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CALCULUS – EVALUATION TEST 12 (solutions)

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Problem 1. Consider the monotone increasing sequence $(a_n)_{n\in\mathbb{N}}$ defined by the *recursive* formula

$$a_0 = 1\,;$$
$$a_n = \sqrt{\frac{2+3a_{n-1}}{2}}\,, \quad \text{with } \; n \geq 1\,.$$

- Prove that the sequence is bounded.
- Calculate $\lim\limits_{n\to\infty} a_n$.

SOLUTION

Let us suppose that the sequence has a finite limit, say $\lim_{n\to\infty} a_n = a \in \mathbb{R}$. Thus, as $n \to \infty$ in both sides of the *recursive* formula, we get

$$a = \sqrt{\frac{2+3a}{2}} \quad \Longrightarrow \quad a^2 = \frac{3}{2}\,a + 1 \quad \Longrightarrow \quad a = -\frac{1}{2},\, 2\,,$$

where the value $a = -1/2$ must be discarded as the sequence is increasing with positive terms. Then, $a = 2$ is the only candidate to be the value of the limit.

Now, let us prove by the *principle of induction* that the given sequence is bounded, namely $0 \leq a_n \leq 2$ for all $n \in \mathbb{N} \cup \{0\}$. First, this property holds for $n = 0$, as $0 \leq a_0 = 1 \leq 2$. Then, assuming that $0 \leq a_k \leq 2$ for $n = k \geq 0$, we get (for $n = k+1$)

$$0 \leq a_{k+1} = \sqrt{\frac{2+3a_k}{2}} \leq \sqrt{\frac{2+6}{2}} = 2\,.$$

Hence, the sequence is bounded and has a finite limit value equal to $a = 2$, as previously calculated.

Problem 2. Consider the function

$$f(x) \;=\; \begin{cases} \arctan\left(\dfrac{1}{x^2}\right) + \dfrac{\pi}{2}\,, & \text{if } \; x \neq 0\,, \\ \pi\,, & \text{if } \; x = 0\,. \end{cases}$$

- Prove that the function $f(x)$ is differentiable for all $x \in \mathbb{R}$.
- Find for which values of $x \in \mathbb{R}$ the function $f(x)$ is increasing.

SOLUTION

For $x \neq 0$, $f(x)$ is differentiable as defined in terms of differentiable functions. In addition, we have

$$f'(x) = \frac{-2x}{x^4+1}\,.$$

On the other hand, at $x = 0$, we get

$$f'(0) \;=\; \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \;=\; \lim_{x\to 0} \frac{\arctan\left(\frac{1}{x^2}\right) + \frac{\pi}{2} - \pi}{x} \;=\; \lim_{x\to 0} \frac{-2x}{x^4+1} \;=\; 0\,,$$

where the l'Hôpital's rule has been applied in the last-but-one identity. Hence, $f(x)$ is differentiable at $x = 0$ as well. Moreover, $f(x)$ is increasing for $x < 0$ since $f'(x) > 0$ in this case.

Problem 3. Let $F(x) = \int_0^{x^3} \ln\left(t^{\frac{1}{3}} + \frac{1}{2}\right) \, dt$.

(a) Find and classify the local extrema of $F(x)$ for $x \in (0, 1)$.

(b) Use the Maclaurin polynomial of degree 3 for $F(x)$ to approximate $F(0.2)$.

SOLUTION

(a) Thanks to the Fundamental Theorem of Calculus, we have $F'(x) = 3x^2 \ln\left(x + \frac{1}{2}\right)$. Hence, the only critical point in the interval $(0, 1)$ is $x = 1/2$. In addition, $F(x)$ is decreasing for $x < 1/2$ (as $F'(x) < 0$) and increasing for $x > 1/2$ (as $F'(x) > 0$). Hence, $x = 1/2$ is a point of local minimum.

(b) The Maclaurin polynomial of degree 3 for $F(x)$ is $P_3(x) = \ln\left(\frac{1}{2}\right) x^3$, hence

$$F(0.2) \approx \ln\left(\frac{1}{2}\right) (0.2)^3 \approx -0.0055\,.$$

Problem 4. Calculate $\displaystyle\int_e^5 \frac{dx}{x\ \ln(x)}$.

SOLUTION

By applying the change of variable $u = \ln(x)$ ($du = dx/x$), we get

$$\int_e^5 \frac{dx}{x\ \ln(x)} \ = \int_1^{\ln(5)} \frac{du}{u} \ = \ \ln(\ln(5)) - 0 \ = \ \ln(\ln(5))\,.$$

Problem 5. Study the convergence of the *improper* integral $\displaystyle\int_0^\infty \frac{|\sin(x)|}{x+x^2}\,dx\,.$

SOLUTION

The given *improper* integral converges as, for instance, we can write

$$\int_0^\infty \frac{|\sin(x)|}{x+x^2}\,dx = \int_0^1 \frac{|\sin(x)|}{x+x^2}\,dx + \int_1^\infty \frac{|\sin(x)|}{x+x^2}\,dx\,,$$

where the first integral in the right-hand-side is not improper and the second (improper) integral converges thanks to the limit comparison test with $\int_1^\infty 1/x^{3/2}\,dx\,.$