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# CALCULUS – EVALUATION TEST 2 (solutions)

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**Problem 1.** Consider the monotone increasing sequence $(a_n)_{n\in\mathbb{N}}$ defined by the following *recursive* formula

$$a_1 = 0\,;$$
$$a_{n+1} = \sqrt{4a_n + 5}\,, \quad \text{with } \; n \geq 1\,.$$

Prove that the sequence is bounded and calculate $\lim\limits_{n\to\infty} a_n$ .

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**SOLUTION**

Let us suppose that the sequence has a finite limit, say $\lim_{n\to\infty} a_n = a \in \mathbb{R}$. Then, as $n \to \infty$ in both sides of the *recursive* formula, we have

$$a = \sqrt{4a+5} \quad \Longrightarrow \quad a^2 = 4a + 5 \quad \Longrightarrow \quad a = -1, 5\,,$$

where the value $a = -1$ must be discarded, since the sequence is increasing with positive terms. Thus, $a = 5$ is the only candidate to be the value of the limit.

Now, let us prove by the *principle of induction* that the sequence is bounded above by 5, namely $0 \leq a_n \leq 5$ for all $n \in \mathbb{N}$. First, such property holds for $n = 1$, namely $0 \leq a_1 = 0 \leq 5$. Then, assuming that $0 \leq a_k \leq 5$ for $n = k \in \mathbb{N}$, we get (for $n = k+1$)

$$0 \;\leq\; a_{k+1} = \sqrt{4a_k + 5} \;\leq\; \sqrt{4\cdot 5 + 5} \;=\; 5\,.$$

Hence, the sequence is bounded and has a finite limit thanks to its increasing behavior. As a consequence, the desired value of the limit is $a = 5$, as previously calculated.

**Problem 2.** Find *all* values of the parameter $x \in \mathbb{R}$ such that the series

$$\sum_{k=1}^{\infty} \frac{3^{2k} x^{3k}}{(2k+1)\,5^k}$$

is convergent.

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**SOLUTION**

Let $a_k = \dfrac{3^{2k} x^{3k}}{(2k+1)\,5^k}$. Then, we have

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{9}{5}|x|^3 \,\frac{2k+1}{2k+3} \quad \longrightarrow \quad \frac{9}{5}|x|^3 \quad \text{as } k \to \infty\,.$$

Thus, thanks to the *ratio test*, the series converges if

$$\frac{9}{5}|x|^3 < 1 \quad \Longleftrightarrow \quad |x|^3 < \frac{5}{9} \quad \Longleftrightarrow \quad x \in \left(-\frac{5^{1/3}}{9^{1/3}}\,,\ \frac{5^{1/3}}{9^{1/3}}\right),$$

while it diverges if

$$\frac{9}{5}|x|^3 > 1 \quad \Longleftrightarrow \quad x \in \left(-\infty\,,\ -\frac{5^{1/3}}{9^{1/3}}\right) \cup \left(\frac{5^{1/3}}{9^{1/3}}\,,\ +\infty\right).$$

On the other hand, let

$$\frac{9}{5}|x|^3 = 1 \quad \Longleftrightarrow \quad |x|^3 = \frac{5}{9} \quad \Longleftrightarrow \quad x^3 = \pm\frac{5}{9}\,.$$

Then, for $x^3 = 5/9$, the series is

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{2k+1}\,,$$

which diverges, while for $x^3 = -5/9$ the series is

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1}\,,$$

which converges by the *Leibniz test*. Hence, we can conclude that the proposed series is convergent if and only if

$$x \in \left[-\frac{5^{1/3}}{9^{1/3}}\,,\ \frac{5^{1/3}}{9^{1/3}}\right).$$

**Problem 3.** Consider the function

$$F(x) = \int_0^{5x} e^{-7t^4}\, dt\,, \quad \text{with } x \in \mathbb{R}\,.$$

- Prove that $F(x)$ is *odd*.
- Prove the existence of the limit $\ell = \lim_{x\to\infty} F(x)\,.$
- Prove that the function $F : \mathbb{R} \to (-\ell, \ell)$ is monotone *increasing*.
- Calculate $(F^{-1})'(0)$.
- Calculate $\lim_{x\to 0} \dfrac{5x - F(x)}{x^5}\,.$

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**SOLUTION**

- The function $F(x)$ is *odd* since

$$F(-x) \,=\, \int_0^{-5x} e^{-7t^4}\, dt \,=\, -\int_0^{5x} e^{-7u^4}\, du \,=\, -F(x)\,,$$

where the second identity is obtained by means of the change of variable $u = -t\,.$

- First, note that

$$\ell \,=\, \lim_{x\to\infty} F(x) \,=\, \lim_{x\to\infty} \int_0^{5x} e^{-7t^4}\, dt \,=\, \int_0^{\infty} e^{-7t^4}\, dt$$

is an *improper* integral of the first kind (of a positive function). Then, we have

$$\lim_{t\to\infty} \frac{e^{-7t^4}}{e^{-t}} \,=\, \lim_{t\to\infty} e^{-7t^4+t} \,=\, 0\,.$$

Thus, since $\int_0^\infty e^{-t}\, dt$ converges, by the limit comparison test we can conclude that also $\int_0^\infty e^{-7t^4}\, dt$ converges, namely the limit $\ell$ exists.

- The function $F(x)$ is *increasing* as

$$F'(x) \;=\; 5\,e^{-7(5x)^4} \;>\; 0$$

for all $x \in \mathbb{R}$. Note that $F'(x)$ has been calculated by means of the Fundamental Theorem of Calculus.

- Thanks to the result in the previous item, $F^{-1}(x)$ exists. Moreover, we have

$$(F^{-1})'(x) \;=\; \frac{1}{F'\big(F^{-1}(x)\big)} \quad\Longrightarrow\quad (F^{-1})'(0) \;=\; \frac{1}{F'\big(F^{-1}(0)\big)} \;=\; \frac{1}{F'(0)} \;=\; \frac{1}{5}\,,$$

where the last-but-one identity holds since $F(0) = 0$, while the last one is obtained from the expression for $F'(x)$ calculated before.

- By applying the l'Hôpital's rule twice, we get

$$\lim_{x\to 0} \frac{5x - F(x)}{x^5} \;=\; \lim_{x\to 0} \frac{5 - F'(x)}{5x^4} \;=\; \lim_{x\to 0} \frac{5 - 5\,e^{-7(5x)^4}}{5x^4} \;=\; \lim_{x\to 0} 7 \cdot 5^4\, e^{-7(5x)^4} \;=\; 4375\,.$$

**Problem 4.** Calculate
$$\int \frac{dx}{(x+1)^{4/3}-(x+1)^{2/3}}\,.$$

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**SOLUTION**

Let us apply the change of variable

$$u = (x+1)^{1/3}; \quad du = \frac{1}{3}(x+1)^{-2/3}dx \implies dx = 3u^2du\,.$$

Then, the given integral is transformed into

$$\int \frac{dx}{(x+1)^{4/3}-(x+1)^{2/3}} = 3\int \frac{u^2}{u^4-u^2}du = 3\int \frac{du}{u^2-1}\,.$$

Now, we can write

$$\frac{1}{u^2-1} = \frac{1}{(u-1)(u+1)} = \frac{1/2}{u-1} - \frac{1/2}{u+1}$$

and finally

$$\int \frac{du}{u^2-1} = \frac{1}{2}\int \frac{du}{u-1} - \frac{1}{2}\int \frac{du}{u+1} = \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + c\,,$$

where c is an arbitrary constant. Thus, we get

$$\int \frac{dx}{(x+1)^{4/3}-(x+1)^{2/3}} = \frac{3}{2}\ln\left|\frac{(x+1)^{1/3}-1}{(x+1)^{1/3}+1}\right| + c\,,$$

with $c \in \mathbb{R}$.