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CALCULUS – EVALUATION TEST 15 (solutions)

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Problem 1. Study the convergence of the series

$$\sum_{n=1}^{\infty} a_n \,,$$

where $(a_n)_{n\in\mathbb{N}}$ is the *recursive* sequence defined as

$$a_1 = 1\,; \quad a_{n+1} = -\frac{a_n}{2}\left(1+\frac{1}{n}\right)^{n/2}, \quad \text{with } n \in \mathbb{N}\,.$$

SOLUTION

By applying the *ratio test*, we get

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2}\left(1+\frac{1}{n}\right)^{n/2} \longrightarrow \frac{1}{2}\sqrt{e} \quad \text{as } n \to \infty\,,$$

where a known result has been taken into account. Then, observing that $2 < e < 3$ implies $\sqrt{2} < \sqrt{e} < \sqrt{3} < 2$, we can conclude that $\sqrt{e}/2 < 1$. Hence, the given series is (absolutely) convergent.

Problem 2. Find the exact number of real solutions of the equation

$$e^x = ax\,,$$

depending on the value of $a \in \mathbb{R}$.

SOLUTION

Let $f(x) = e^x - ax$, whose first derivative is equal to $f'(x) = e^x - a$. Now, we can distinguish three different cases.

- Let $a = -|a| < 0$. Then, $f'(x) > 0$ for all $x \in \mathbb{R}$ and the function $f(x)$ is increasing. Moreover, we have
$$\lim_{x\to\pm\infty} f(x) = \pm\infty\,.$$
Thus, the given equation $f(x) = 0$ has a unique real solution.

- Let $a = 0$. Then, the given equation reads $e^x = 0$, which has no solution.

- Let $a = |a| > 0$. Then, we have
$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} (e^x - |a|x) = +\infty.$$
In addition, the only critical point for $f(x)$ is equal to $x^* = \ln|a|$ (which is the solution of $f'(x) = 0$), yielding $f(x^*) = |a|(1 - \ln|a|)$. Hence, if $f(x^*) > 0$ (namely, if $0 < a < e$) the given equation has no solution, if $f(x^*) = 0$ (namely, if $a = e$) the equation has a unique real solution, if $f(x^*) < 0$ (namely, if $a > e$) the equation has two real solutions.

Problem 3. Consider the function

$$f(x) = (36 + x^3)^{-1/2}, \quad \text{with } \; x \neq -6^{2/3}.$$

(a) Write the Taylor polynomial of degree 6 about $a = 0$ for $f(x)$.

(b) Find a rational number that approximates $f(-1)$ within an error smaller than 10^{-2}.

SOLUTION

(a) Noting that

$$f(x) = (6^2 + x^3)^{-1/2} = \frac{1}{6}\left(1 + \frac{x^3}{6^2}\right)^{-1/2},$$

we can write

$$f(x) = \frac{1}{6}\left(1 - \frac{x^3}{72} + \frac{x^6}{3456} + o(x^6)\right) = \frac{1}{6} - \frac{x^3}{432} + \frac{x^6}{20736} + o(x^6),$$

as $x \to 0$.

(b) It can be checked that the rational number

$$\frac{1}{6} + \frac{1}{432}$$

is an approximation of $f(-1)$ within an error smaller than 10^{-2}.

Problem 4. Let $f, F : [0,7] \longrightarrow \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1\,, & \text{if} \quad 0 \le x \le 4\,, \\ 5 - x\,, & \text{if} \quad 4 < x \le 5\,, \\ -1\,, & \text{if} \quad 5 < x \le 7\,, \end{cases} \qquad\qquad F(x) = \int_0^x f(t)\,dt\,.$$

(a) Calculate the values $F(4)$, $F(5)$, and $F(7)$.

(b) Study the continuity and differentiability of $F(x)$.

SOLUTION

(a) Let $0 \le x \le 4$. Then, we have

$$F(x) \;=\; \int_0^x f(t)\,dt \;=\; \int_0^x 1\,dt \;=\; x\,.$$

Hence, $F(4) = 4$. Now, let $4 < x \le 5$. In this case, we can write

$$F(x) \;=\; \int_0^4 f(t)\,dt + \int_4^x f(t)\,dt \;=\; F(4) + \int_4^x (5 - t)\,dt \;=\; \frac{9}{2} - \frac{1}{2}(x-5)^2\,.$$

Hence, $F(5) = 9/2$. Finally, for $5 < x \le 7$, we get

$$F(x) \;=\; \int_0^5 f(t)\,dt + \int_5^x f(t)\,dt \;=\; F(5) + \int_5^x (-1)\,dt \;=\; \frac{19}{2} - x\,.$$

Hence, $F(7) = 5/2$.

(b) By the Fundamental Theorem of Calculus, we can deduce that $F(x)$ is continuous for all $x \in [0,7]$ (as f is integrable) and $F(x)$ is differentiable for all $x \in [0,7]$, with $x \neq 5$ (as f is continuous). Indeed, note that $F'(5)$ does not exist.

Problem 5. Calculate

$$\lim_{x \to 0} \frac{\sin(x)\cos(x) - \arctan(x)}{\ln(1 + x^3)}$$

by using appropriate Taylor polynomials.

SOLUTION

By using appropriate Taylor polynomials about $a = 0$ for the involved functions, we can write

$$\lim_{x \to 0} \frac{\sin(x)\cos(x) - \arctan(x)}{\ln(1 + x^3)} = \lim_{x \to 0} \frac{\left(x - \frac{x^3}{6} + o(x^3)\right)\left(1 - \frac{x^2}{2} + o(x^2)\right) - \left(x - \frac{x^3}{3} + o(x^3)\right)}{x^3 + o(x^3)}$$

$$= \lim_{x \to 0} \frac{x - \frac{2}{3}x^3 - x + \frac{x^3}{3} + o(x^3)}{x^3 + o(x^3)} = -\frac{1}{3},$$

where all powers of x up to 3 have been retained.

Problem 6. Calculate $\int x \arctan(x)\, dx$.

SOLUTION

We can write

$$\int x \arctan(x)\, dx = \frac{x^2}{2}\arctan(x) - \frac{1}{2}\int \frac{x^2}{1+x^2}\, dx = \frac{x^2}{2}\arctan(x) - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx$$

$$= \frac{x^2}{2}\arctan(x) - \frac{x}{2} + \frac{1}{2}\arctan(x) + k,$$

with $k \in \mathbb{R}$. Note that the first identity has been obtained upon integration by parts.