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CALCULUS – EVALUATION TEST 10 (solutions)

Filippo Terragni, Eduardo Sánchez Villaseñor, Manuel Carretero Cerrajero

Problem 1. Consider the *recursive* sequence $(a_n)_{n\in\mathbb{N}}$ defined as

$$a_1 = 1\,; \quad a_{n+1} = 3 - \frac{1}{a_n}\,, \quad \text{with } n = 1, 2, \ldots\,.$$

(a) Prove that the sequence is increasing and bounded above by 3.

(b) Calculate $\lim\limits_{n\to\infty} a_n$.

SOLUTION

(a) Let us prove that the sequence is increasing, namely $a_{n+1} > a_n$ for all $n \in \mathbb{N}$, by applying the *principle of induction*. First, $a_2 = 2 > a_1 = 1$. Now, assuming that $a_{k+1} > a_k$ for $n = k \in \mathbb{N}$, we get $-1/a_{k+1} > -1/a_k$. Then, we obtain (for $n = k+1$)

$$a_{k+2} = 3 - \frac{1}{a_{k+1}} > 3 - \frac{1}{a_k} = a_{k+1}\,.$$

By the same principle, we can prove that $a_n < 3$ for all $n \in \mathbb{N}$. First, $a_1 = 1 < 3$. Moreover, assuming that $a_k < 3$ for $n = k \in \mathbb{N}$, we get $-1/a_k < -1/3$. Then, we can write (for $n = k+1$)

$$a_{k+1} = 3 - \frac{1}{a_k} < 3 - \frac{1}{3} = \frac{8}{3} < 3\,.$$

(b) Let $\lim_{n\to\infty} a_n = L \in \mathbb{R}$, which exists since the sequence is increasing and bounded above. Thus

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(3 - \frac{1}{a_n}\right) \implies L = 3 - \frac{1}{L} \implies L = \frac{3}{2} \pm \frac{\sqrt{5}}{2}.$$

Due to the behavior of the sequence, the desired limit value is $L = \dfrac{3}{2} + \dfrac{\sqrt{5}}{2}$.

Problem 2. Analyze the convergence of the series $\sum_{n=1}^{\infty} \frac{3\ \ln(n^2)}{(n+1)!}$.

SOLUTION

Let a_n be the general term of the series. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3\ \ln((n+1)^2)}{(n+2)!} \frac{(n+1)!}{3\ \ln(n^2)}\right| = \frac{\ln(n+1)}{(n+2)\ln(n)} \longrightarrow 0$$

as $n \to \infty$. Thus, thanks to the *ratio test*, the series converges.

Problem 3. Let $f(x) = \sin(x)$.

(a) Use the Taylor polynomial of degree 2 about $a = \pi/2$ for $f(x)$ to approximate $\sin(\pi/2 + 0.1)$ and find an *upper bound* for the involved error.

(b) Consider the Taylor polynomial of degree $n \in \mathbb{N}$ about $a = \pi/2$ for $f(x)$ and apply the change of variable $s = x - \pi/2$. Then, do you recognize the resulting Taylor polynomial?

SOLUTION

(a) We can write

$$\sin(x) = 1 - \frac{(x - \frac{\pi}{2})^2}{2} + R_2(x)\,,$$

which yields the approximation

$$\sin(\pi/2 + 0.1) \approx 1 - \frac{(0.1)^2}{2} = 0.995\,.$$

Furthermore, the involved error can be upper bounded by using the remainder $R_2(x)$ as

$$|R_2(\pi/2 + 0.1)| \leq \frac{(0.1)^4}{24} \approx 4 \times 10^{-6}\,.$$

(b) By means of the suggested change of variable, we get

$$\sin(s + \pi/2) \approx 1 - \frac{s^2}{2!} + \frac{s^4}{4!} + \ldots + (-1)^n \frac{s^{2n}}{(2n)!}\,,$$

whose right-hand-side is the Taylor polynomial of degree $n \in \mathbb{N}$ about $a = 0$ for the function $\cos(s)$.

Problem 4. Calculate the exact number of real solutions of the equation

$$\arctan(x) - \frac{1}{2}\ln(1 + x^2) + \alpha = 0\,,$$

depending on the value of the parameter $\alpha \in \mathbb{R}$.

SOLUTION

Let $f_\alpha(x) = \arctan(x) - \frac{1}{2}\ln(1 + x^2) + \alpha$, which is continuous and differentiable in $\mathbb{R}$. Its first derivative is

$$f'_\alpha(x) = \frac{1}{1 + x^2} - \frac{1}{2}\frac{2x}{1 + x^2} = \frac{1 - x}{1 + x^2}\,.$$

Hence, independently of α, $f_\alpha(x)$ is increasing in the interval $(-\infty, 1)$ (positive derivative) and decreasing in the interval $(1, +\infty)$ (negative derivative). In addition, we have

- $\lim_{x\to\pm\infty} f_\alpha(x) = -\infty\,;$
- $f_\alpha(1) = \arctan(1) - \frac{\ln(2)}{2} + \alpha = \alpha + \frac{\pi}{4} - \frac{\ln(2)}{2}\,.$

Thus, the exact number of real solutions of the equation $f_\alpha(x) = 0$ depends on the value of $f_\alpha(1)$. Specifically, if $\alpha > \frac{\ln(2)}{2} - \frac{\pi}{4}$ ($f_\alpha(1) > 0$) there are two solutions, if $\alpha < \frac{\ln(2)}{2} - \frac{\pi}{4}$ ($f_\alpha(1) < 0$) there is no solution, if $\alpha = \frac{\ln(2)}{2} - \frac{\pi}{4}$ ($f_\alpha(1) = 0$) there is a unique solution.