OpenCourseWare

CALCULUS – EVALUATION TEST 6 (solutions)

Filippo Terragni, Eduardo Sánchez Villaseñor, Manuel Carretero Cerrajero

Problem 1. Consider the sequence of real numbers defined by the general term

$$a_n = \frac{n + \sin(\pi n/2)}{3n+5}, \quad \text{with } n = 1, 2, 3, \dots .$$

Prove whether the sequence is bounded, monotone, and convergent.

SOLUTION

First, note that

$$(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{4}, \frac{2}{11}, \frac{1}{7}, \frac{4}{17}, \dots\right),$$

with $a_1 = \frac{1}{4} > a_2 = \frac{2}{11} > a_3 = \frac{1}{7} < a_4 = \frac{4}{17}$. Hence, the sequence is not monotone.

Moreover, for all $n \in \mathbb{N}$, we can write

$$0 \leq \frac{n-1}{3n+5} \leq \frac{n + \sin(\pi n/2)}{3n+5} \leq \frac{n+1}{3n+5} \leq 1,$$

which implies that the sequence is bounded. Finally, using the previous inequalities and applying the *sandwich theorem*, we can conclude that the sequence is convergent, with $a_n \to 1/3$ as $n \to \infty$.

Problem 2. Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^2+5}\,, \qquad \sum_{n=1}^{\infty} \frac{\ln(n)}{n}\,.$$

SOLUTION

Let a_n be the general term of the first series. Then, considering $b_n = 1/n^2$, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\arctan(n)}{n^2+5}\, n^2 = \frac{\pi}{2} > 0\,.$$

Hence, by the limit comparison test, the first series is convergent since $\sum_{n=1}^{\infty} 1/n^2$ is so (note that both considered series have positive terms).

Regarding the second series, for all $n \geq 3$, we can write

$$\frac{\ln(n)}{n} \geq \frac{1}{n} > 0\,.$$

Hence, by the comparison test, the second series is divergent since $\sum_{n=1}^{\infty} 1/n$ is so.

Problem 3. Determine the exact number of real solutions of the equation $\cos(x) = x$.

SOLUTION

Let us consider the function $f(x) = x - \cos(x)$, which is continuous and differentiable in $\mathbb{R}$. In addition, we have that $f'(x) = 1 + \sin(x) \geq 0$ for all $x \in \mathbb{R}$ (namely, $f(x)$ is increasing) and

$$\lim_{x \to +\infty} f(x) = +\infty, \qquad \lim_{x \to -\infty} f(x) = -\infty.$$

As a consequence, the graph of $f(x)$ crosses the x-axis at only one point, namely the given equation $f(x) = 0$ has a unique real solution.

Problem 4. Calculate the angle formed by the tangent lines from the right and from the left, at $x_0 = 0$, to the graph of the function

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x}, & \text{if } x < 0, \\ xe^x + 1, & \text{if } x \geq 0. \end{cases}$$

SOLUTION

The slope of the tangent line from the right, at $x_0 = 0$, is

$$f'(0^+) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} e^x = 1.$$

On the other hand, the slope of the tangent line from the left, at $x_0 = 0$, is

$$f'(0^-) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{\sin(x) - x}{x^2} = 0.$$

Hence, the desired angle is $\pi/4$.

Problem 5. Find the family of polynomials $\mathrm{P}(x)$ such that

$$\lim_{x\to 0} \frac{\sqrt{1-x^4} - \mathrm{P}(x)}{x^7} = 0\,.$$

SOLUTION

Thanks to the *Taylor Theorem*, we can write $\sqrt{1-z} = 1 - z/2 - z^2/8 + o(z^2)$, hence

$$\sqrt{1-x^4} = 1 - \frac{x^4}{2} + o(x^7)\,.$$

Then, if we need that

$$\lim_{x\to 0} \frac{\sqrt{1-x^4} - \mathrm{P}(x)}{x^7} = \lim_{x\to 0} \frac{1 - \frac{x^4}{2} + o(x^7) - \mathrm{P}(x)}{x^7} = 0\,,$$

we can choose $\mathrm{P}(x)$ in the family of polynomials of the form

$$\mathrm{P}(x) = 1 - \frac{x^4}{2} + a_8 x^8 + a_9 x^9 + \ldots + a_n x^n\,,$$

where $a_8, \ldots, a_n$ are arbitrary real coefficients (with $n \in \mathbb{N}$).

Problem 6. Calculate

$$\lim_{x\to 0} \frac{1}{x^3}\int_0^x t^2\cos(t^2)\,dt\,.$$

SOLUTION

According to the *Taylor Theorem*, we can write

$$t^2\cos(t^2) \;=\; t^2\left(1-\frac{t^4}{2}+\frac{t^8}{24}+o(t^8)\right) \;=\; t^2-\frac{t^6}{2}+\frac{t^{10}}{24}+o(t^{10})\,.$$

Hence

$$\int_0^x t^2\cos(t^2)\,dt \;=\; \int_0^x \left(t^2-\frac{t^6}{2}+\frac{t^{10}}{24}+o(t^{10})\right)dt \;=\; \frac{x^3}{3}-\frac{x^7}{14}+o(x^7)\,.$$

Finally, we can calculate the desired limit as

$$\lim_{x\to 0}\frac{1}{x^3}\int_0^x t^2\cos(t^2)\,dt \;=\; \lim_{x\to 0}\frac{\frac{x^3}{3}-\frac{x^7}{14}+o(x^7)}{x^3} \;=\; \frac{1}{3}\,.$$

Problem 7. Study the convergence of the *improper* integral

$$\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx\,.$$

SOLUTION

We can write

$$\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx = \int_0^1 e^{-x}\,x^{-1/2}\,dx + \int_1^\infty e^{-x}\,x^{-1/2}\,dx\,. \tag{1}$$

Regarding the first integral in the right-hand-side of eq.(1), we have that

$$\lim_{x\to 0^+} \frac{e^{-x}\,x^{-1/2}}{x^{-1/2}} = 1\,.$$

Hence, by the limit comparison test, the integral converges since $\int_0^1 x^{-1/2}\,dx$ is convergent (all involved functions are positive for $x \in (0,1]$). Regarding the second integral in the right-hand-side of eq.(1), we have that

$$\lim_{x\to\infty} \frac{e^{-x}\,x^{-1/2}}{e^{-x}} = 0\,.$$

Hence, by the limit comparison test, the integral converges since $\int_1^\infty e^{-x}\,dx$ is convergent (all involved functions are positive for $x \geq 1$). As a consequence, the given improper integral converges since it is the sum of two convergent improper integrals.