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CALCULUS – EVALUATION TEST 13 (solutions)

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Problem 1. Prove that 5 is an upper bound for the *recursive* sequence defined as

$$a_1 = 0; \quad a_{n+1} = 4 + \frac{1}{5}a_n, \quad \text{for } n \geq 1.$$

Then, for $n \geq 1$, check that

$$a_n = 5 - \frac{1}{5^{n-2}}.$$

SOLUTION

Let us prove by the *principle of induction* that the given sequence is bounded above by 5. First, we have $a_1 = 0 \leq 5$. Then, supposing that $a_k \leq 5$ for $n = k \in \mathbb{N}$, we get

$$\frac{1}{5}a_k \leq 1 \implies 4 + \frac{1}{5}a_k \leq 5 \implies a_{k+1} \leq 5.$$

Hence, we can conclude that 5 is an upper bound for $(a_n)_{n\in\mathbb{N}}$.

Now, from $a_n = 5 - \frac{1}{5^{n-2}}$, we get

$$a_1 = 5 - \frac{1}{5^{-1}} = 5 - 5 = 0$$

for $n = 1$. In addition, we have

$$a_{n+1} = 5 - \frac{1}{5^{n-1}}$$

and

$$4 + \frac{1}{5}a_n = 5 - \frac{1}{5^{n-1}}$$

for $n \geq 1$. Thus, we can conclude that the indicated explicit expression for a_n is the solution of the *recursive* formula for $n \geq 1$.

Problem 2. Determine the exact number of real solutions of the equation

$$e^{-x} - e^{x} - \ln(x) = 0\,, \quad \text{with } \ x \in (0, +\infty)\,.$$

SOLUTION

Let us define $f(x) = e^{-x} - e^{x} - \ln(x)$, which is continuous and differentiable for $x > 0$. Thus, we have

$$f'(x) = -e^{-x} - e^{x} - \frac{1}{x} \, < \, 0$$

for $x \in (0, +\infty)$, which means that $f(x)$ is decreasing in the definition interval. Finally, since

$$\lim_{x \to 0^+} f(x) = +\infty, \quad \lim_{x \to +\infty} f(x) = -\infty,$$

we can guarantee that the equation $f(x) = 0$ has a unique real solution for $x \in (0, +\infty)$.

Problem 3. Let

$$F(x) = \int_0^x e^{-t^2} dt\,.$$

Use a Taylor polynomial of degree 3 for $F(x)$ to approximate the value $F(1/10)$ and find an *upper bound* for the involved error.

SOLUTION

After writing $e^{-t^2} = 1 - t^2 + o(t^2)$, we get

$$F(x) = \int_0^x e^{-t^2} dt \;=\; \int_0^x [\,1 - t^2 + o(t^2)\,]dt \;=\; x - \frac{x^3}{3} + o(x^3)\,,$$

for values of x in a neighborhood of the origin. Hence, the Taylor polynomial of degree 3 about $a = 0$ for $F(x)$ is given by $P_3(x) = x - x^3/3$, which can be used to approximate the desired value as

$$F(1/10) \approx P_3(1/10) = \frac{1}{10} - \frac{1}{3000} = \frac{299}{3000}\,.$$

On the other hand, noting that $F^{(4)}(x) = 4x(3 - 2x^2)e^{-x^2}$, the remainder associated with $P_3(x)$ is given by

$$R_3(x) = \frac{4c(3 - 2c^2)e^{-c^2}}{4!}\, x^4\,,$$

with $c \in (0, x)$ when $x > 0$. Thus, an *upper bound* for the involved error at $x = 1/10$ can be found as

$$|R_3(1/10)| = \left| \frac{c(3 - 2c^2)e^{-c^2}}{6}\, \frac{1}{10^4} \right| \;\leq\; \frac{1}{6 \cdot 10^4}(3c + 2c^3)e^{-c^2} \;\leq\; \frac{1}{6 \cdot 10^4}\left(\frac{3}{10} + \frac{2}{10^3} \right),$$

where the last inequality holds as $e^{-c^2} < 1$, being $c \in (0, 1/10)$.

Problem 4. Find *all* differentiable functions $F : (0, +\infty) \to \mathbb{R}$ that satisfy

$$F'(x) = \ln^2(x)\,, \quad F(1) = 0\,.$$

SOLUTION

Integrating by parts twice, we get

$$F(x) = \int \ln^2(x)dx \;=\; x\,\ln^2(x) - 2\int \ln(x)dx \;=\; x\,\ln^2(x) - 2x\,\ln(x) + 2\int dx$$

$$= x\,\ln^2(x) - 2x\,\ln(x) + 2x + k\,,$$

with $k \in \mathbb{R}$. Finally, after imposing $F(1) = 0$, which yields $k = -2$, we obtain

$$F(x) = x\,\ln^2(x) - 2x\,\ln(x) + 2x - 2\,.$$

Problem 5. Find *all* values of $a, b \in \mathbb{R}$ that make the function

$$f(x) = \begin{cases} a + \displaystyle\int_0^{2x} \frac{\sin(t)}{t} dt\,, & \text{if} \quad x < 0\,, \\ \sqrt{2} + b\cos(2x)\ln(1+3x)\,, & \text{if} \quad x \geq 0\,, \end{cases}$$

continuous and differentiable in the domain.

SOLUTION

For $x < 0$, $f(x)$ is continuous and differentiable thanks to the Fundamental Theorem of Calculus (all assumptions are readily seen to be satisfied). Moreover, for $x > 0$, $f(x)$ is continuous and differentiable since defined in terms of continuous and differentiable elementary functions.

Now, continuity of $f(x)$ at $x = 0$ holds if $\lim_{x\to 0} f(x) = f(0) = \sqrt{2}$. Since

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \left[a + \int_0^{2x} \frac{\sin(t)}{t} dt \right] = a\,,$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left[\sqrt{2} + b\cos(2x)\ln(1+3x) \right] = \sqrt{2}\,,$$

we need $a = \sqrt{2}$ to ensure the continuity of $f(x)$ at $x = 0$, hence in its domain.

On the other hand, taking $a = \sqrt{2}$, $f(x)$ is differentiable at $x = 0$ if the following lateral limits

$$f'_-(0) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{\sin(2x)}{x} = 2\,,$$

$$f'_+(0) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{b\cos(2x)\ln(1+3x)}{x} = 3b\,,$$

provide the same finite result. Note that, in the first limit, the l'Hôpital's rule has been applied, together with the Fundamental Theorem of Calculus. Thus, $b = 2/3$ ensures the differentiability of $f(x)$ at $x = 0$, hence in its domain.