

COMMUNICATION THEORY

LAB 1: STOCHASTIC PROCESSES

Objectives

In this activity the student will learn to:

- Generate and study random variables and stochastic processes in MATLAB.
- Estimate the mean, the autocorrelation function and the power spectral density of the processes from realizations.
- Draw conclusions about the properties of random processes.

Exercise 1

The file `P1_Demo_LGN.m` generates several realizations of Gaussian random variables with unit mean and variance and makes histograms of the average of the realizations of several random variables.

a) For independent Gaussian random variables, of unit mean and variance:

- I) Obtain and compare the averages of 10,000 realizations and the histograms of the averages of N independent variables, for

$$N \in \{1, 2, 5, 10, 50, 100\}.$$

- II) Calculate the mean and variance of the averages from the previous section.
- III) Explain the behavior observed when averaging realizations of several independent random variables with the given characteristics.

b) Repeat the previous section for uniform random variables between 0 and 2.

Exercise 2

Consider the stochastic process $X(t)$ associated with the random experiment of rolling a die and seeing which of the 6 faces is up. The 6 signals associated with each of the possible outputs of the experiment are

$$X(t, \lambda_i) = \frac{1}{2} + \sin(\omega_c t - \theta_i)$$

with $\omega_c = \frac{2\pi}{32}$ rad/s and with

$$\theta_i = (i - 1) \frac{2\pi}{6}, \text{ for } i \in \{1, 2, 3, 4, 5, 6\}$$

The file `P1_Demo_SP_Die.m` generates the 6 signals, represents them (in time and frequency, in this case it represents the module of the Fourier transform of each signal), and calculates

estimates of the mean, $m_X(t)$, and autocorrelation function, $R_X(t_1, t_2)$. These estimates are based on the definition of these statistical averages:

$$m_X(t) = E[X(t)], \quad R_X(t_1, t_2) = E[X(t_1)X(t_2)].$$

Furthermore, it represents the autocorrelation function $R_X(t + \tau, t)$ as a function of the separation between the time instants, τ , for $t = 1$ seconds.

a) For the random process $X(t)$ defined above:

- I) Represent the estimates of the mean $m_X(t)$ and the autocorrelation function $R_X(t_1, t_2)$ of the random process $X(t)$. Consider for t, t_1 and t_2 a range between 1 and 128 seconds in steps of 1 second.
- II) Represent the autocorrelation function $R_X(t + \tau, t)$ as a function of the separation τ , in a range between 0 and 100 seconds, for the following values of t

$$t \in \{1, 5, 10, 15, 20\} \text{ seconds.}$$

- III) Discuss what the average value of the squared modulus of the Fourier transforms of the 6 signals is like, and discuss the relationship of said average with the theoretical value of the power spectral density. Draw the estimate of the power spectral density that you would obtain from the Fourier transform of the 6 signals. Does this estimate adjust to the expected theoretical value?
- IV) Given the results, discuss the type of random process that $X(t)$ is.

b) Repeat the previous section for the next random process:

$$X(t, \lambda_i) = \frac{1}{2} \text{sen}(\omega_c t - \theta_i) + \frac{1}{2} \cos(\omega_c t)$$

with $\omega_c = \frac{2\pi}{32}$ rad/s and with

$$\theta_i = (i - 1) \frac{2\pi}{6}, \text{ for } i \in \{1, 2, 3, 4, 5, 6\}$$

Exercise 3

In this section we will study thermal noise. The MATLAB model, discrete time model, is carried out by modeling the process at different instants as independent realizations of a Gaussian random variable, with zero mean, and a variance equal to $N_0/2$ (an infinite variance cannot be simulated in discrete time, what you actually have are samples of the process, and not the complete process). For example, to model the samples in 100 time instants (whatever instants they may be), the samples, for a given variance, would be obtained as

```
>> n = sqrt(varNoise)*randn(1,100);
```

NOTE: Regarding its frequency representation, for this type of signals (taking into account the relationship between FT and DFT), the DFT result must be divided by the square root of the number of DFT samples (see Appendix B).

In the experiments, for simplicity, a unit variance will be considered.

a) From 50 realizations of thermal noise signals:

- I) Draw the signal $n(t)$ and the magnitude of its Fourier transform, $|N(j\omega)|$ for 5 of the total realizations.
- II) Represent the estimates of the mean $m_n(t)$ and the autocorrelation function $R_n(t_1, t_2)$ of the random process $n(t)$. Consider for t, t_1 and t_2 a range between 1 and 128 seconds in steps of 1 second.
- III) Represent the autocorrelation function $R_n(t+\tau, t)$ as a function of τ , in a range between 0 and 50 seconds, for the following values of t

$$t \in \{1, 5, 10, 15, 20\} \text{ seconds.}$$

- IV) Discuss what the average value of the squared modulus of the Fourier transforms of the 50 signals is like, and discuss the relationship of said average with the theoretical value of the power spectral density.
 - v) Given the results, discuss the type of random process that $n(t)$ is.
- b) Repeat the previous section using 1000 realizations of thermal noise signals, and discuss the differences with the previous case¹.

References

- *Communication Systems Engineering*. J.G. Proakis y M. Salehi. Prentice-Hall. 1994.

¹To facilitate the comparison, it is recommended to use the same scale in the figures to be compared, using the command `axis`.

Annex A : Some MATLAB functions

Function	Description
help	Matlab help function
doc	Extended help document, with examples
length	Length of a vector
size	Dimensions of a matrix or vector
rand	Random numbers from a uniform distribution
randn	Random numbers from a Gaussian distribution
mean	Sample mean
var	Sample variance
cov	Sample covariance
xcorr	Sample cross-correlation
fft	Discrete Fourier Transform
fftshift	Centers the FT around 0
sin,cos,exp	Mathematical functions sine, cosine and exponential
figure	Creates an empty figure
plot	Represents a vector or matrix in a 2D graph
hist	Calculates and represents a histogram
bar	Represents a histogram
imagesc	Represents an array as a 2D image
mesh	3D representation of a matrix in the form of a mesh
contour	3D representation of a matrix with contour lines
hold on	Allows you to paint several graphs on the same figure
xlabel,ylabel	Adds text to the axes of a figure
title	Add a title to the figure
legend	Add a legend to the figure
print	Print a figure by printer or to file

To find more information about these functions or any other functions, you can use the MATLAB `help` command.

Annex B : Relationship among samples of the DFT and the FT of a continuous signal

This appendix discusses some aspects related to the relationship between samples of the discrete Fourier transform of a discrete sequence of N samples and the Fourier transform of a continuous-time signal. These aspects are important for handling continuous-time signals with MATLAB.

Relationships among samples

The N -sample discrete Fourier transform (DFT) of a N -sample discrete signal is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}.$$

The Fourier transform (TF) of a continuous-time signal $x(t)$ is

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt,$$

while the Fourier transform of a discrete-time signal is

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}.$$

If the discrete-time signal $x[n]$ is obtained by sampling a continuous-time signal $x(t)$ at a sampling rate $f_s = \frac{1}{T_s}$ samples/s, so that

$$x[n] = x(nT_s),$$

The Fourier transform of this discrete signal $x[n]$ is related to the Fourier transform of the continuous-time signal $x(t)$ in the following way (result of the Sampling Theorem)

$$X(e^{j\omega}) = \frac{1}{T_s} \sum_k X\left(j\frac{\omega}{T_s} - j\frac{2\pi}{T}k\right).$$

Taking into account the previous expressions, it is evident that if we have a discrete-time signal $x[n]$ of duration N samples, the DFT of N samples of it provides samples (values at specific frequencies) of the TF of the discrete sequence, which is implicitly equivalent to scaled samples of the TF of the continuous-time signal. Specifically, it can be seen that

$$X[k] = X(e^{j\omega}) \Big|_{\omega=\frac{2\pi}{N}k} = \begin{cases} \frac{1}{T_s} X(j\omega) \Big|_{\omega=\frac{2\pi}{NT_s}k} & \text{if } k \leq \frac{N}{2} \\ \frac{1}{T_s} X(j\omega) \Big|_{\omega=\frac{2\pi}{NT_s}(k-N-1)} & \text{if } k > \frac{N}{2} \end{cases}$$

Which means that when using the DFT to try to represent the FT of a signal in continuous time, in addition to rearranging the DFT samples (the `fftshift` function allows you to rearrange the DFT samples so that the null frequency is in the central position), a scale factor T_s must be introduced.

Windowing effect

When the duration of the signal $x(t)$ is greater than $N \times T_s$, the N samples from which the DFT is obtained do not represent $x(t)$, but rather a truncated or windowed version from it

$$x_w(t) = x(t) w_{NT_s}(t), \quad \text{with } w_T(t) = \begin{cases} 1, & \text{if } t \leq T \\ 0 & \text{in other case} \end{cases}$$

since the N discrete-time samples only represent that part of the signal $x(t)$. The fact of multiplying in the time domain by a window of duration NT_s seconds (windowed), assumes that the frequency response is affected by the convolution with a *sinc* function, which is the Fourier transform of a rectangular time window (see Figure 1), which can distort the estimate of the FT from the DFT.

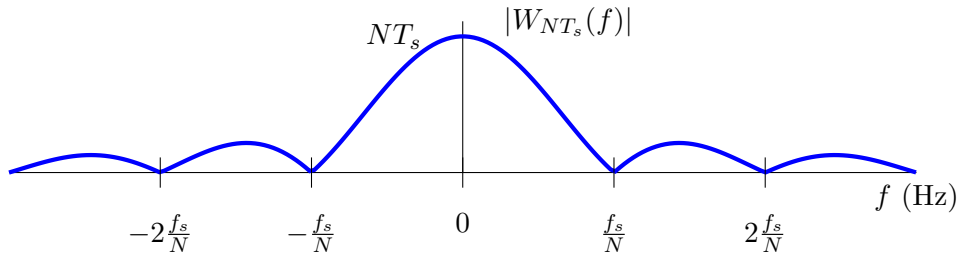


Figura 1: Frequency response of a window of NT_s seconds.

Spectrum of sinusoidal signals

When the frequency response of sinusoidal signals is to be represented, some peculiarities of this type of signals must be taken into account. The frequency response of a sinusoid of infinite length is Dirac deltas located at the frequency of the sinusoid. The particular properties of delta functions, and in particular their infinitesimal support, have some implications. On the one hand, normally the amplitude of the FT is the same whether the frequency response is being calculated in f (Hz) or $j\omega$ (rad/s), but for sinusoids this is not the case since the amplitude is $\frac{1}{2}$, or this quantity multiplied by 2π when working in $j\omega$ (a factor necessary to compensate for the fact that the delta support does not scale by that factor when going from f to $j\omega$). Furthermore, these functions do not comply with the scaling property of the Fourier transform, since a scaling of the time axis (change in frequency) does not affect the scaling in the amplitude of the transform, as happens with the rest of the functions (again this is due to the infinitesimal support of the delta in frequency; it can also be related to the fact that the power of the sinusoid is not modified when changing the frequency).

The fact that a sinusoid has infinite duration implies that the representation of its spectrum using DFT is affected by the windowing effect. The Fourier transform of a windowed sinusoid using a DFT of N samples and a sampling rate of f_s samples/s is plotted in Figure 2.

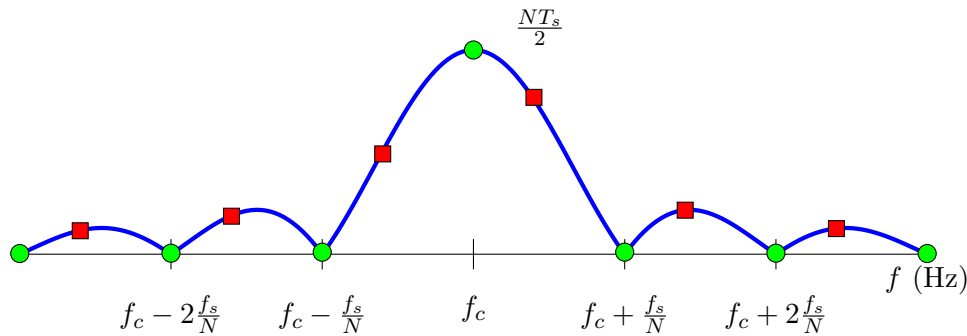


Figura 2: Frequency response of a windowed sinusoid in NT_s seconds.

It is important to note that the zero crossings of the **sinc** are spaced exactly the same amount as the frequencies at which the DFT samples of the FF are separated (remember that these frequencies are $\omega = \frac{f_s}{N}k$). This means that if the frequency of the sinusoid is a multiple of $\frac{f_s}{N}$, the frequencies where the FT is sampled will coincide with the zeros of the **sinc** and with the center of the main lobe (green circles in the figure), so the DFT result returns a delta. However, if this is not the case, it will be sampling at other points in the different lobes (red squares in the figure), which can distort the representation of the FT of the continuous time signal.