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CALCULUS – EVALUATION TEST 7 (solutions)

Filippo Terragni, Eduardo Sánchez Villaseñor, Manuel Carretero Cerrajero

Problem 1. Consider the sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ where

$$a_n = \sqrt{n}\,\frac{2\cos(\pi(n+1)/2)}{1+n}, \quad \text{with } n = 1, 2, 3, \ldots .$$

(a) Study whether the sequence is monotone and bounded.

(b) Calculate $\lim_{n\to\infty} a_n$.

SOLUTION

(a) Note that $a_1 = -1$, $a_2 = 0$, $a_3 = \sqrt{3}/2$, $a_4 = 0$. Hence, the sequence is not monotone. On the other hand, we can write

$$|a_n| \le \frac{2\sqrt{n}}{1+n} \le \frac{2\sqrt{n}}{n} = \frac{2}{\sqrt{n}} \le 2$$

for all $n \in \mathbb{N}$, namely the sequence is bounded.

(b) The desired limit can be calculated as

$$\lim_{n\to\infty} 2\cos(\pi(n+1)/2)\,\frac{\sqrt{n}}{1+n} = 0\,,$$

being the product of a bounded term and $\sqrt{n}/(1+n)$, which tends to zero as $n \to \infty$.

Problem 2. Find *all* values of the parameter $\alpha \in \mathbb{R}$ such that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{(2\alpha)^{3n}}{7^n \sqrt[3]{n^2+n}}$$

is convergent.

SOLUTION

Let a_n be the general term of the series. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(2\alpha)^{3n+3}}{7^{n+1}\sqrt[3]{(n+1)^2+n+1}} \frac{7^n \sqrt[3]{n^2+n}}{(-1)^n (2\alpha)^{3n}}\right| = \frac{8}{7}|\alpha|^3 \frac{\sqrt[3]{n^2+n}}{\sqrt[3]{n^2+3n+2}} \longrightarrow \frac{8}{7}|\alpha|^3$$

as $n \to \infty$. Hence, thanks to the *ratio test*, the series converges if $8|\alpha|^3/7 < 1$, namely $|\alpha| < \sqrt[3]{7}/2$. On the other hand, the series is divergent if $8|\alpha|^3/7 > 1$, namely $|\alpha| > \sqrt[3]{7}/2$. For $\alpha = \sqrt[3]{7}/2$, the series is

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt[3]{n^2+n}},$$

which is convergent thanks to the *Leibniz test* (indeed, it is an alternating series, where $1/\sqrt[3]{n^2+n}$ is positive, decreasing, and tending to zero as $n \to \infty$). For $\alpha = -\sqrt[3]{7}/2$, the series is

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+n}},$$

which diverges by the limit comparison test with the divergent series $\sum_{n=1}^{\infty} 1/n^{2/3}$. Thus, we can conclude that the given series is convergent for $-\sqrt[3]{7}/2 < \alpha \leq \sqrt[3]{7}/2$.

Problem 3. Approximate the value

$$\ln\left(\frac{3}{2}\right)$$

by a polynomial of suitable degree such that the involved error is smaller than 10^{-2}.

SOLUTION

Note that

$$\ln\left(\frac{3}{2}\right) = \ln\left(1+\frac{1}{2}\right)$$

can be calculated by evaluating the function $f(x) = \ln(1+x)$ at $x = 1/2$. Thanks to the *Taylor Theorem*, this function can be expressed as

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1}\frac{x^n}{n} + R_n(x)\,,$$

where the remainder $R_n(x)$ is

$$R_n(x) = (-1)^n \frac{1}{(n+1)(1+c)^{n+1}}\, x^{n+1}\,,$$

with $n \in \mathbb{N}$ and $c \in (0, x)$ when $x > 0$. Hence, at $x = 1/2$, we can find an *upper bound* for the error involved in the approximation as

$$\left|R_n\left(\frac{1}{2}\right)\right| = \frac{1}{2^{n+1}(n+1)(1+c)^{n+1}} < \frac{1}{2^{n+1}\,(n+1)}\,,$$

where the inequality is obtained by recalling that $c \in (0, 1/2)$. Finally, after imposing

$$\frac{1}{2^{n+1}\,(n+1)} < 10^{-2},$$

we can deduce that the degree of the considered Maclaurin polynomial must be $n \geq 4$. Using $n = 4$, the desired approximation is

$$\ln\left(\frac{3}{2}\right) \approx 0.5 - \frac{(0.5)^2}{2} + \frac{(0.5)^3}{3} - \frac{(0.5)^4}{4}\,,$$

within an error smaller than 10^{-2}.

Problem 4. Consider the function

$$f(x) = \begin{cases} \sqrt{1-x}\,\arctan\left(\dfrac{1}{x}\right), & \text{if } 0 < x \leq 1, \\ 0, & \text{if } x = 0, \\ \dfrac{\cos(x) - 1}{x}, & \text{if } x < 0. \end{cases}$$

(a) Study whether $f(x)$ is continuous at $x = 0$.

(b) Find the exact number of real solutions of the equation $f(x) = -1$ in the interval $(0, 1/2]$.

SOLUTION

(a) We have

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt{1-x}\,\arctan\left(\frac{1}{x}\right) = \frac{\pi}{2},$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\cos(x)-1}{x} = 0.$$

Since the values of these lateral limits are distinct, $\lim_{x\to 0} f(x)$ does not exist and the function $f(x)$ is not continuous at $x = 0$.

(b) For $x \in (0, 1/2]$, the given equation can be written as

$$F(x) \equiv f(x) + 1 = 0 \quad\Longrightarrow\quad F(x) = \sqrt{1-x}\,\arctan\left(\frac{1}{x}\right) + 1 = 0.$$

Noting that $F(x)$ is continuous and differentiable in the considered interval (as $f(x)$ is so), we can calculate its derivative as

$$F'(x) = -\left[\frac{\arctan\left(\frac{1}{x}\right)}{2\sqrt{1-x}} + \frac{\sqrt{1-x}}{x^2+1}\right].$$

Hence, $F'(x) < 0$ and $F(x)$ is decreasing in $(0, 1/2]$, with $\lim_{x\to 0^+} F(x) = \pi/2 + 1 > 0$ (see the previous item) and $F(1/2) = \arctan(2)/\sqrt{2} + 1 > 0$. As a consequence, the equation $f(x) = -1$ has no real solution in the indicated interval.