OpenCourseWare

CALCULUS – EVALUATION TEST 1 (solutions)

Filippo Terragni, Eduardo Sánchez Villaseñor, Manuel Carretero Cerrajero

Problem 1. Consider the sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers defined by

$$\begin{aligned} a_n &= -4 + \frac{a_{n-1}}{3}, \quad \text{with } n \geq 2; \\ a_1 &= 0. \end{aligned}$$

- Prove that the sequence is decreasing.
- Prove that the sequence is bounded.
- Calculate $\lim_{n\to\infty} a_n$.

SOLUTION

Observe that

$$a_2 = -4, \quad a_3 = -4 - \frac{4}{3}, \quad a_4 = -4 - \frac{4}{3} - \frac{4}{9}, \quad \ldots,$$

hence, for all $n \geq 2$, we have

$$a_n = -4 - \frac{4}{3} - \ldots - \frac{4}{3^{n-2}}. \tag{1}$$

Then

$$a_n - a_{n-1} = -\frac{4}{3^{n-2}} < 0,$$

namely the sequence is (strictly) decreasing. In addition, using eq.(1), we can write a_n as the *geometric sum*

$$a_n = -4 \sum_{k=0}^{n-2} \left(\frac{1}{3}\right)^k = -4 \, \frac{1 - \left(\frac{1}{3}\right)^{n-1}}{1 - \frac{1}{3}} = -6 \left\{ 1 - \left(\frac{1}{3}\right)^{n-1} \right\} , \tag{2}$$

where a known result has been used. From eq.(2), we deduce that $|a_n| \leq 6$ for all $n \geq 2$, hence the sequence is bounded. In addition, we get

$$\lim_{n \to \infty} a_n = -6 \, .$$

Problem 2. Find *all* values of the parameter $\alpha \in \mathbb{R}$ such that the series

$$\sum_{n=0}^{\infty} \frac{(\alpha-2)^n}{n^2+1}$$

is convergent.

SOLUTION

Given $a_n = (\alpha-2)^n/(n^2+1)$, note that

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(\alpha-2)^{n+1}}{(n+1)^2+1}\,\frac{n^2+1}{(\alpha-2)^n}\right| = |\alpha-2|\,\frac{n^2+1}{n^2+2n+2} \longrightarrow |\alpha-2| \quad \text{as} \ \ n\to\infty\,.$$

Hence, thanks to the *ratio test*, the series is convergent if $|\alpha-2|<1$, namely if $1<\alpha<3$ (if $\alpha>3$ or $\alpha<1$ the series diverges). If $\alpha=3$ the series is given by

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1}\,,$$

which is convergent since $1/(n^2+1) \ < \ 1/n^2$, for all $n \geq 1$, and the series with general term equal to $1/n^2$ is convergent (by the *comparison test*). On the other hand, if $\alpha = 1$ the series is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}\,,$$

which is convergent by the *Leibniz test* (indeed, it is an alternating series, where $1/(n^2+1)$ is positive, decreasing, and tending to zero as $n\to\infty$).

Problem 3. Consider the function

$$f(x) = \begin{cases} -x^2 - 7\cos\left(\frac{\pi}{2}x\right) & \text{if } x > 2, \\ a(x+1) + b & \text{if } -1 < x \leq 2, \\ x^3 - 12x + 5 & \text{if } x \leq -1, \end{cases}$$

where a and b are real numbers.

- Find the values of a and b such that $f(x)$ is continuous in $\mathbb{R}$.
- Calculate (if any) the local maxima and minima of $f(x)$ for $x < -1$.

SOLUTION

The function $f(x)$ is continuous for $x \in (-\infty, -1) \cup (-1, 2) \cup (2, +\infty)$ as defined in terms of continuous elementary functions. The continuity at $x = -1$ is ensured by imposing

$$f(-1) = \lim_{x \to -1^-} \left[x^3 - 12x + 5 \right] = \lim_{x \to -1^+} \left[a(x+1) + b \right],$$

which requires that $b = 16$. On the other hand, the continuity at $x = 2$ is guaranteed by imposing

$$f(2) = \lim_{x \to 2^-} \left[a(x+1) + b \right] = \lim_{x \to 2^+} \left[-x^2 - 7\cos\left(\frac{\pi}{2}x\right) \right],$$

which implies $3a + b = 3$, namely $a = -13/3$. Thus, the calculated values for a and b make $f(x)$ continuous in $\mathbb{R}$. Now, the only *critical point* for $f(x)$ in the interval $(-\infty, -1)$ is obtained by imposing

$$f'(x) = 3x^2 - 12 = 0 \qquad \Longrightarrow \qquad x = -2,$$

which is a point of local maximum (since $f''(-2) = -12 < 0$) where $f(-2) = 21$. The function $f(x)$ does not have any local minimum for $x < -1$.

Problem 4. Approximate the value

$$\ln\left(\frac{4}{3}\right)$$

by using a polynomial of degree 3 and find an appropriate *upper bound* for the involved error.

SOLUTION

Note that

$$\ln\left(\frac{4}{3}\right) = \ln\left(1+\frac{1}{3}\right)$$

can be obtained by evaluating the function $f(x) = \ln(1+x)$ at $x = 1/3$. Such function can be expressed by the *Taylor Theorem* as

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + R_3(x),$$

where the remainder $R_3(x)$ can be written as

$$R_3(x) = \frac{f^{(4)}(c)}{4!}\, x^4,$$

with $f^{(4)}(x) = -6\,(1+x)^{-4}$ and $c \in (0,x)$ when $x > 0$. Then, we can approximate the desired value as

$$\ln\left(\frac{4}{3}\right) \approx \frac{1}{3} - \frac{1}{18} + \frac{1}{81} = \frac{47}{162} \approx 0.2901$$

and find an *upper bound* for the involved error as

$$\left|R_3\left(\frac{1}{3}\right)\right| = \left|\frac{-6}{(1+c)^4}\,\frac{(1/3)^4}{4!}\right| = \frac{(1/3)^4}{4(1+c)^4} \le \frac{(1/3)^4}{4} \approx 0.003\,.$$

Note that the previous inequality holds since $c \in (0, 1/3)$, which gives $c+1 \in (1, 4/3)$, ensuring that $(1+c)^{-4} < 1$.