dfrac{f(x)f'(x) + g(x)g'(x)}{\sqrt{f^2(x) + g^2(x)}}\,.$

4) $h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{f^2(x) + g^2(x)}$.

5) $h'(x) = \dfrac{g'(x)}{g(x)} - f'(x)\tan(f(x))$.

Problem 4.8. In each case, calculate the needed derivatives of $f(x)$ and substitute them in the left-hand-side of the given *differential* equation.

Problem 4.9. In each case, calculate the first derivative of the function in the left-hand-side of the identity and check that it is equal to zero for all considered values of x. As a consequence, the involved function must be constant for those values of x. Hence, in order to prove the identity, just evaluate the function at some of the indicated values of x.

Problem 4.10. The slope of the tangent line from the right can be calculated as $\lim_{h\to 0^+} [f(0+h) - f(0)]/h = 0$, hence such line is parallel to the x-axis. On the other hand, the slope of the tangent line from the left can be calculated as $\lim_{h\to 0^-} [f(0+h) - f(0)]/h = 1$, hence such line is parallel to the line $y = x$. As a consequence, the two tangent lines form an angle equal to $\pi/4$.

Problem 4.11.

- For $x \neq 0$, we have $f_1'(x) = k\,x\,|x|^{k-2}$ and $f_2'(x) = k\,|x|^{k-1}$.
- Using the definition of differentiability, we get $f_1'(0) = f_2'(0) = 0$.
- From $0 \leq |f(x)| \leq |x|^k$ for $x = 0$, we deduce that $f(0) = 0$. In addition, we have $0 \leq |f(x)\,/\,x| \leq |x|^{k-1}$ for all $x \neq 0$ in a neighborhood of $x_0 = 0$, which implies that $f'(0) = \lim_{x\to 0} f(x)\,/\,x = 0$ (apply the sandwich theorem, recalling that $k > 1$).

Problem 4.12. The function $f(x)$ is continuous in $\mathbb{R}$, with $f(1) = 1$. In addition, it is differentiable in $\mathbb{R}$, with $f'(1) = -1$. On the other hand, in the interval $[0, 2]$, all hypotheses of the *Lagrange mean-value Theorem* are satisfied and the points of the theorem statement (namely, points $c \in (0, 2)$ such that $f'(c) = [f(2) - f(0)]/2$) are $c = 1/2,\ \sqrt{2}$.

Problem 4.13. The function $f(x)$ verifies all hypotheses of *Rolle Theorem*, except for the differentiability of $f(x)$ in the interval $(-1, 1)$. Indeed, $f'(0)$ does not exist. Hence, such theorem cannot be applied in this case.

Problem 4.14. The desired values are $h(0) = 0$, $h'(0) = 0$, $h''(0) = 2$. All of them are obtained by observing that $\lim_{x\to 0} h(x)/x^2 = f(0) = 1$ (since $f(x)$ is continuous at $x = 0$) and using this limit in the definitions of continuity and (first and second) differentiability of $h(x)$ at $x = 0$.

Problem 4.15. Since $f(x)$ is continuous at $x = 0$, we have that $\lim_{x\to 0} f(2x^3) = f(0)$. Hence, as the given limit is finite, $f(0) = 0$. The value $f'(0) = 5/2$ can be obtained by using the same limit in the definition of differentiability of $f(x)$ at $x = 0$. On the other hand, we can write

$$\lim_{x\to 0} \frac{f(f(2x))}{3f^{-1}(x)} = \frac{2}{3} \lim_{x\to 0} \frac{f(f(2x))}{f(2x)} \lim_{x\to 0} \frac{f(2x)}{2x} \lim_{x\to 0} \frac{x}{f^{-1}(x)},$$

where each limit in the right-hand-side can be transformed (by means of a change of variable) into $\lim_{t\to 0} f(t)/t = f'(0) = 5/2$. The value of the desired limit is then $125/12$.

Problem 4.16.

THEOREM 1. Suppose that $f(x)$ vanishes at $k \geq 2$ points, say $\bar{x}_1, \dots, \bar{x}_k$, in the interval $[x_1, x_2]$. Hence, we have $k - 1$ subintervals $[\bar{x}_1, \bar{x}_2], [\bar{x}_2, \bar{x}_3], \dots, [\bar{x}_{k-1}, \bar{x}_k]$ in which $f(x)$ satisfies the assumptions of *Rolle Theorem*. Thus, there are at least $k - 1$ points in $[x_1, x_2]$ (at least one in each subinterval) where $f'(x)$ vanishes.

THEOREM 2. The statement can be proved by repeatedly applying THEOREM 1 to $f(x), f'(x), f''(x), \dots, f^{(k-1)}(x)$.

Problem 4.17. Write the given equations as $f(x) = 0$, where $f(x)$ is a function to be properly defined in each case. Then, study the sign of $f'(x)$ where needed and determine the *exact* number of roots of $f(x)$.

1) 1 real solution.
2) 1 real solution.
3) 2 real solutions.
4) 1 real solution.
5) No real solutions.

Problem 4.18.

- The value of the limit is $1/2$ (apply the l'Hôpital's rule twice).

- The value of the limit is 1 (apply the l'Hôpital's rule twice).

5. The Newton-Raphson method

Problem 5.1. Let N be a natural number and $f(x) = x^2 - N$, where $f'(x) = 2x$. Then, consider the Newton-Raphson *recursive* sequence defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{x_n}{2} + \frac{N}{2x_n},$$

with $n = 0, 1, 2, \dots$. Given a proper initial guess x_0, the latter formula provides an effective means to approximate the square root of $N \in \mathbb{N}$.

Problem 5.2. The true root in the interval $[1, 2]$ is given by

$$x_{\text{true}} = 1.532088886237956070404785301 11 \dots .$$

If we define

$$x_{n+1} = x_n - \frac{x_n^3 - 3x_n + 1}{3x_n^2 - 3} = \frac{2x_n^3 - 1}{3x_n^2 - 3}, \quad \text{for } n = 0, 1, 2, \dots; \quad x_0 = 2,$$

then we get $x_1 = 1.66666666666666666666\dots$, $x_2 = 1.54861111111111111111\dots$, $x_3 = 1.53239016186538002148\dots$, $x_4 = 1.53208898939722427968\dots$, etc.

Problem 5.3. We can write

$$x_{n+1} = x_n + \frac{\cos(x_n)}{\sin(x_n)},$$

for $n = 0, 1, 2, \dots$. For instance, if $x_0 = 1$, we get $x_1 = 1.64209\dots$, $x_2 = 1.57067\dots$, $x_3 = 1.57079\dots$, etc. Clearly, the *recursive* sequence $(x_n)_{n\in\mathbb{N}}$ converges to $\pi/2$.

Problem 5.4.

(a) For $n = 0, 1, 2, \dots$, we get

$$x_{n+1} = x_n - 1 \implies x_{n+2} = x_n - 2 \implies x_{n+k} = x_n - k,$$

with $k \in \mathbb{N}$. Hence, the Newton-Raphson *recursive* sequence approaches $-\infty$.

(b) Note that
$$f(x) = x(x^2 + 2e^x) = 0 \iff x = 0\,.$$
In this case, the Newton-Raphson method yields the *recursive* formula
$$x_{n+1} = x_n - \frac{x_n^3 + 2x_n e^{x_n}}{3x_n^2 + 2(x_n+1)e^{x_n}}\,,$$
for $n = 0, 1, 2, \ldots$. Thus, taking into account the initial guess $x_0 = 1$, we get the values $x_1 = 0.536041$, $x_2 = 0.21108$, $x_3 = 0.0412345$, $x_4 = 0.00169358$, $x_5 = 2.86819 \cdot 10^{-6}$, $x_6 = 8.22654 \cdot 10^{-12}$, etc.

6. Taylor polynomial

Problem 6.1.

- We get $\sin(1) \approx 0.8415$ by using the Maclaurin polynomial of degree 7 for the function $\sin(x)$ at $x = 1$.
- We get $\sqrt[5]{\frac{3}{2}} \approx 1.08$ by using the Maclaurin polynomial of degree 2 for the function $(1+x)^{1/5}$ at $x = 1/2$.

Problem 6.2.

a) $P_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3\,.$

b) $P_n(x) = 3x^2 - \frac{9}{2}x^6 + \frac{81}{40}x^{10} + \ldots + (-1)^k \frac{3^{2k+1}}{(2k+1)!}\,x^{4k+2}\,, \quad n = 4k+2\,.$

c) $P_5(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5\,.$

d) $P_3(x) = 1 - \frac{3}{2}x^2$ (the coefficient of x^3 is equal to zero).

e) $P_n(x) = 4 + 4x + 3x^2 + \frac{5}{3}x^3 + \ldots + \frac{2^n+2}{n!}\,x^n\,.$

Problem 6.3. The indicated polynomial can be (exactly) expressed by its Taylor polynomial of degree 4 about $a = 4$, namely

$$x^4 - 5x^3 + x^2 - 3x + 4 \;=\; -56 + 21(x-4) + 37(x-4)^2 + 11(x-4)^3 + (x-4)^4 \,.$$

Problem 6.4. Using the principle of induction, we can prove that

$$f(x) \;=\; -1 - (x+1) - (x+1)^2 - \ldots - (x+1)^n + \frac{1}{c}\left(-\frac{x+1}{c}\right)^{n+1},$$

where the last term is the Taylor remainder, with $c \in (-1, x)$ or $(x, -1)$.

Problem 6.5. $P_5(x) \;=\; x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 \,.$

Problem 6.6. The desired coefficient is $\frac{f^{(4)}(0)}{4!} \;=\; -\frac{1}{12}\,.$

Problem 6.7.

- $P_3(x) \;=\; 2x - \frac{4}{3}x^3 \,.$
- $P_3(x) \;=\; 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 \,.$
- $P_3(x) \;=\; x - x^2 + \frac{1}{2}x^3 \,.$
- $P_3(x) \;=\; -x - \frac{3}{2}x^2 - \frac{4}{3}x^3 \,.$
- $P_3(x) \;=\; x^2$ (the coefficient of x^3 is equal to zero).
- $P_3(x) \;=\; x - x^2 + \frac{11}{6}x^3 \,.$

Problem 6.8.

- $P_n(x) \;=\; 1 - \frac{a^2}{2!}x^2 + \frac{a^4}{4!}x^4 - \frac{a^6}{6!}x^6 + \ldots + (-1)^k \frac{a^{2k}}{(2k)!}x^{2k}, \;\; n = 2k \,.$
- $P_n(x) \;=\; ax + \frac{a^3}{3!}x^3 + \frac{a^5}{5!}x^5 + \ldots + \frac{a^{2k+1}}{(2k+1)!}x^{2k+1}, \;\; n = 2k+1 \,.$
- $P_n(x) \;=\; 1 + ax^2 + \frac{a^2}{2!}x^4 + \frac{a^3}{3!}x^6 + \ldots + \frac{a^k}{k!}x^{2k}, \;\; n = 2k \,.$

- $P_n(x) = 1 + 2x + 2x^2 + \ldots + 2x^n$.

Problem 6.9.

- An equation for the desired tangent line is $y = 0$.
- The value of the limit is 2.
- $f^{(4)}(1) = -72$.

Problem 6.10. For each case, prove that the indicated limit of the function in the left-hand-side divided by the power of x in $o(\cdot)$ is equal to zero (use suitable Taylor polynomials in the first three cases and the l'Hôpital's rule in the last case).

Problem 6.11. We can choose $P(x)$ in the *family* of polynomials of the form

$$P(x) = 1 - \frac{x^4}{2} + a_8 x^8 + a_9 x^9 + \ldots + a_n x^n,$$

where $a_8, \ldots, a_n$ are arbitrary real coefficients (with $n \in \mathbb{N}$).

Problem 6.12. The desired polynomial is $P_3(x) = 2 + x + x^3/6$. Moreover, the involved approximation error can be *upper bounded* as

$$|R_3(x)| = \left| \frac{\cos(c) + e^c}{4!} x^4 \right| \leq \frac{1 + e^{1/4}}{4!} \left(\frac{1}{4} \right)^4,$$

where $c \in (-1/4, 1/4)$.

Problem 6.13. We need to consider a Maclaurin polynomial of degree $n \geq 7$.

Problem 6.14. We get $1/\sqrt{1.1} \approx 0.9534375$ by using the Maclaurin polynomial of degree 3 for $f(x) = (1+x)^{-1/2}$ at $x = 0.1$. An *upper bound* for the involved error is

$$\frac{35\,(0.1)^4}{2^7} \approx 0.000027.$$

Problem 6.15.

- We get an approximation of $\cos(1)$ by considering a Maclaurin polynomial of degree $n \geq 6$ for $\cos(x)$ at $x = 1$.
- We get an approximation of e^{-2} by using a Maclaurin polynomial of degree $n \geq 9$ for e^x at $x = -2$.

- We get an approximation of $\ln(2)$ by using a Maclaurin polynomial of degree $n \geq 1000$ for $\ln(1+x)$ at $x = 1$.

Problem 6.16. At least all terms up to $-x^{11}/11!$ (included) must be considered, with $x = 1/2$.

Problem 6.17. The values of the requested limits are the following.

a) $1/2$.

b) $1/120$.

c) $1/2$.

d) $1/2$.

e) $1/27$.

f) $1/6$.

g) 0.

h) $1/3$.

i) $-1/4$ (apply the change of variable $t = 1/x$).

j) $1/2$ (apply the change of variable $t = 1/x$).

Problem 6.18. The values of the given limits are the following.

a) 0 (apply the l'Hôpital's rule).

b) $+\infty$ (apply the l'Hôpital's rule).

c) 1 (apply the l'Hôpital's rule).

d) 0 (apply the l'Hôpital's rule).

e) 0 (apply the l'Hôpital's rule).

f) 1 (apply the l'Hôpital's rule).

g) e (consider a suitable Taylor polynomial).

7. Local & global behavior of a real function

Problem 7.1.

- $x = 2$ is a point of local minimum, $x = -1$ is a point of local maximum.
- No local extrema.
- $x = 0$ is a point of local minimum, $x = 1$ is a point of local maximum.

Problem 7.2.

- The function $f(x)$ is (strictly) increasing in $(0, 3) \cup (4, +\infty)$ and decreasing in $(-\infty, 0) \cup (3, 4)$.
- $x = 0, 4$ are points of local minima, while $x = 3$ is a point of local maximum.
- The solution is unique since $f(x)$ is strictly increasing in the interval $(0, 1)$ and $f(0) < 0$, $f(1) > 0$.

Problem 7.3. The point $x = 0$ is an inflection point and $f(x)$ is *locally* concave down/up at the left/right of it.

Problem 7.4.

- $f(x)$ is concave up in $(-2/5, 0) \cup (0, +\infty)$ and concave down in $(-\infty, -2/5)$; hence $x = -2/5$ is an inflection point.
- $f(x)$ is concave up in $(2, +\infty)$.
- $f(x)$ is concave up in $\mathbb{R}$.
- $f(x)$ is concave down in $(-\infty, 2) \cup (4, +\infty)$.

Problem 7.5. The point $x = 0$ is of *local* minimum and $f(x)$ is concave up in a neighborhood of it.

Problem 7.6.

- The function $f(x)$ is decreasing in $(-\infty, -1/2)$.
- The values $\alpha = 0$ and $\beta = 1$ make $f(x)$ differentiable at $x = 0$, hence in $\mathbb{R}$.

- The global minimum is $-5/4$ and is attained at $x = -1/2$. On the other hand, there is no global maximum.

Problem 7.7.

- The critical points are $x = 1$ (point of local minimum) and $x = 0$ (inflection point).
- $f(x)$ is increasing in $(1, +\infty)$ and decreasing in $(-\infty, 0) \cup (0, 1)$.
- The inflection points are $x = 0, 2/3$.
- $f(x)$ is concave down in $(0, 2/3)$ and concave up in $(-\infty, 0) \cup (2/3, +\infty)$.

Problem 7.8.

- The global minimum is $\dfrac{3\pi - 4}{4\sqrt{2}}$ and is attained at $x = \pm\dfrac{3}{4}\pi$, while the global maximum is $\dfrac{\pi + 4}{4\sqrt{2}}$ and is attained at $x = \pm\dfrac{\pi}{4}$.
- The global minimum is 0 and is attained at $x = 0$, while the global maximum is 7 and is attained at $x = 1$.

8. Integration: fundamental theorems & techniques

Problem 8.1. The area is equal to 1 .

Problem 8.2. The area is equal to $2 - \ln(3)$.

Problem 8.3. We get

$$F(x) \;=\; \begin{cases} \sin(x)\,, & \text{if} \;\; 0 \le x \le \pi/2\,, \\ 1 + \pi/2 - x\,, & \text{if} \;\; \pi/2 < x \le \pi\,. \end{cases}$$

In addition, thanks to the Fundamental Theorem of Calculus, we have $F'(x) = f(x)$ for all $x \in (0, \pi)$, with $x \neq \pi/2$ (indeed, $F(x)$ is not differentiable at $x = \pi/2$).

Problem 8.4. An equation for the desired tangent line is

$$y = F(1) + F'(1)(x-1) \;=\; -\frac{1}{3}\,x + \frac{1}{3}\,.$$

Problem 8.5. The function $F(x)$ is strictly increasing in $\mathbb{R}$ since $F'(x) > 0$. Thus, $F(x)$ is one-to-one for all $x \in \mathbb{R}$.

Problem 8.6.

a) The value of the limit is 1 (for instance, apply the l'Hôpital's rule).

b) The value of the limit is 0 (for instance, use the sandwich theorem).

Problem 8.7. We get

$$H'(x) \;=\; \int_{2x}^{3x} e^{-t^2}\,dt \;+\; x\,\left\{ 3e^{-9x^2} - 2e^{-4x^2} \right\},$$

$$H''(x) \;=\; 2\left\{ 3e^{-9x^2} - 2e^{-4x^2} \right\} \;+\; 2x^2\left\{ 8e^{-4x^2} - 27e^{-9x^2} \right\}.$$

Problem 8.8. The function $H(x)$ is decreasing in $[0, 1/2]$ as $H'(x) = \ln(1-x^2) < 0$ in that interval.

Problem 8.9. The global maximum is attained at $x = 3$ and the global minimum is attained at $x = 1$. In addition

$$H(3) \;=\; 2\int_0^1 e^{-t^4}\,dt \;>\; 2\int_0^1 e^{-1}\,dt \;=\; \frac{2}{e} \;>\; \frac{2}{3}\,.$$

Problem 8.10. Using suitable Taylor polynomials, we get the following values.

a) 0.

b) $1/3$.

Problem 8.11.

- We have $F'(x) = 1 + \sin(\sin(x)) > 0$ for all $x \in \mathbb{R}$. Thus, $F(x)$ is strictly increasing and one-to-one in $\mathbb{R}$. Then, a property of limits yields $F(0) = 0$, hence $F^{-1}(0) = 0$. As a consequence, we obtain

$$(F^{-1})'(0) = \frac{1}{F'(0)} = \frac{1}{1+\sin(\sin(0))} = 1\,.$$

- Note that

$$G(x) = \int_1^0 \sin(\sin(t))\,dt + \int_0^x \sin(\sin(t))\,dt\,,$$

where the first integral is independent of x (say, equal to $G_0 \in \mathbb{R}$) and the second one is a function $H(x)$ that is invariant under changing x with $-x$ (this can be proved by applying the change of variable $u = -t$). As a consequence, we have $G(-x) = G_0 + H(-x) = G_0 + H(x) = G(x)\,.$

Problem 8.12. The desired Taylor polynomial is $P_3(x) = x^3/3$ and the value of the limit is $1/3\,.$

Problem 8.13.

a) $$H'(x) = \sin^3(x)\left\{1 + \left(\int_1^x \sin^3(t)\,dt\right)^2 + \sin^6\left(\int_1^x \sin^3(t)\,dt\right)\right\}^{-1}.$$

b) $$K'(x) = \cos\left(\int_0^x \sin\left(\int_0^t \sin^3(s)\,ds\right)dt\right)\sin\left(\int_0^x \sin^3(s)\,ds\right).$$

Problem 8.14. In each case, the integral $I(x)$ has the indicated expression ($k \in \mathbb{R}$).

- $I(x) = x\arctan(3x) - \dfrac{1}{6}\ln(1+9x^2) + k$ (integration by parts).
- $I(x) = \dfrac{1}{2}e^x\big(\sin(x) - \cos(x)\big) + k$ (integration by parts).
- $I(x) = \dfrac{1}{2}x\big(\cos(\ln(x)) + \sin(\ln(x))\big) + k$ (change of variable $t = \ln(x)$ and integration by parts).
- $I(x) = \dfrac{x}{2} + \dfrac{x}{10}\cos(2\ln(x)) + \dfrac{x}{5}\sin(2\ln(x)) + k$ (change of variable $t = \ln(x)$, identity $\cos(2\alpha) = 2\cos^2(\alpha) - 1$, and integration by parts).

- $I(x) = \arctan\left(\frac{1}{2}\sqrt{e^x - 4}\right) + k$ (change of variable $t = \sqrt{e^x - 4}$).
- $I(x) = \arctan\left(\sqrt{x^2 - 1}\right) + k$ (change of variable $t = \sqrt{x^2 - 1}$).

Problem 8.15. In each case, the integral I or $I(x)$ has the given value or expression ($k \in \mathbb{R}$).

- $I = \sqrt{3} - \frac{\pi}{3}$ (change of variable $u = \sqrt{t^2 - 1}$).
- $I = 2 - \frac{\pi}{2}$ (change of variable $u = \sqrt{e^t - 1}$).
- $I(x) = 2\arctan\left(\sqrt{1 + x}\right) + k$ (change of variable $t = \sqrt{1 + x}$).
- $I(x) = -\frac{3}{2}(1 - x)^{2/3} + 3(1 - x)^{1/3} - 3\ln\left(|(1 - x)^{1/3} + 1|\right) + k$ (change of variable $t = (1 - x)^{1/3}$).

Problem 8.16. In each case, the integral $I(x)$ has the indicated expression ($k \in \mathbb{R}$).

- $I(x) = \frac{1}{\sqrt{2}} \arctan\left(\frac{3}{\sqrt{2}}x + \sqrt{2}\right) + k\,.$
- $I(x) = \frac{1}{4}\frac{1}{x - 1} - \frac{1}{4}\frac{1}{x + 1} + \frac{1}{2}x^2 + k\,.$
- $I(x) = \frac{1}{x} + \ln(|x - 1|) - \ln(|x + 1|) + k\,.$
- $I(x) = \frac{3}{2}\ln(x^2 + 4x + 13) + \frac{47}{3}\arctan\left(\frac{x + 2}{3}\right) + \frac{1}{2}x^2 - 4x + k\,.$
- $I(x) = \frac{3}{2}\ln(|x - 1|) - \frac{1}{2}\ln(|x - 3|) - \frac{13}{x - 3} + k\,.$

Problem 8.17. In each case, the integral $I(x)$ has the given expression ($k \in \mathbb{R}$).

a) $I(x) = \sin(x) - \frac{1}{3}\sin^3(x) + k\,.$

b) $I(x) = \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + k\,.$

c) $I(x) = \frac{1}{2}e^{2x} - 2e^x + \ln(e^{2x} + 2e^x + 2) + 2\arctan(e^x + 1) + k\,.$

d) $I(x) = \cos(x) - 2\arctan(\cos(x)) + k$.

e) $I(x) = \dfrac{a^2}{2}\arcsin\left(\dfrac{x}{a}\right) + \dfrac{x}{2}\sqrt{a^2 - x^2} + k$.

9. Improper integrals

Problem 9.1.

- Divergent (by using the definition of improper integral).
- Divergent (by the limit comparison test with $\int_1^{+\infty} dx/x$).
- Convergent (by the comparison test with $\int_1^{+\infty} dx/x^3$ to prove the absolute convergence).
- Convergent (by the limit comparison test with $\int_1^{+\infty} dx/x^{\alpha+1}$).
- Convergent (by the limit comparison test with $\int_1^{+\infty} dx/x^{3/2}$).
- Divergent (by the limit comparison test with $\int_2^7 dx/(x-2)$).
- It is not an improper integral.
- Convergent (by the limit comparison test with $\int_1^2 dx/(x-1)^{1/2}$).
- Divergent. We can write the given improper integral as

$$\int_1^2 \frac{x}{\sqrt{x^4-1}}\,dx + \int_2^{+\infty} \frac{x}{\sqrt{x^4-1}}\,dx,$$

  where the first improper integral converges (by the limit comparison test with $\int_1^2 dx/(x-1)^{1/2}$) and the second one diverges (by the limit comparison test with $\int_2^{+\infty} dx/x$).

- Divergent. We can write the given improper integral as

$$\int_0^{1/2} f(x)\,dx + \int_{1/2}^{1} f(x)\,dx + \int_1^{3/2} f(x)\,dx + \int_{3/2}^{+\infty} f(x)\,dx,$$

  where $f(x) = (1-\cos(x))/(x^3 \ln(x))$ and the third improper integral diverges (by the limit comparison test with $\int_1^{3/2} dx/(x-1)$).

Problem 9.2. We can prove that all given improper integrals converge by using, for instance, the following hints.

- Apply the definition of improper integral.
- Use the principle of induction and the definition of improper integral.
- Apply the definition of improper integral.
- Use the change of variable $t = \lambda x$ to reduce the improper integral to the case $\int_0^{+\infty} t^n e^{-t}\, dt$ and proceed as in the second item.

Problem 9.3. We can prove that all given improper integrals converge using, for instance, the following hints.

- Note that e^{-x^2} is even and the improper integral is equal to $2\int_0^{+\infty} e^{-x^2}\, dx$, which converges by the limit comparison test with $\int_0^{+\infty} e^{-x}\, dx$.
- If n is odd, then the improper integral is equal to zero. If n is even, then the improper integral is equal to $2\int_0^{+\infty} x^n e^{-x^2}\, dx$ and converges by the limit comparison test with $\int_0^{+\infty} x^n e^{-x}\, dx$, which is studied in Problem 9.2.
- Apply the change of variable $t = (x-3)/2$ and use the result in the first item.
- Apply the change of variable $t = (x-3)/2$ and use the result in the second item.
- Apply the change of variable $t = (x-\mu)/(\sqrt{2}\sigma)$ and use the result in the first item.
- Upon the change of variable $t = (x-\mu)/(\sqrt{2}\sigma)$, we get

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\left(\sqrt{2}\sigma t+\mu\right)^n e^{-t^2}\, dt \;=\; \frac{1}{\sqrt{\pi}}\sum_{k=0}^{n}\binom{n}{k}2^{k/2}\sigma^k\mu^{n-k}\int_{-\infty}^{+\infty} t^k\, e^{-t^2}\, dt\,,$$

  where the *Newton binomial formula* has been used. Thus, the given improper integral converges since $\int_{-\infty}^{+\infty} t^k e^{-t^2}\, dt$ converges (for $k = 0, 1, 2, \ldots$), as seen before.

Problem 9.4. We can prove that all given improper integrals converge using, for instance, the following hints.

- Apply the change of variable $t = (\ln(x)-\mu)/(\sqrt{2}\sigma)$ to reduce the improper integral to a case studied in Problem 9.3.

- Use the change of variable $t = (\ln(x) - \mu)/(\sqrt{2}\sigma)$ and the limit comparison test with $\int_{-\infty}^{+\infty} e^{-t}\,dt$.

Problem 9.5. We can prove that all given improper integrals converge using, for instance, the following hints.

- As in the next item, with $\alpha = \beta = 1/2$.
- The improper integral can be written as

$$\int_0^{1/2} x^{\alpha-1}(1-x)^{\beta-1}\,dx + \int_{1/2}^{1} x^{\alpha-1}(1-x)^{\beta-1}\,dx\,,$$

  where the first improper integral converges by the limit comparison test with $\int_0^{1/2} dx/x^{1-\alpha}$ and the second one converges by the limit comparison test with $\int_{1/2}^{1} dx/(1-x)^{1-\beta}$.

- As in the previous item, by the limit comparison test with $\int_0^{1/2} dx/x^{1-\alpha-n}$ for the first resulting integral and with $\int_{1/2}^{1} dx/(1-x)^{1-\beta}$ for the second one.

Problem 9.6. The improper integral can be written as

$$\int_0^1 x^{\frac{n_1}{2}-1}\left(1+\frac{n_1}{n_2}x\right)^{-\frac{n_1+n_2}{2}} dx + \int_1^{+\infty} x^{\frac{n_1}{2}-1}\left(1+\frac{n_1}{n_2}x\right)^{-\frac{n_1+n_2}{2}} dx\,,$$

where the first improper integral converges by the limit comparison test with $\int_0^1 dx/x^{1-n_1/2}$ and the second improper integral converges by the limit comparison test with $\int_1^{+\infty} dx/x^{1+n_2/2}$.