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CALCULUS – EVALUATION TEST 5 (solutions)

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Problem 1. Consider the monotone sequence $(a_n)_{n\in\mathbb{N}}$ defined by the *recursive* formula

$$a_1 = 0\,; \qquad a_n = \sqrt{a_{n-1}+20}\,, \quad \text{with } n \geq 2\,.$$

Prove that the sequence is bounded and calculate $\lim\limits_{n\to\infty} a_n$.

SOLUTION

Let us suppose that the sequence has a finite limit, say $\lim_{n\to\infty} a_n = a \in \mathbb{R}$. Then, as $n \to \infty$ in both sides of the *recursive* formula, we get

$$a = \sqrt{a+20} \quad\Longrightarrow\quad a^2 = a+20 \quad\Longrightarrow\quad a = -4, 5\,,$$

where the value $a = -4$ must be discarded, since the sequence is increasing with positive terms. Hence, $a = 5$ is the unique candidate to be the value of the limit.

Now, let us prove by the *principle of induction* that the sequence is bounded above by 5, namely $0 \leq a_n \leq 5$ for all $n \in \mathbb{N}$. First, this property holds for $n = 1$, as $0 \leq a_1 = 0 \leq 5$. Then, assuming that $0 \leq a_k \leq 5$ for $n = k \in \mathbb{N}$, we get (for $n = k+1$)

$$0 \leq a_{k+1} = \sqrt{a_k + 20} \leq \sqrt{5+20} = 5\,.$$

Hence, the sequence is bounded and has a finite limit thanks to its increasing behavior. As a consequence, the limit value is $a = 5$, as previously calculated.

Problem 2. Calculate

$$\lim_{x\to 0} \frac{\sqrt{1+x^2}+2x+x\arctan(x)-e^{3x}\big[1-\ln(1+x)\big]}{x\big[\ln(1+5x)+\arctan(2x)\big]}\,.$$

SOLUTION

In the given limit, we have $x \to 0$, hence we can approximate all involved elementary functions by the corresponding Maclaurin polynomials of a suitable degree, namely

$$\lim_{x\to 0} \frac{1+\frac{1}{2}x^2+o(x^2)+2x+x\big[\,x+o(x)\,\big]-\big[\,1+3x+\frac{9}{2}x^2+o(x^2)\,\big]\,\big[\,1-x+\frac{1}{2}x^2+o(x^2)\,\big]}{x\big[\,5x+o(x)+2x+o(x)\,\big]}\,.$$

Finally, after simplifying the previous expression and retaining all terms with powers of x up to 2, we obtain

$$\lim_{x\to 0} \frac{-\frac{1}{2}x^2+o(x^2)}{7x^2+o(x^2)} = -\frac{1}{14}\,.$$

Problem 3. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \int_0^x e^{1-\sqrt{1+t^2}}\, dt\,.$$

- Prove that $f(x)$ is *odd*.
- Prove that $f(x)$ is *increasing*.
- Find the Taylor polynomial of degree 3 about $x_0 = 0$ for $f(x)$.
- Study the convergence of the *improper* integral

$$\lim_{x\to+\infty} f(x) = \int_0^\infty e^{1-\sqrt{1+t^2}}\, dt\,.$$

SOLUTION

- The function $f(x)$ is *odd* since

$$f(-x) = \int_0^{-x} e^{1-\sqrt{1+t^2}}\, dt = -\int_0^x e^{1-\sqrt{1+u^2}}\, du = -f(x)\,,$$

where the second identity is obtained by the change of variable $u = -t$.

- The function $f(x)$ is *increasing* as

$$f'(x) = e^{1-\sqrt{1+x^2}} > 0$$

for all $x \in \mathbb{R}$. Note that $f'(x)$ has been calculated thanks to the Fundamental Theorem of Calculus.

- The desired Taylor polynomial is

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3\,.$$

In addition, we have $f(0) = 0$ and $f'(0) = 1$ (see the previous item). After calculating the second and third derivatives of $f(x)$, we obtain $f''(0) = 0$ and $f'''(0) = -1$. Thus, we can finally write

$$P_3(x) = x - \frac{1}{6}x^3\,.$$

- We have

$$\lim_{t\to\infty} \frac{e^{1-\sqrt{1+t^2}}}{e^{-t}} = e > 0 .$$

Hence, being $\int_0^\infty e^{-t}\, dt$ convergent, the limit comparison test concludes that $\int_0^\infty e^{1-\sqrt{1+t^2}}\, dt$ converges as well.

Problem 4. Calculate

$$\int \frac{\sin(x^{1/3})}{x^{1/3}}\,dx$$

in terms of elementary functions.

SOLUTION

Let us consider the change of variable

$$u = x^{1/3}; \quad du = \frac{1}{3}x^{-2/3}dx \implies dx = 3u^2du\,.$$

Then, the given integral can be written as

$$\int \frac{\sin(x^{1/3})}{x^{1/3}}\,dx = 3\int u\,\sin(u)\,du = -3u\,\cos(u) + 3\sin(u) + k\,,$$

where k is an arbitrary constant and the last identity has been obtained integrating by parts. Finally, in terms of the original variable, we get

$$\int \frac{\sin(x^{1/3})}{x^{1/3}}\,dx = -3x^{1/3}\,\cos(x^{1/3}) + 3\sin(x^{1/3}) + k\,,$$

with $k \in \mathbb{R}$.