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CALCULUS – EVALUATION TEST 4 (solutions)

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Problem 1. Consider the monotone decreasing sequence $(a_n)_{n\in\mathbb{N}}$ defined by the *recursive* formula

$$a_1 = 1;$$
$$a_n = -8 + \frac{a_{n-1}}{3}, \quad \text{with } n \geq 2.$$

- Prove that the sequence is bounded.
- Calculate $\lim_{n\to\infty} a_n$.

SOLUTION

Let us suppose that the sequence has a finite limit, say $\lim_{n\to\infty} a_n = a \in \mathbb{R}$. Then, as $n \to \infty$ in both sides of the *recursive* formula, we get

$$a = -8 + \frac{a}{3} \implies a = -12.$$

Hence, if the sequence converges, $a = -12$ must be the value of its limit.

Now, let us prove by the *principle of induction* that the sequence is bounded, namely $-12 \leq a_n \leq 1$ for all $n \in \mathbb{N}$. This property holds for $n = 1$, as $-12 \leq a_1 = 1 \leq 1$. Then, assuming that $-12 \leq a_k \leq 1$ for $n = k \in \mathbb{N}$, we get (for $n = k+1$)

$$-12 = -8 - \frac{12}{3} \leq a_{k+1} = -8 + \frac{a_k}{3} \leq -8 + \frac{1}{3} \leq 1.$$

Thus, the sequence is bounded, hence it has a finite limit thanks to its decreasing behavior. As a consequence, the desired limit value is $a = -12$, as previously calculated.

Problem 2. Find *all* values of the parameter $\alpha \in \mathbb{R}$ such that the series

$$\sum_{k=1}^{\infty} (-1)^k \frac{3^k \alpha^{2k}}{k+1}$$

is convergent.

SOLUTION

Let a_k be the general term of the given series. Then

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(-1)^{k+1} 3^{k+1} \alpha^{2k+2}}{k+2} \frac{k+1}{(-1)^k 3^k \alpha^{2k}}\right| = 3\alpha^2 \frac{k+1}{k+2} \longrightarrow 3\alpha^2 \quad \text{as } k \to \infty\,.$$

Hence, thanks to the *ratio test*, we can say that the series converges if $3\alpha^2 < 1$, namely for $-\sqrt{3}/3 < \alpha < \sqrt{3}/3$ (it diverges for $\alpha > \sqrt{3}/3$ or $\alpha < -\sqrt{3}/3$). On the other hand, if $\alpha = \pm\sqrt{3}/3$, the series is

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k+1}\,,$$

which is convergent by the *Leibniz test* (indeed, it is an alternating series, where $1/(k+1)$ is positive, decreasing, and tending to zero as $k \to \infty$).

Problem 3. Approximate the value

$$\sqrt[3]{1.1}$$

by a polynomial of degree 2 and find an appropriate *upper bound* for the involved error.

SOLUTION

Note that

$$\sqrt[3]{1.1} = (1+0.1)^{1/3}$$

can be calculated by evaluating the function $f(x) = (1+x)^{1/3}$ at $x = 0.1$. Thanks to the *Taylor Theorem*, such function can be expressed as

$$\begin{aligned}(1+x)^{1/3} &= 1 + \frac{1}{3}x + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2}x^2 + R_2(x) \\ &= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + R_2(x)\,,\end{aligned}$$

where the remainder $R_2(x)$ is

$$R_2(x) = \frac{f'''(c)}{3!}\,x^3\,,$$

with

$$f'''(c) = \frac{10}{27}(1+c)^{-8/3}\,, \quad \text{for } c \in (0,x) \text{ when } x > 0\,.$$

Thus, we can approximate the desired value as

$$\sqrt[3]{1.1} \approx 1 + \frac{0.1}{3} - \frac{(0.1)^2}{9} \approx 1.03222$$

and find an *upper bound* for the involved error as

$$|R_2\,(0.1)| = \left|\frac{10}{27\cdot 3!}\,\frac{1}{(1+c)^{8/3}}\,(0.1)^3\right| < \frac{10}{27\cdot 6}\,(0.1)^3 \approx 6\cdot 10^{-5}\,,$$

where the inequality is obtained recalling that $c \in (0, 0.1)$.

Problem 4. Given the function

$$f(x) = x^x\,,$$

find the exact number of real solutions of the equation $f(x) = 2$ in the interval $[1, +\infty)$.

SOLUTION

In the interval $[1, +\infty)$, the function $f(x)$ is continuous and differentiable, since it is defined as

$$f(x) = x^x = e^{x \ln(x)}\,.$$

On the other hand, the given equation can be written as

$$f(x) - 2 = 0 \iff F(x) = 0\,,$$

where $F(x) = f(x) - 2$ is also continuous and differentiable in $[1, +\infty)$, with

$$F'(x) = e^{x \ln(x)} \left(\ln(x) + 1\right) > 0\,.$$

Thus, $F(x)$ is increasing in $[1, +\infty)$, $F(1) = -1 < 0$, and $\lim_{x\to\infty} F(x) = +\infty$. Hence, we can conclude that the given equation has a unique real solution in the indicated interval.