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# CALCULUS – EVALUATION TEST 14 (solutions)

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**Problem 1.** Knowing that the *recursive* sequence defined as

$$a_1 = 0\,; \quad a_{n+1} = \sqrt{\frac{2+3\,a_n}{2}}, \quad \text{with } n \in \mathbb{N}\,,$$

is bounded, prove that it is increasing and calculate $\lim\limits_{n\to\infty} a_n$ .

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## SOLUTION

Let us prove that the sequence is monotone increasing by the *principle of induction*. For $n = 1$, we have $a_2 = 1 \geq 0 = a_1$ . Then, supposing that $a_{k-1} \leq a_k$ for $n = k \in \mathbb{N}$ (with $k \geq 2$), we can write

$$\frac{3}{2}a_{k-1} \leq \frac{3}{2}a_k \implies 1 + \frac{3}{2}a_{k-1} \leq 1 + \frac{3}{2}a_k \implies \sqrt{\frac{2+3\,a_{k-1}}{2}} \leq \sqrt{\frac{2+3\,a_k}{2}} \implies a_k \leq a_{k+1}\,,$$

which proves the statement. Thus, since the sequence is bounded and monotone, it is also convergent, namely $\lim_{n\to\infty} a_n = L \in \mathbb{R}$. Now, as $n \to \infty$ in both sides of the *recursive* formula, we get

$$L = \sqrt{1 + \frac{3}{2}L}\,,$$

namely $L = -1/2$ or $L = 2$ . As the sequence is increasing, the value of the limit must be $L = 2$ .

**Problem 2.** Given $n \in \mathbb{N}$, consider

$$f(x) = \begin{cases} -\arctan(\ln(x^{2n}))\,, & \text{if} \quad x \neq 0\,, \\ \pi/2\,, & \text{if} \quad x = 0\,. \end{cases}$$

(a) Study the continuity and differentiability of $f(x)$.

(b) Find the intervals where $f(x)$ is increasing.

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**SOLUTION**

(a) First, for all $x \in \mathbb{R}$, with $x \neq 0$, the function is continuous and differentiable as given in terms of continuous and differentiable elementary functions. On the other hand, $f(x)$ is also continuous at $x = 0$ since $f(0) = \pi/2$ and we have

$$\lim_{x \to 0} f(x) \;=\; -\arctan(\ln(0^+)) = -\arctan(-\infty) = \pi/2\,.$$

In addition, we have

$$f'_+(0) \;=\; \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} \;=\; \lim_{x \to 0^+} -\frac{2n}{x(1 + \ln^2(x^{2n}))} \;=\; -\infty\,,$$

$$f'_-(0) \;=\; \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} \;=\; \lim_{x \to 0^-} -\frac{2n}{x(1 + \ln^2(x^{2n}))} \;=\; +\infty\,,$$

hence the function is not differentiable at $x = 0$.

(b) Observing that

$$f'(x) \;=\; -\frac{2n}{x(1 + \ln^2(x^{2n}))}$$

for $x \neq 0$, we can conclude that $f(x)$ is increasing for $x \in (-\infty, 0)$ (where $f'(x) > 0$) and $f(x)$ is decreasing for $x \in (0, +\infty)$ (where $f'(x) < 0$).

**Problem 3.** Let $n \in \mathbb{N}$. Then, calculate

$$\lim_{x\to 0} \frac{x\displaystyle\int_0^x e^{t^2}\,dt}{\displaystyle\int_0^x e^{t^2}\sin(t)\,dt}\,; \qquad\qquad \lim_{x\to 0} \frac{e^x - 1 - x - \frac{x^2}{2!} - \ldots - \frac{x^n}{n!}}{x^{n+1}}\,.$$

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**SOLUTION**

Regarding the first limit, by applying the l'Hôpital's rule twice, we get

$$\lim_{x\to 0} \frac{x\displaystyle\int_0^x e^{t^2}\,dt}{\displaystyle\int_0^x e^{t^2}\sin(t)\,dt} = \lim_{x\to 0} \frac{x\,e^{x^2} + \displaystyle\int_0^x e^{t^2}\,dt}{e^{x^2}\sin(x)} = \lim_{x\to 0} \frac{2e^{x^2} + 2\,x^2 e^{x^2}}{e^{x^2}\cos(x) + 2x\,e^{x^2}\sin(x)} = 2\,,$$

where the Fundamental Theorem of Calculus has been used in order to calculate the derivatives of the involved integral functions. On the other hand, for $n \in \mathbb{N}$, we can write

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + o(x^{n+1})\,.$$

Hence, regarding the second limit, we get

$$\lim_{x\to 0} \frac{e^x - 1 - x - \frac{x^2}{2!} - \ldots - \frac{x^n}{n!}}{x^{n+1}} = \lim_{x\to 0} \frac{\frac{x^{n+1}}{(n+1)!} + o(x^{n+1})}{x^{n+1}} = \frac{1}{(n+1)!}\,.$$

**Problem 4.** Let $G : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ be defined by $G(x) = \displaystyle\int_{1/x}^{x^2+x} \frac{1}{2+t^2}\, dt$.

(a) Study the continuity and differentiability of $G(x)$.

(b) Find the function $G'(x)$.

(c) Calculate the indefinite integral

$$\int \frac{1}{2+t^2}\, dt$$

and use the result to determine the value of $\lim_{x \to +\infty} G(x)$.

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**SOLUTION**

(a) Let $f(t) = 1/(2+t^2)$. Then, thanks to the Fundamental Theorem of Calculus, the function $G(x)$ is continuous and differentiable since $f(t)$ is continuous, and both functions $x^2 + x$ and $1/x$ are differentiable for all $x \neq 0$.

(b) By the Fundamental Theorem of Calculus and the chain rule, we get

$$G'(x) = \frac{2x+1}{2+(x^2+x)^2} + \frac{1}{2+(1/x)^2}\frac{1}{x^2} = \frac{2x+1}{2+(x^2+x)^2} + \frac{1}{1+2x^2}.$$

(c) A direct integration yields

$$\int \frac{1}{2+t^2}\, dt = \frac{1}{\sqrt{2}} \arctan\left(\frac{t}{\sqrt{2}}\right) + k,$$

with $k \in \mathbb{R}$. Thus, we have

$$G(x) = \frac{1}{\sqrt{2}} \arctan\left(\frac{x^2+x}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}} \arctan\left(\frac{1}{\sqrt{2}\,x}\right).$$

Hence

$$\lim_{x \to +\infty} G(x) = \frac{\pi}{2\sqrt{2}}.$$

**Problem 5.** Calculate $\int \ln^2(x)\,dx$.

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**SOLUTION**

Integrating by parts twice, we get

$$\int \ln^2(x)\,dx = x\ln^2(x) - 2\int \ln(x)\,dx = x\ln^2(x) - 2\left[x\ln(x) - \int dx\right].$$

Hence

$$\int \ln^2(x)\,dx = x\ln^2(x) - 2x\ln(x) + 2x + k,$$

with $k \in \mathbb{R}$.