uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL CALCULUS

Degree in Applied Mathematics and Computation

Chapter 1

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General index

1

REAL VARIABLE FUNCTIONS

This first chapter is dedicated to basic concepts: the different kinds of numbers with the properties of the real numbers and the real variable functions.

Contents

1.1 The real line

1.1.1 Different kinds of numbers

A. **Natural numbers**: $\mathbb{N} = \{(0), 1, 2, 3, \dots\}$. Operations and properties:

$(+)$ associative, commutative, additive identity (0),

$(\cdot)$ associative, commutative, multiplicative identity (1),

$(+, \cdot)$ distributive.

> **Induction principle**
> The property P is true for every $n \in \mathbb{N}$ if
>
> 1. P is true for $n = 1$;
> 2. when P is true for n then it is also true for $n + 1$.

Example: 1.1. $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ can be proved using induction.

Also they are **unbounded above**. *Problem*: $x + 3 = 0$ does not have any solution $x \in \mathbb{N}$.

B. **Integers**: $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$. New property: $(+)$ inverse element.

Problem: $2x - 3 = 0$ does not have any solution $x \in \mathbb{Z}$.

C. **Rationals**: $\mathbb{Q} = \left\{\frac{p}{q} : p,\, q \in \mathbb{Z},\, q \neq 0\right\}$. New property: $(\cdot)$ inverse element (but not for 0).

They are **countable**, as $\mathbb{N}$ and $\mathbb{Z}$. *Problem*: $x^2 - 2 = 0$ does not have any solution $x \in \mathbb{Q}$.

D. **Reals:** $\mathbb{R} = \mathbb{Q} \cup \{\text{numbers with non periodic infinite figures}\}$ (in a first view). New and important property:

> **Completeness** (*continuity property*):
> Any non empty set in $\mathbb{R}$ which is bounded from above has a minimum upper bound.

Two other useful properties are:

Trichotomy property:
For any number a one and only one of the following holds:

$$a = 0, \qquad a > 0, \qquad \text{or} \qquad a < 0.$$

Arquimedian property:
For any number a in $\mathbb{R}$ there exists an integer k such that

$$k \leq a < k + 1.$$

Problem: $x^2 + 1 = 0$ does not have any solution $x \in \mathbb{R}$, this leads to the **complex numbers**!!

The **real line** is the representation of the elements of $\mathbb{R}$ as points.

1.1.2 Inequalities, absolute value and subsets

The inequalities: $>, <, \geq, \leq$ satisfy the following:

Properties of inequalities

1. $a > b \quad \Longrightarrow \quad a + c > b + c \quad \forall c.$
2. $a > b$ and $c > 0 \quad \Longrightarrow \quad ac > bc.$
3. $a > b$ and $c < 0 \quad \Longrightarrow \quad ac < bc.$

Another important concept is the **absolute value**, $|a|$, defined by:

$$|a| = \begin{cases} a & \text{if} \quad a \geq 0, \\ -a & \text{if} \quad a < 0. \end{cases}$$

Properties of absolute values

1. $|a| \geq 0, \ \forall\, a \in \mathbb{R}; \ \ |a| = 0 \Longleftrightarrow a = 0.$
2. $|ab| = |a| \cdot |b|.$
3. $|a + b| \leq |a| + |b|$ (**triangle inequality**).
4. $\sqrt{a^2} = |a|.$

With this we define the **distance**: $\operatorname{dist}(a, b) = |a - b|.$

Example: 1.2. $\{x \in \mathbb{R} : \text{dist}(x, -1) \leq 4\} = \{x \in \mathbb{R} : |x + 1| \leq 4\} = \{x \in \mathbb{R} : -5 \leq x \leq 3\} = [-5, 3]$.

We need some more definitions, the intervals on $\mathbb{R}$ can be:

1. **open**: $(a, b) = \{x \in \mathbb{R} : a < x < b\}$.

2. **closed**: $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.

3. **semi-open** or **semi-closed**: $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$, $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$.

4. **infinite**: $(a, \infty) = \{x \in \mathbb{R} : a < x\}$, $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$, $[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$, $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$.

Also we say that the set $A \subset \mathbb{R}$ is **bounded above** if $\exists\, M \in \mathbb{R} : x \leq M,\ \forall\, x \in A$. The number M is called **upper bound**. Analogously, it is **bounded below** if $\exists\, K \in \mathbb{R} : x \geq K,\ \forall\, x \in A$ and the number K is called **lower bound**. It is **bounded** if it is bounded above and below.

The minimum upper bound is called **supremum** and it is denoted by $\sup(A)$. The maximum lower bound is called **infimum** and it is denoted by $\inf(A)$.

If $\sup(A) \in A$ then it is called **maximum**, $\max(A)$, and if $\inf(A) \in A$ it is called **minimum**, $\min(A)$.

Example: 1.3. The set $A = (-\infty, 1) \cup (1, 2)$ is bounded above but not below. The least upper bound is 2, and it is not in the set, so $2 = \sup(A)$.

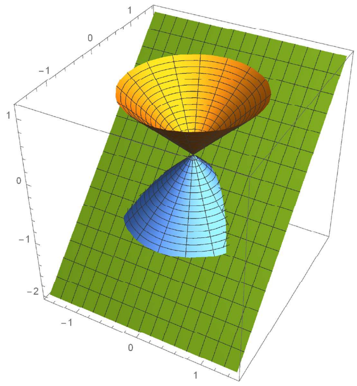
1.1.3 Elementary curves

A **straight line** has the equation: $y = mx + d$, or more generally, $ax + by + c = 0$, and even more: $y = y_0 + m(x - x_0)$ is the line through the point (x_0, y_0) with slope m.

Remember that the **distance** between $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, is

$$\text{dist}(P, Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

With this we define the curves known as **conics** (sections of a double cone by a plane), that include:



1. **Parabola**: Points with the same distance to a fixed point, the focus, and a fixed straight line, the directrix, $y = \frac{1}{4c}x^2$ has the focus $(0, c)$ and the directrix $y = -c$. A parabola is usually:

$$y = ax^2 + bx + c, \qquad a \neq 0,$$

and also $y = \sqrt{ax + b}$ is a half parabola.

2. **Circumference**: Points with the same distance, the radius r, to a fixed point, the center (x_0, y_0):

$$(x - x_0)^2 + (y - y_0)^2 = r^2, \qquad r > 0.$$

3. **Ellipse**: Points with the same sum of distances to two fixed points, the focuses, placed at $(x_0 + c, y_0)$ and $(x_0 - c, y_0)$, the sum is $2a$, where:

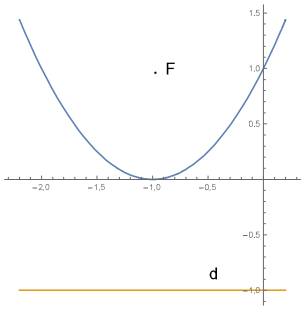
$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1, \quad a, b \neq 0, \quad b^2 + c^2 = a^2 \quad (\text{if } a > b).$$

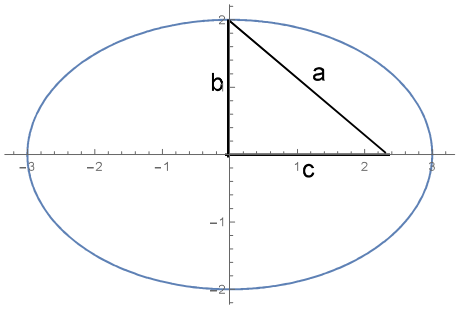
4. **Hyperbola**: Points with the same difference of distances to two fixed points, the focuses, placed at $(x_0 + c, y_0)$ and $(x_0 - c, y_0)$ if the equation is:

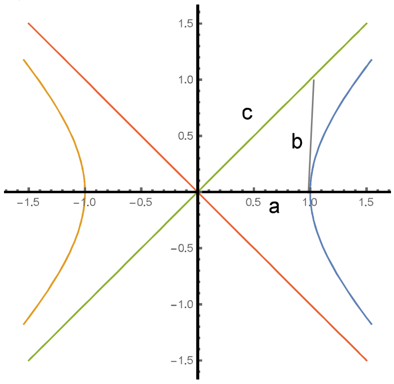
$$\frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2} = 1, \qquad a, b \neq 0, \qquad a^2 + b^2 = c^2.$$

Also, we can have negative sign in the first term and positive in the second. How is the picture in this case?

5. A **point** and a **straight line** are also conics!!







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1.2 Elementary functions

1.2.1 First definitions

A **function** is any rule that relates a value to every point of a certain set $A \subset \mathbb{R}$. The notation is $f : A \to \mathbb{R}$ and $f(x)$ is the value of the function f at the point x.

The **domain** of f is the set of points for which it is defined, and it is denoted by $\mathrm{Dom}(f)$. The **image** is the set $\mathrm{Img}(f) = \{f(x) : x \in \mathrm{Dom}(f)\}$.

The **graph** of f is the set of points $\{(x, f(x)) : x \in \mathrm{Dom}(f)\}$, and we can map those points in the XY-plane.

The **elementary functions** we are going to consider are polynomials, quotients of polynomials (rational functions), roots, trigonometric functions, logarithms and exponentials.

Example: 1.4. The domain of these functions is:

1. Polynomials: $\mathrm{Dom} = \mathbb{R}$.
2. Quotients: $\mathrm{Dom} = \{\text{denominator different from zero}\}$.
3. $\sqrt{x}$: $\mathrm{Dom} = \{x \geq 0\}$.
4. $\sin x$ and $\cos x$: $\mathrm{Dom} = \mathbb{R}$.
5. $\tan x$: $\mathrm{Dom} = \{x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\}$ (where $\cos x$ does not vanish).
6. e^x: $\mathrm{Dom} = \mathbb{R}$.
7. $\log x$: $\mathrm{Dom} = \{x > 0\}$.

Observation 1.1. We use the notation $\log x = \ln x$. Every logarithm is a natural logarithm unless we specify a different basis.

A function is **bounded** if the set $\mathrm{Img}(f)$ is bounded (above or below or both).

The function f is **increasing** if

$$a, b \in \mathrm{Dom}(f), \quad a < b \quad \Longrightarrow \quad f(a) \leq f(b)$$

and **decreasing** if $f(a) \geq f(b)$. Even more, f is **strictly increasing** if

$$a, b \in \mathrm{Dom}(f), \quad a < b \quad \Longrightarrow \quad f(a) < f(b)$$

and it is **strictly decreasing** if $f(a) > f(b)$.

1.2.2 Properties

1. **Injectivity:**

   a) $f : A \to B$ is **injective**, or **one to one**, if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

   b) $f : A \to B$ is **surjective** if for every $y \in B$ there exists at least one $x \in A$ with $f(x) = y$.

   c) $f : A \to B$ is **bijective** if it is injective and surjective, i.e., if $\forall\, y \in B \;\; \exists!\, x \in A \; : \; f(x) = y$.

   In the graph of the function this means in a) that every horizontal line passing through points of B **intersects at most once** the graph, in b) that it **intersects at least once** and in c) that it **intersects exactly once** the graph.

   Example: 1.5. x^3 is bijective, x^2 is not injective.

2. A function is **periodic** if

$$\exists K > 0 \; : \; f(x) = f(x + K) \qquad \forall x \in \mathrm{Dom}(f).$$

   The constant K is the **period**, but nK is also a period $\forall\, n \in \mathbb{Z}$, so we usually consider the **minimum period**.

   Example: 1.6. All the trigonometric functions are periodic: $\sin x$, $\cos x$, $\sec x$ and $\operatorname{cosec} x$ have period 2π, $\tan x$ and $\operatorname{cotg} x$ have period π.

3. **Symmetries:**

   a) f is **even** if $f(-x) = f(x)$, $\forall x \in \mathrm{Dom}(f)$.

   b) f is **odd** if $f(-x) = -f(x)$, $\forall x \in \mathrm{Dom}(f)$.

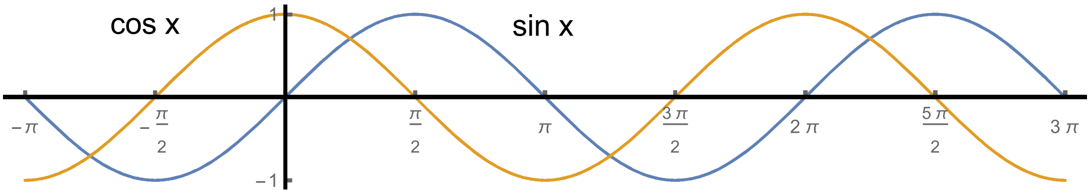
   If f is even, its graph is symmetric with respect to the Y axis. If it is odd it is symmetric with respect to the origin.

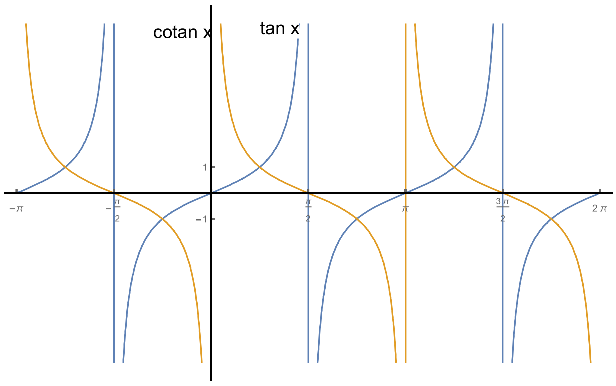
   Example: 1.7. x^2 and $\cos x$ are even, x^3 and $\sin x$ are odd.

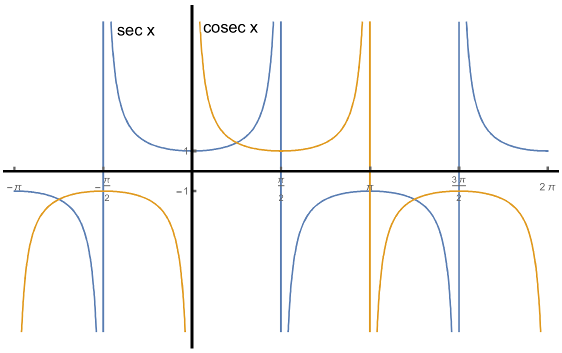
1.2.3 Trigonometric functions

The functions $\sin x$ and $\cos x$ are the most famous and appear in the unit circumference. The arguments are always in radians (not in degrees). Related to them we have:

$$\tan x = \frac{\sin x}{\cos x}, \qquad \operatorname{cotg} x = \frac{\cos x}{\sin x}, \qquad \sec x = \frac{1}{\cos x}, \qquad \operatorname{cosec} x = \frac{1}{\sin x}.$$







Besides, there are very useful relations, the most important are:

$$
\begin{aligned}
&\sin^2 x + \cos^2 x = 1,\\
&\sin(2x) = 2\sin x \cos x,\\
&\cos(2x) = \cos^2 x - \sin^2 x,\\
&\tan(2x) = \frac{2\tan x}{1-\tan^2 x},\\
&\sin(x+y) = \sin x \cos y + \cos x \sin y,\\
&\cos(x+y) = \cos x \cos y - \sin x \sin y,\\
&\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y},\\
&\sin(x+\tfrac{\pi}{2}) = \cos x, \qquad \sin(x+\pi) = -\sin x.
\end{aligned}
$$

1.2.4 Logarithm and exponential

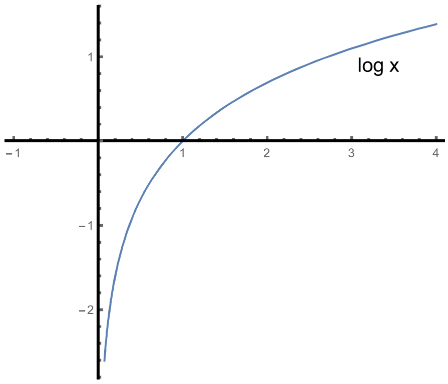
Every logarithm is related to the natural logarithm by a constant, if $a > 0$ (in our notation, $\log x = \ln x$):

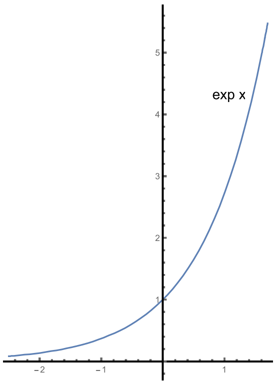
$$\log_a x = b \iff a^b = x \iff e^{b \log a} = x \iff b \log a = \log x$$
$$\implies \log_a x = \frac{\log x}{\log a}.$$

All the logarithms we consider are natural, unless another basis is specified.

Properties of logarithms

1. $\log_a 1 = 0$.
2. $\log_a a = 1$.
3. $\log_a x + \log_a y = \log_a (xy)$.
4. $\log_a x - \log_a y = \log_a \frac{x}{y}$.
5. $\log_a (x^c) = c \log_a x$.





Exponentials are defined for positive bases, and their properties are:

Properties of exponentials

1. $a^0 = 1$.
2. $a^1 = a$.
3. $a^{x+y} = a^x a^y$.

4. $a^{x-y} = \frac{a^x}{a^y}$.
5. $a^{xc} = (a^x)^c$.
6. $a^{x/y} = \sqrt[y]{a^x}$.

The basis we prefer is e and we can use the notation $\mathrm{e}^x = \exp(x)$.

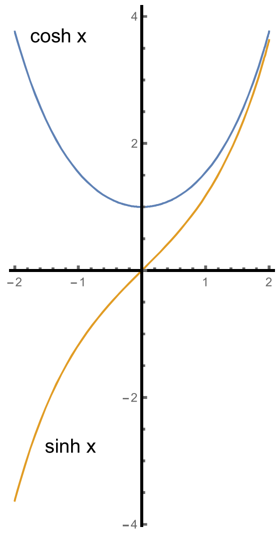
Now we can define the **hyperbolic functions** as follows:

a) **hyperbolic sine** of x is $\sinh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}$.

b) **hyperbolic cosine** of x is $\cosh x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}$.

c) **hyperbolic tangent** of x is $\operatorname{tgh} x = \frac{\sinh x}{\cosh x}$.

c) **hyperbolic cotangent** of x is $\operatorname{cotgh} x = \frac{\cosh x}{\sinh x}$.



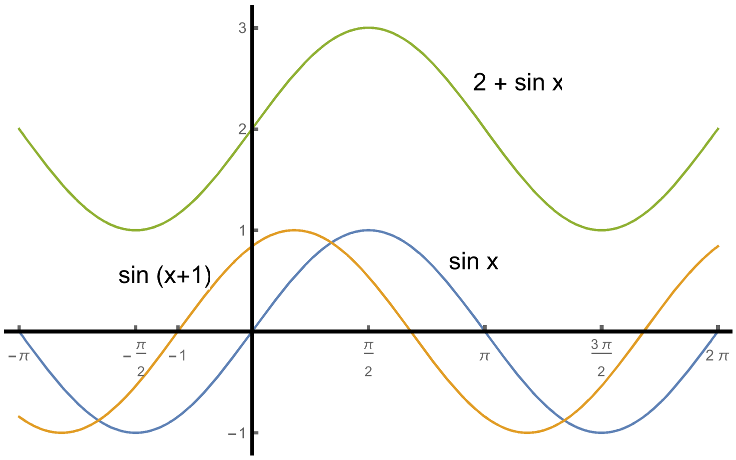
They are related to the trigonometric functions, for example they satisfy:

$$\cosh^2 x - \sinh^2 x = 1.$$

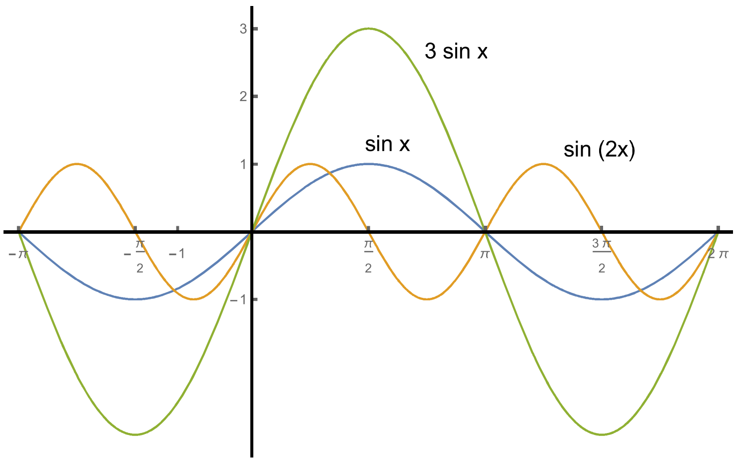
1.2.5 Operations with functions

Changes in graphs:

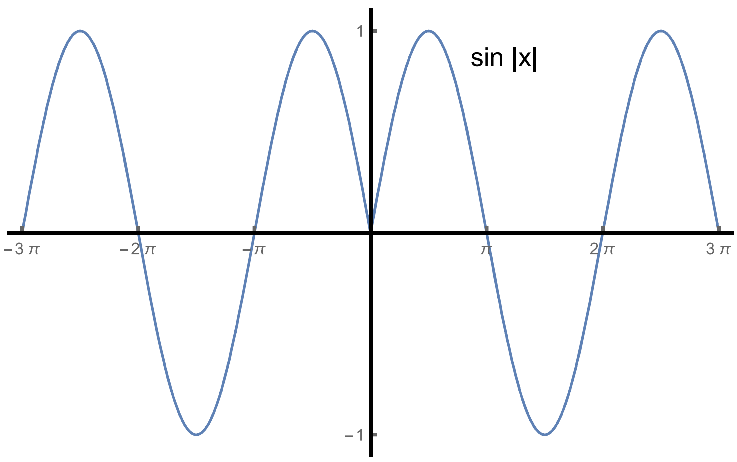
a) **Translations**: horizontal $y = f(x + c)$, vertical $y = f(x) + c$.



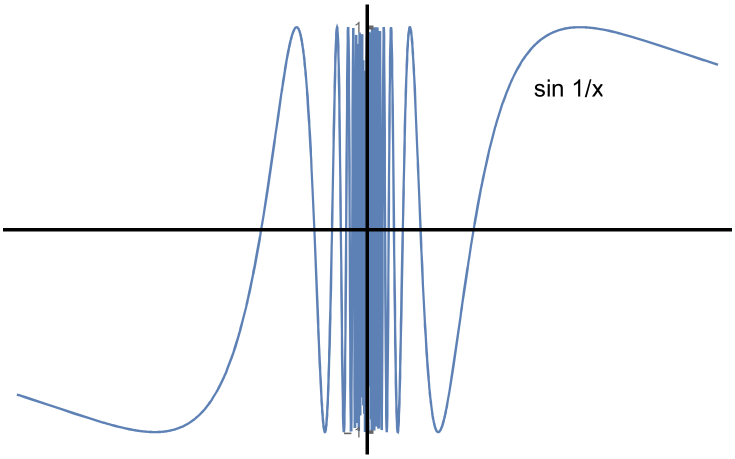
b) **Dilations**: horizontal $y = f(cx)$, vertical $y = cf(x)$.



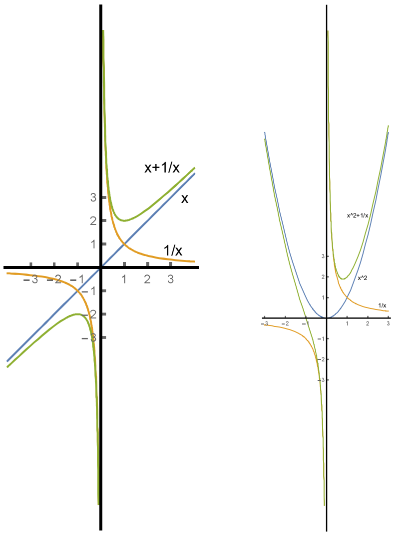
c) **Symmetries**: horizontal $y = f(|x|)$, vertical $y = |f(x)|$.



d) **Inversions**: horizontal $y = f(1/x)$, vertical $y = 1/f(x)$.



e) **Sums**: We can obtain graphically the sum of functions if they are easy:



The **composition** of two functions f and g where $f\,:\,A \to B$ and $g\,:\,B \to C$ is a function h defined by

$$h = g \circ f : A \to C, \qquad h(x) = g \circ f(x) = g(f(x)).$$

In general $\mathrm{Dom}(g \circ f) = \{x \in \mathrm{Dom}(f)\,:\, f(x) \in \mathrm{Dom}(g)\}$.

Example: 1.8. Given the functions $f(x) = \cos x$, $g(x) = x^2$ and $h(x) = 1 + \log |x|$, we have that:

$$f \circ g(x) = f(g(x)) = \cos(x^2), \qquad g \circ h \circ f(x) = g\left(h(f(x))\right) = \left(1 + \log|\cos x|\right)^2.$$

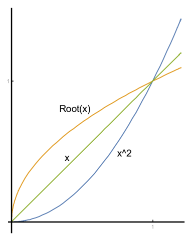
1.2.6 Inverse functions

Two functions $f : A \to B$ and $g : B \to A$ are **inverse functions** of each other if their composition is the identity function,

$$g \circ f = Id : A \to A, \qquad f \circ g = Id : B \to B,$$
$$g \circ f(x) = x \quad \forall x \in A, \qquad f \circ g(x) = x \quad \forall x \in B.$$

They are denoted by $g = f^{-1}$ and $f = g^{-1}$. The inverse function of $f : A \to B$ exists if and only if f is injective. But for functions which are not injective not everything is lost! we can consider only a part of the function where it is injective. Some famous examples of this lead to the definition of **arcsine**, **arccosine** and **arctangent**:

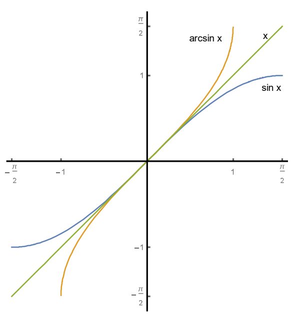
Example: 1.9. 1. $f(x) = x^2$, $f : [0, \infty) \to [0, \infty)$, $f^{-1}(x) = \sqrt{x}$.

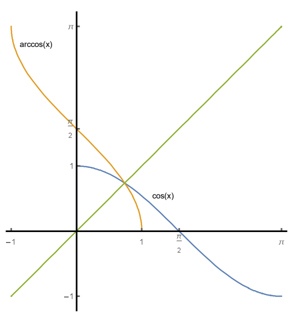


2. $f(x) = \mathrm{e}^x$, $f : \mathbb{R} \to (0, \infty)$, $f^{-1}(x) = \log x$.

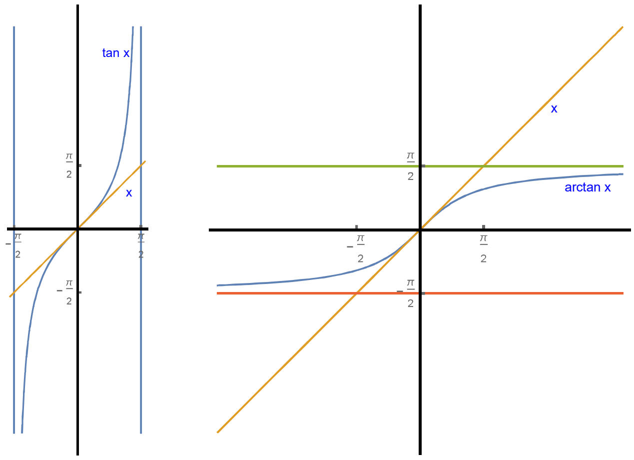
3. $f(x) = \sin x$, $f : [-\pi/2, \pi/2] \to [-1, 1]$, $f^{-1}(x) = \arcsin x$.

4. $f(x) = \cos x$, $f : [0, \pi] \to [-1, 1]$, $f^{-1}(x) = \arccos x$.





5. $f(x) = \tan x$, $f : (-\pi/2, \pi/2) \to \mathbb{R}$, $f^{-1}(x) = \arctan x$.

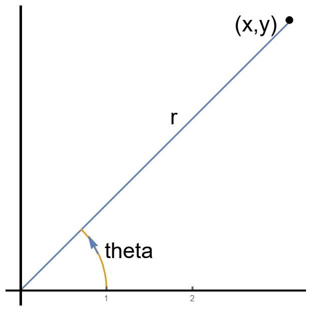


Observe that the graphs of a function and its inverse are symmetric with respect to the bisectrix of the first and third quadrants: $y = x$.

1.2.7 Polar coordinates

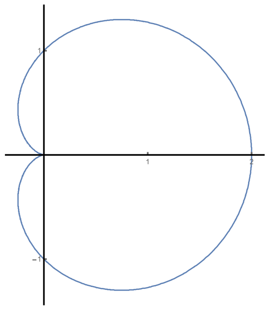
Some graphs are easier in terms of the radius, r, and the angle from the positive X-axis, θ, which are known as **polar coordinates**, defined by:

$$\begin{cases} r = \sqrt{x^2 + y^2}, \\ \theta = \arctan \dfrac{y}{x}. \end{cases} \iff \begin{cases} x = r\cos\theta, & (x, y) \in \mathbb{R}^2, \\ y = r\sin\theta. & r \in [0, \infty),\ \theta \in [0, 2\pi). \end{cases}$$



Example: 1.10. $r = 2$ is the circumference centered at the origin with radius 2. This cannot be described with a single function in cartesian coordinates.

Example: 1.11. The graph shows the curve $r = 1 + \cos\theta$, $\theta \in [0, 2\pi]$, named **cardioid**.



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