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CALCULUS – EVALUATION TEST 11

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Problem 1. The paradox of Achilles and the tortoise

Achilles and a tortoise are in a footrace. Trusting his superiority, Achilles allows the tortoise a head start. However, when he reaches the position where the tortoise started from, the latter has run some distance. Thus, Achilles keeps on running and gets to that new position, but he soon realizes that the tortoise has advanced further.

Repeating this reasoning infinite times, it seems that Achilles will never reach the tortoise. Nevertheless, Achilles does catch up with his competitor if the total time spent during the infinite steps is a finite value.

Let us suppose that the tortoise starts a distance L ahead, the velocity of Achilles is v_A, and the velocity of the tortoise is v_T (with $L, v_A, v_T \in \mathbb{R}$). Hence, the total time t spent in the infinite steps is

$$t = \frac{L}{v_A}\left(1 + r + r^2 + r^3 + r^4 + \dots\right), \quad \text{with } r = \frac{v_T}{v_A}.$$

- Which type of series do you get?
- Discuss the convergence of the series depending on the values of v_A and v_T.
- In the case Achilles reaches the tortoise, calculate the time t it takes him to do so.

Problem 2. Prove that the function

$$f(x) = \begin{cases} (x^3 - 3x + 1)\ \ln(1 + x^2), & \text{if } -1 \le x \le 0, \\ e^{-1/x^2} - \arctan(3^x - 1), & \text{if } 0 < x \le 1, \end{cases}$$

is bounded.

Problem 3. Let $F(x) = \int_0^{\sin(x)} \frac{1-\arcsin(t)}{\sqrt{1-t^2}}\,dt$, with $x \in (0, \pi/2)$.

(a) Calculate and classify the local extrema of $F(x)$.

(b) Using a Taylor polynomial of degree 2 for $F(x)$, calculate $\lim_{x \to 0^+} \frac{F(x) - x}{7x^2}$.

Problem 4. Calculate $\int \frac{3e^{2x} + 7e^{x}}{e^{2x} + 4e^{x} + 5}\, dx$, using the change of variable $u = e^{x}$.

Problem 5. Analyze the convergence of the *improper* integral $\displaystyle\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx$.
