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CALCULUS – EVALUATION TEST 8 (solutions)

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Problem 1. Solve the following issues.

(a) Consider the *recursive* sequence $(a_n)_{n\in\mathbb{N}}$ defined as

$$a_1 = \frac{1}{2}\,; \qquad a_{n+1} = (a_n)^2 + \frac{4}{25}\,, \quad \text{with } \; n \geq 1\,.$$

Then, prove that $\lim\limits_{n\to\infty} a_n$ exists and calculate its value.

(b) Study the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan\left(n^4\right)}{\sqrt{n^4+1}}$.

SOLUTION

(a) Let us apply the *principle of induction* to prove that the sequence is bounded below for all $n \in \mathbb{N}$. First, we have $a_1 = 1/2 > 1/5$. Then, assuming that $a_k > 1/5$ for $n = k \in \mathbb{N}$, we get (for $n = k+1$)

$$a_{k+1} = (a_k)^2 + \frac{4}{25} > \left(\frac{1}{5}\right)^2 + \frac{4}{25} = \frac{1}{5}\,.$$

On the other hand, by means of the same principle, we can prove that the sequence is decreasing for all $n \in \mathbb{N}$. First, we have $a_1 = 1/2 > a_2 = 41/100$. Then, assuming that $a_k > a_{k+1}$ for $n = k \in \mathbb{N}$, we get (for $n = k+1$)

$$a_{k+1} = (a_k)^2 + \frac{4}{25} > (a_{k+1})^2 + \frac{4}{25} = a_{k+2}\,.$$

Thus, the sequence is bounded below and decreasing, hence convergent. Now, let $\lim_{n\to\infty} a_n = L \in \mathbb{R}$. Then, if $n \to \infty$ in both sides of the *recursive* formula, we get $L = L^2 + 4/25$, namely $L = 1/5$ or $L = 4/5$. As the sequence is decreasing and $a_1 = 1/2$, the limit value must be $L = 1/5$.

(b) The series converges by the limit comparison test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Problem 2. Consider

$$f(x) \;=\; \begin{cases} e^{\frac{1}{x}} + \beta\, x\,, & \text{if } \; x < 0\,, \\ \beta \sin(x) - \dfrac{1}{2}\sin^2(x)\,, & \text{if } \; x \geq 0\,, \end{cases}$$

where $\beta \in \mathbb{R}$ is a parameter.

(a) Find for which values of β the function $f(x)$ is differentiable in $\mathbb{R}$.

(b) Find (if any) the values of β such that the tangent line to the graph of $f(x)$ at $x_0 = 0$ is parallel to the line with equation $y = 3x - 7$.

SOLUTION

(a) If $x \neq 0$, $f(x)$ is defined in terms of differentiable elementary functions, hence it is differentiable independently of the value of β. Moreover, $f(x)$ is also differentiable at $x = 0$, for all $\beta \in \mathbb{R}$, since

$$f'(0^+) \;=\; \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} \;=\; \lim_{x\to 0^+} \frac{\beta \sin(x) - \frac{1}{2}\sin^2(x)}{x} \;=\; \beta\,,$$

$$f'(0^-) \;=\; \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} \;=\; \lim_{x\to 0^-} \frac{e^{\frac{1}{x}} + \beta\, x}{x} \;=\; \beta\,.$$

(b) Being $f'(0) = \beta$ (see the previous item), the desired value is $\beta = 3$.

Problem 3. Approximate the value $\sqrt[3]{1010}$ by means of the Taylor polynomial of degree 3 for the function $f(x) = \sqrt[3]{x}$ about $a = 1000$. Then, find an *upper bound* for the involved error.

SOLUTION

Thanks to the *Taylor Theorem*, we can write (at $x = 1010$)

$$\sqrt[3]{1010} \approx 10 + \frac{1}{3 \cdot 10^2}(1010-1000) - \frac{2}{2 \cdot 9 \cdot 10^5}(1010-1000)^2 + \frac{10}{6 \cdot 27 \cdot 10^8}(1010-1000)^3 .$$

On the other hand, the involved approximation error can be upper bounded as

$$|R_3(1010)| = \left| \frac{80}{4!\, 81\, c^{11/3}}(1010-1000)^4 \right| \le \frac{80}{4!\, 81}\, 10^{-7} ,$$

where $R_3(x)$ is the remainder and the inequality is obtained as $1000 < c < 1010$.

Problem 4. Let $F(x) = \int_0^{e^{-x}} \frac{1}{\ln(t)}\, dt$.

(a) Find the global maximum and minimum of $F(x)$ in the interval $x \in [1,2]$.

(b) Calculate $\lim_{x\to+\infty} x\, F(x)$.

SOLUTION

(a) Thanks to the Fundamental Theorem of Calculus, we can write $F'(x) = e^{-x}/x > 0$ for all $x \in [1,2]$, hence $F(x)$ is strictly increasing in that interval. Thus, the global maximum and minimum of $F(x)$ are attained at $x = 2$ and $x = 1$, respectively.

(b) We have

$$\lim_{x\to+\infty} x\, F(x) = \lim_{x\to+\infty} \frac{F(x)}{1/x} = \lim_{x\to+\infty} \frac{e^{-x}/x}{-1/x^2} = \lim_{x\to+\infty} -\frac{x}{e^x} = 0\,,$$

where the l'Hôpital's rule has been applied in the second identity.

Problem 5. Calculate the following indefinite integrals.

$$\text{(a)} \quad \int x^2\, e^{-3x}\, dx\,. \qquad \text{(b)} \quad \int \frac{x}{x^2 - x + 1}\, dx\,.$$

SOLUTION

(a) $\displaystyle\int x^2\, e^{-3x}\, dx \;=\; \text{[integration by parts]} \;=\; -\frac{1}{3}e^{-3x}\left(x^2 + \frac{2}{3}x + \frac{2}{9}\right) + c \quad (\text{with } c \in \mathbb{R})\,.$

(b) $\displaystyle\int \frac{x}{x^2 - x + 1}\, dx \;=\; \text{[integral of a rational function; partial fraction decomposition]} \;=$

$$= \frac{1}{2}\ln|x^2 - x + 1| + \frac{\sqrt{3}}{3}\arctan\left(\frac{2x - 1}{\sqrt{3}}\right) + c \quad (\text{with } c \in \mathbb{R})\,.$$

Problem 6. Consider the *improper* integral $\int_0^\infty (x+1)^p \, e^{-x^2} dx$, with $p \in \mathbb{N}$.

(a) Study its convergence for $p = 2$.

(b) Knowing that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$, calculate its value for $p = 1$.

SOLUTION

(a) The given integral is convergent by the limit comparison test with $\int_0^\infty e^{-x} \, dx$.

(b) Using the definition of improper integral (of the first kind), we can write

$$\int_0^\infty (x+1) \, e^{-x^2} dx \;=\; \lim_{b\to\infty} \int_0^b (x+1) \, e^{-x^2} dx \;=\; \lim_{b\to\infty} \int_0^b x \, e^{-x^2} dx \;+\; \int_0^\infty e^{-x^2} dx$$

$$= \frac{1}{2} \lim_{b\to\infty} \left[1 - e^{-b^2}\right] + \frac{\sqrt{\pi}}{2} \;=\; \frac{\sqrt{\pi}+1}{2} \, ,$$

where $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ has been used.