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CALCULUS – EVALUATION TEST 11 (solutions)

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Problem 1. The paradox of Achilles and the tortoise

Achilles and a tortoise are in a footrace. Trusting his superiority, Achilles allows the tortoise a head start. However, when he reaches the position where the tortoise started from, the latter has run some distance. Thus, Achilles keeps on running and gets to that new position, but he soon realizes that the tortoise has advanced further.

Repeating this reasoning infinite times, it seems that Achilles will never reach the tortoise. Nevertheless, Achilles does catch up with his competitor if the total time spent during the infinite steps is a finite value.

Let us suppose that the tortoise starts a distance L ahead, the velocity of Achilles is v_A, and the velocity of the tortoise is v_T (with $L, v_A, v_T \in \mathbb{R}$). Hence, the total time t spent in the infinite steps is

$$t = \frac{L}{v_A}\left(1 + r + r^2 + r^3 + r^4 + \dots\right), \quad \text{with } r = \frac{v_T}{v_A}.$$

- Which type of series do you get?
- Discuss the convergence of the series depending on the values of v_A and v_T.
- In the case Achilles reaches the tortoise, calculate the time t it takes him to do so.

SOLUTION

The resulting series is *geometric*, which converges if $r = \frac{v_T}{v_A} < 1$, namely if Achilles runs faster than the tortoise. In this case, the time it takes Achilles to reach the tortoise is

$$t = \frac{L}{v_A} \sum_{n=0}^{\infty} \left(\frac{v_T}{v_A} \right)^n = \frac{L}{v_A - v_T} \,.$$

Note that this is consistent with the solution of the problem formulated by means of the equation

$$L + v_T t = v_A t \,.$$

Problem 2. Prove that the function

$$f(x) \;=\; \begin{cases} (x^3 - 3x + 1)\,\ln(1 + x^2)\,, & \text{if} \;\; -1 \le x \le 0\,, \\ e^{-1/x^2} - \arctan(3^x - 1)\,, & \text{if} \;\; 0 < x \le 1\,, \end{cases}$$

is bounded.

SOLUTION

Since the domain of $f(x)$ is the interval $[-1, 1]$, which is closed and bounded, we can prove that $f(x)$ is continuous in $[-1, 1]$. Thus, for $x \neq 0$, $f(x)$ is continuous as defined in terms of continuous elementary functions. On the other hand, the continuity at $x = 0$ holds as

$$\lim_{x \to 0^-} f(x) \;=\; \lim_{x \to 0^+} f(x) \;=\; f(0) \;=\; 0\,.$$

Hence, thanks to its continuity in $[-1, 1]$, $f(x)$ is bounded in the same interval.

Problem 3. Let $F(x) = \int_0^{\sin(x)} \frac{1 - \arcsin(t)}{\sqrt{1-t^2}}\, dt$, with $x \in (0, \pi/2)$.

(a) Calculate and classify the local extrema of $F(x)$.

(b) Using a Taylor polynomial of degree 2 for $F(x)$, calculate $\lim_{x \to 0^+} \frac{F(x) - x}{7x^2}$.

SOLUTION

(a) Thanks to the Fundamental Theorem of Calculus, we have $F'(x) = 1 - x$. Hence, the unique critical point is $x = 1$, which is a point of local maximum.

(b) The Taylor polynomial of degree 2 about $a = 0$ for $F(x)$ is $P_2(x) = x - x^2/2$. Thus, we can write

$$\lim_{x \to 0^+} \frac{F(x) - x}{7x^2} = \lim_{x \to 0^+} \frac{\left[\, x - x^2/2 + o(x^2) \,\right] - x}{7x^2} = -\frac{1}{14}\,.$$

Problem 4. Calculate $\displaystyle\int \frac{3e^{2x}+7e^{x}}{e^{2x}+4e^{x}+5}\,dx$, using the change of variable $u = e^{x}$.

SOLUTION

By applying the change of variable $u = e^{x}$ ($du = e^{x}dx$), we get

$$\int \frac{3e^{2x}+7e^{x}}{e^{2x}+4e^{x}+5}\,dx = \int \frac{3u+7}{u^2+4u+5}\,du = \frac{3}{2}\int \frac{2u+4}{u^2+4u+5}\,du + \int \frac{1}{u^2+4u+5}\,du$$

$$= \frac{3}{2}\ln|u^2+4u+5| + \arctan(u+2) + k,$$

where k is an arbitrary constant. Hence

$$\int \frac{3e^{2x}+7e^{x}}{e^{2x}+4e^{x}+5}\,dx = \frac{3}{2}\ln|e^{2x}+4e^{x}+5| + \arctan(e^{x}+2) + k,$$

with $k \in \mathbb{R}$.

Problem 5. Analyze the convergence of the *improper* integral $\displaystyle\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx$.

SOLUTION

The given *improper* integral converges as, for instance, we can write

$$\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx = \int_0^1 \frac{e^{-x^2}}{\sqrt{x}}\,dx + \int_1^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx\,,$$

where the first integral in the right-hand-side is convergent by the comparison test with $\int_0^1 1/\sqrt{x}\,dx$ and the second one converges by the limit comparison test with $\int_1^\infty 1/x^2\,dx$.