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CALCULUS – EVALUATION TEST 9 (solutions)

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Problem 1. Let $(x_n)_{n\in\mathbb{N}}$ be the *recursive* sequence defined as $x_{n+1} = \sqrt{2x_n+3}$, for $n \geq 1$, with $x_1 = 1$. Prove that $\lim\limits_{n\to\infty} x_n$ exists and calculate its value.

SOLUTION

Let us prove by the *principle of induction* that the sequence is increasing. For $n = 1$, we have $x_2 = \sqrt{5} \geq 1 = x_1$. Then, assuming that $x_{k-1} \leq x_k$ for $n = k \in \mathbb{N}$ ($k \geq 2$), we get (for $n = k+1$)

$$2x_{k-1} \leq 2x_k \implies 2x_{k-1}+3 \leq 2x_k+3 \implies \sqrt{2x_{k-1}+3} \leq \sqrt{2x_k+3} \implies x_k \leq x_{k+1}\,.$$

Now, we can prove by the same principle that the sequence is bounded above by 3. If $n = 1$, we have $x_1 = 1 \leq 3$. In addition, assuming that $x_k \leq 3$ for $n = k \in \mathbb{N}$, we get (for $n = k+1$)

$$2x_k \leq 6 \implies 2x_k+3 \leq 9 \implies \sqrt{2x_k+3} = x_{k+1} \leq \sqrt{9} = 3\,.$$

Hence, the sequence is convergent, namely $\lim_{n\to\infty} x_n = L \in \mathbb{R}$. Now, as $n \to \infty$ in both sides of the *recursive* formula, we get $L = \sqrt{2L+3}$, namely $L = -1$ or $L = 3$. Since the sequence is increasing, the limit value must be $L = 3$.

Problem 2. Study the convergence of the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3 + n^{-3}}{n^a}$$

depending on the value of the parameter $a \in \mathbb{N}$.

SOLUTION

If $a = 1, 2, 3$, the series is divergent since its general term does not tend to zero as $n \to \infty$. If $a \geq 4$, the series is convergent by the *Leibniz test*.

Problem 3. Consider the function

$$f(x) \;=\; \begin{cases} 6\sqrt{x} - x\sqrt{x}\,, & \text{if} \;\; 0 \le x \le 4\,, \\ (x-4)\, e^{16-x^2} \left[2 - \beta \,\sin\left(\dfrac{\pi}{8}\, x\right) \right] + 4\,, & \text{if} \;\; x > 4\,, \end{cases}$$

where $\beta \in \mathbb{R}$ is a parameter.

(a) Find the value of β such that $f(x)$ is differentiable at $x = 4$.

(b) Find the global maximum and minimum of $f(x)$ in the interval $[0, 4]$, if they exist.

SOLUTION

(a) Note that

$$\lim_{x \to 4^+} \frac{f(x) - f(4)}{x - 4} \;=\; \lim_{x \to 4^+} e^{16-x^2} \left[2 - \beta \,\sin\left(\frac{\pi}{8}\, x\right) \right] \;=\; 2 - \beta\,,$$

$$\lim_{x \to 4^-} \frac{f(x) - f(4)}{x - 4} \;=\; \lim_{x \to 4^-} \frac{6\sqrt{x} - x\sqrt{x} - 4}{x - 4} \;=\; -\frac{3}{2}\,.$$

By definition of differentiability of a real function at a point, the values of these lateral limits should coincide. Hence, $f(x)$ is differentiable at $x = 4$ if $\beta = 7/2$.

(b) In the closed and bounded interval $[0, 4]$, the function $f(x)$ is continuous, hence its global maximum and minimum exist. Moreover, the only critical point in $(0, 4)$ is $x = 2$, which is calculated by imposing

$$f'(x) \;=\; \frac{6 - 3x}{2\sqrt{x}} \;=\; 0\,.$$

Thus, the global extrema of $f(x)$ must be attained at certain points among $x = 2$, $x = 0$, and $x = 4$. Observing that $f(2) = 4\sqrt{2}$, $f(0) = 0$, and $f(4) = 4$, we can conclude that 0 is the global minimum (attained at $x = 0$) and $4\sqrt{2}$ is the global maximum (attained at $x = 2$).

Problem 4. Let $F(x) = \int_0^{x^2} \cos(\sqrt{t})\, dt$.

(a) Using the Maclaurin polynomial of degree 2 for $F(x)$, approximate the value

$$\int_0^{0.01} \cos(\sqrt{t})\, dt\,.$$

(b) Calculate $\lim_{x\to 0} \frac{F(x) - x^2}{x^4}$.

SOLUTION

(a) We have $F(x) \approx P_2(x) = x^2$ and we can write

$$\int_0^{0.01} \cos(\sqrt{t})\, dt \;\approx\; P_2(0.1) \;=\; 0.01\,.$$

(b) We have $F(x) \approx P_4(x) = x^2 - \frac{x^4}{4}$. Then, we get

$$\lim_{x\to 0} \frac{F(x) - x^2}{x^4} \;=\; \lim_{x\to 0} \frac{\left[x^2 - \frac{x^4}{4} + o(x^4)\right] - x^2}{x^4} \;=\; -\frac{1}{4}\,.$$

Problem 5. Calculate the following indefinite integrals.

$$\text{(a)} \int e^x \cos(2x)\, dx\,. \qquad \text{(b)} \int \frac{2x-3}{x^2+2x+2}\, dx\,.$$

SOLUTION

(a) Integration by parts: $\displaystyle\int e^x \cos(2x)\, dx \;=\; \frac{1}{5} e^x \Big[\, \cos(2x) + 2\sin(2x) \Big] + c \quad (c \in \mathbb{R})\,.$

(b) Integral of a rational function: $\displaystyle\int \frac{2x-3}{x^2+2x+2}\, dx \;=\; \ln|x^2+2x+2| - 5\arctan(x+1) + c$ $(c \in \mathbb{R})\,.$

Problem 6. Study for which values of the parameter $k \in \mathbb{R}$ the *improper* integral

$$\int_0^2 \frac{1}{x^k}(1-x)^{k-1}dx$$

is convergent.

SOLUTION

Let $f(x)$ be the function to be integrated and $\lambda \in (0,1)$. Then, the given *improper* integral can be written as

$$\int_0^2 f(x)\,dx = \int_{0^+}^{\lambda} f(x)\,dx + \int_{\lambda}^{1^-} f(x)\,dx + \int_{1^+}^{2} f(x)\,dx\,.$$

Hence, by the limit comparison test, we can prove that $\int_{0^+}^{\lambda} f(x)\,dx$ converges if $k < 1$, while $\int_{\lambda}^{1^-} f(x)\,dx$ and $\int_{1^+}^{2} f(x)\,dx$ converge if $k > 0$. Thus, the proposed integral converges for $k \in (0,1)$.