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CALCULUS – EVALUATION TEST 3 (solutions)

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Problem 1. Find *all* values of the parameter $x \in \mathbb{R}$ such that the series

$$\sum_{k=1}^{\infty} \frac{\sin^k(x/3)}{k^{1/5} + k^{1/6}}$$

converges.

SOLUTION

Let $a_k = \dfrac{\sin^k(x/3)}{k^{1/5} + k^{1/6}}$. Then, we have

$$\left|\frac{a_{k+1}}{a_k}\right| = |\sin(x/3)| \, \frac{k^{1/5} + k^{1/6}}{(k+1)^{1/5} + (k+1)^{1/6}} \quad \longrightarrow \quad |\sin(x/3)| \quad \text{as} \quad k \to \infty\,.$$

Thus, thanks to the *ratio test*, the series converges if

$$|\sin(x/3)| < 1 \quad \Longleftrightarrow \quad \forall\, x \in \mathbb{R}\,, \quad \text{with} \ \ \frac{x}{3} \neq \frac{\pi}{2} + n\pi\,, \ \ n = 0, 1, 2, \ldots\,.$$

Note that the series would be divergent for those values of x satisfying $|\sin(x/3)| > 1$, which never holds. On the other hand, if

$$\sin(x/3) = 1 \quad \Longleftrightarrow \quad \frac{x}{3} = \frac{\pi}{2} + j\pi\,, \ \ j = 0, 2, 4, \ldots\,,$$

then $a_k = 1/(k^{1/5} + k^{1/6})$. In this case, considering $b_k = 1/k^{1/5}$, we get

$$\lim_{k\to\infty} \frac{a_k}{b_k} = 1 > 0 .$$

Thus, by the limit comparison test, the series diverges since $\sum_{k=1}^{\infty} b_k$ is divergent (both series have positive terms). Finally, if

$$\sin(x/3) = -1 \quad \iff \quad \frac{x}{3} = \frac{3\pi}{2} + j\pi\,, \;\; j = 0, 2, 4, \ldots\,,$$

then

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{1/5} + k^{1/6}}\,,$$

which converges thanks to the *Leibniz test*. Hence, we can conclude that the given series is convergent for all $x \in \mathbb{R}$ such that $x \neq 3\,(\pi/2 + j\pi)\,,\;\; j = 0, 2, 4, \ldots$.

Problem 2. Consider the function $f : [-1, 1] \longrightarrow \mathbb{R}$ such that

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if} \quad -1 \le x < 0 \,, \\ x & \text{if} \quad 0 \le x \le 1 \,. \end{cases}$$

- Prove that $f(x)$ is bounded and calculate its image.
- Study the differentiability of $f(x)$ in the interval $(-1, 1)$.

SOLUTION

In order to prove that $f(x)$ is bounded in $[-1, 1]$, we can prove that $f(x)$ is continuous in that interval (closed and bounded). Indeed, the function is continuous in $[-1, 0) \cup (0, 1]$ as defined in terms of continuous functions. Now, let us prove its continuity at $x = 0$. We can write

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0 \,,$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} e^{-1/x^2} = 0 \,,$$

hence $\lim_{x \to 0} f(x) = f(0)$. As a consequence, $f(x)$ is continuous and bounded in $[-1, 1]$. Moreover, its image is the interval $[0, 1]$. Finally, the function $f(x)$ is differentiable for all $x \in (-1, 1)$, with $x \neq 0$, as

$$\lim_{h \to 0^+} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1 \,,$$

$$\lim_{h \to 0^-} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0^-} \frac{e^{-1/h^2}}{h} = 0 \,.$$

Problem 3.

- Approximate the value $\sqrt[7]{6/5}$ by using a polynomial of degree 2.
- Find a proper *upper bound* for the error involved in the previous approximation.

SOLUTION

Note that

$$\sqrt[7]{\frac{6}{5}} = \left(1+\frac{1}{5}\right)^{1/7}$$

can be calculated by evaluating the function $f(x) = (1+x)^{1/7}$ at $x = 1/5$. Such function can be expressed, thanks to the *Taylor Theorem*, as

$$(1+x)^r = 1 + r\,x + \frac{1}{2}r(r-1)x^2 + R_2(x)\,,$$

with $r = 1/7$. The previous expression is the sum of a polynomial of degree 2 and the remainder $R_2(x)$ given by

$$R_2(x) = \frac{f'''(c)}{3!}\,x^3\,,$$

where $f'''(c) = 78/343\,(1+c)^{-20/7}$ and $c \in (0,x)$ when $x > 0$. Thus, we can approximate the desired value as

$$\sqrt[7]{\frac{6}{5}} \approx 1 + \frac{1}{35} - \frac{3}{49}\frac{1}{25} \approx 1.0261\,.$$

Finally, an *upper bound* for the involved error at $x = 1/5$ can be obtained as

$$\left|R_2\left(\frac{1}{5}\right)\right| = \frac{78}{343}\,\frac{1}{(1+c)^{20/7}}\,\frac{(1/5)^3}{3!} < \frac{78}{343}\,\frac{(1/5)^3}{3!} \approx 3\cdot 10^{-4}\,,$$

where the inequality holds since $c \in (0, 1/5)$, which implies $1/(1+c)^{20/7} < 1$.

Problem 4. Calculate

$$\lim_{x\to 0}\left(\frac{1}{x^4}-\frac{1}{x^7}\int_0^x \sin(3t^2)\,dt\right).$$

SOLUTION

The given limit can be expressed as

$$\lim_{x\to 0}\frac{x^3-F(x)}{x^7}, \quad \text{with } F(x)=\int_0^x \sin(3t^2)\,dt\,.$$

Thus, by applying the l'Hôpital's rule and the Fundamental Theorem of Calculus, we get

$$\begin{aligned}\lim_{x\to 0}\frac{x^3-F(x)}{x^7} &= \lim_{x\to 0}\frac{3x^2-F'(x)}{7x^6} = \lim_{x\to 0}\frac{3x^2-\sin(3x^2)}{7x^6}\\ &= \lim_{x\to 0}\frac{3x^2-\left[3x^2-9/2\,x^6+o(x^6)\right]}{7x^6} = \frac{9}{14},\end{aligned}$$

where, in the last-but-one identity, the function $\sin(3x^2)$ has been approximated by the corresponding Maclaurin polynomial of degree 6.

Problem 5. Calculate the definite integral

$$\int_0^{\ln\sqrt{2}} \sqrt{e^{2t}-1}\,dt\,.$$

SOLUTION

By applying the change of variable $u = \sqrt{e^{2t}-1}$ (yielding $dt = u(u^2+1)^{-1}du$), we get

$$\int_0^{\ln\sqrt{2}} \sqrt{e^{2t}-1}\,dt \;=\; \int_0^1 \frac{u^2}{u^2+1}\,du \;=\; \int_0^1 \left[1-\frac{1}{u^2+1}\right]\,du$$

$$=\; \Big[u-\arctan(u)\Big]_0^1 \;=\; 1-\frac{\pi}{4}\,.$$

Problem 6. Study the convergence of the family of *improper* integrals given by

$$I_n(\lambda) = \int_0^{+\infty} x^n e^{-\lambda x}\, dx\,, \quad \text{with } \; n = 0, 1, 2, \ldots\,,$$

where $\lambda > 0$.

SOLUTION

Note that $I_0(\lambda)$ is convergent. In addition, the integration by parts yields

$$I_1(\lambda) \;=\; \lim_{b\to\infty} \int_0^b x\, e^{-\lambda x}\, dx \;=\; \lim_{b\to\infty} \left[\frac{1}{\lambda^2} - e^{-\lambda b} \left(\frac{1}{\lambda^2} + \frac{b}{\lambda} \right) \right] \;=\; \frac{1}{\lambda^2}\,,$$

hence $I_1(\lambda)$ is convergent as well. Now, in order to apply the *principle of induction*, suppose that $I_k(\lambda)$ converges for $n = k \in \mathbb{N}$ and prove that $I_{k+1}(\lambda)$ is also convergent (for $n = k+1$). Then, we have

$$\begin{aligned} I_{k+1}(\lambda) \;&=\; \lim_{b\to\infty} \int_0^b x^{k+1} e^{-\lambda x}\, dx \;=\; \lim_{b\to\infty} \left[-\frac{b^{k+1}}{\lambda}\, e^{-\lambda b} + \frac{k+1}{\lambda} \int_0^b x^k e^{-\lambda x}\, dx \right] \\ &=\; \frac{k+1}{\lambda}\, I_k(\lambda)\,, \end{aligned}$$

using the integration by parts. This means that $I_{k+1}(\lambda)$ converges, since $I_k(\lambda)$ does so thanks to the induction hypothesis. Hence, we can conclude that all improper integrals of the given family are convergent for $\lambda > 0$ (with $n = 0, 1, 2, \ldots$).