uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL CALCULUS

Degree in Applied Mathematics and Computation

Chapter 2

Elena Romera

Open Course Ware, UC3M

2

LIMITS AND CONTINUITY

In this chapter we study the concepts of limits with the calculation techniques and continuity, including the most important theorems.

Contents

2.1 Limits

2.1.1 Definitions and basic properties

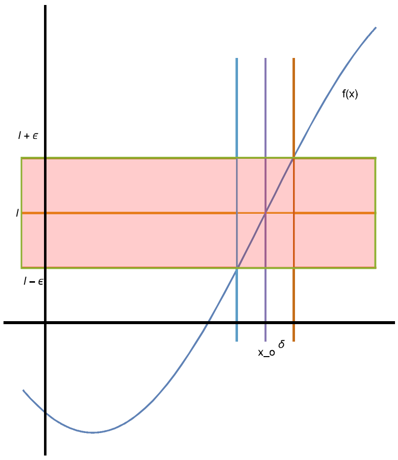
We say that f **tends to the limit** ℓ when x approaches x_0, and write

$$\lim_{x \to x_0} f(x) = \ell,$$

if $\forall \varepsilon > 0\ \exists \delta > 0$ such that $|f(x) - \ell| < \varepsilon$ provided $0 < |x - x_0| < \delta$.

An alternative notation is $\lim\limits_{h \to 0} f(x + h) = \ell$.

We do not care about the values of f far from $x = x_0$, we only look at the values x close to x_0 but not at that point x_0.



Example: 2.1. $\lim\limits_{x \to 4}(3x - 7) = 5$ since $|3x - 7 - 5| = 3|x - 4| < \varepsilon$ is obtained if we take $\delta = \varepsilon/3$.

The function f **does not tend** to the limit ℓ when x approaches x_0 if $\exists \varepsilon > 0$ such that $\forall \delta > 0$ exists some x satisfying $0 < |x - x_0| < \delta$ and nevertheless $|f(x) - \ell| > \varepsilon$.

Example: 2.2. $\frac{1}{x}$ does not have a limit at $x_0 = 0$, but $\lim\limits_{x \to 2} x^2 = 4$.

Example: 2.3. The function $\sin \frac{1}{x}$ does not have a limit at $x_0 = 0$, and the **characteristic function** of $\mathbb{R} \setminus \mathbb{Q}$, defined by

$$\chi_{\mathbb{R} \setminus \mathbb{Q}}(x) = \begin{cases} 1, & x \in \mathbb{R} \setminus \mathbb{Q}, \\ 0, & x \in \mathbb{Q}, \end{cases}$$

has no limit at any point.

Theorem 2.1. *(Uniqueness)*
If the limit exists it is unique, i.e., if ℓ and m both satisfy the definition, then $\ell = m$.

Properties of limits

If there exist $\lim_{x\to x_0} f(x) = l$ and $\lim_{x\to x_0} g(x) = m$, then:

1. $\lim_{x\to x_0} \big(cf(x)\big) = c\ell\,, \quad \text{if } c \in \mathbb{R}\,.$
2. $\lim_{x\to x_0} \big(f(x) + g(x)\big) = \ell + m\,.$
3. $\lim_{x\to x_0} \big(f(x)g(x)\big) = \ell m\,.$
4. $\lim_{x\to x_0} \dfrac{f(x)}{g(x)} = \dfrac{\ell}{m}\,, \quad \text{if } m \neq 0\,.$

The **right hand limit** of f at the point x_0 is ℓ if: $\forall\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ provided $x_0 < x < x_0 + \delta$, and we write:

$$\lim_{x\to x_0^+} f(x) = \ell.$$

Analogously, the **left hand limit** of f at the point x_0 is ℓ if: $\forall\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ provided $x_0 - \delta < x < x_0$, and we write:

$$\lim_{x\to x_0^-} f(x) = \ell.$$

Theorem 2.2. *(Side limits)*
The $\lim_{x\to x_0} f(x)$ exists if and only if both side limits exist and $\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = \ell$. Then, $\lim_{x\to x_0} f(x) = \ell$.

The **limit of a composition** is obtained directly if all the limits involved exist and the operations make sense, that is,

$$\lim_{x\to x_0} f(x) = \ell \quad \text{and} \quad \lim_{x\to \ell} h(x) = m \quad \Longrightarrow \quad \lim_{x\to x_0} h(f(x)) = m.$$

Example: 2.4. If $\lim_{x\to x_0} f(x) = \ell$ and $\lim_{x\to x_0} g(x) = m$, then:

1. $\lim_{x\to x_0} \sqrt{f(x)} = \sqrt{\ell}$, if $\ell \geq 0$

2. $\lim\limits_{x\to x_0} \big(f(x)\big)^{h(x)} = \ell^m$, if the result is not 0^0.

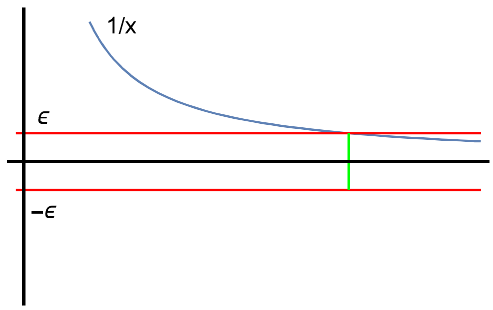
3. $\lim\limits_{x\to x_0} \log_a f(x) = \log_a(\ell)$, if $a > 0$, $\ell > 0$.

But what happens with the **infinite limits** or the **limits at infinity**? the corresponding definitions are:

a) We say that $\lim\limits_{x\to x_0} f(x) = \infty$ if for every real number M there exists a $\delta > 0$ such that $f(x) > M$ provided $0 < |x - x_0| < \delta$.

b) We say that $\lim\limits_{x\to \infty} f(x) = \ell$ if for every $\varepsilon > 0$ there exists a real number N such that $|f(x) - \ell| < \varepsilon$ provided $x > N$.

c) We say that $\lim\limits_{x\to \infty} f(x) = \infty$ if for every real number M there exists a real number N such that $f(x) > M$ provided $x > N$.



Exercise: Write the corresponding definitions for

$$\lim_{x\to -\infty} = \ell, \qquad \lim_{x\to x_0} f(x) = -\infty, \quad \text{and} \quad \lim_{x\to \infty} f(x) = -\infty.$$

The properties of the limits (of the sum, product, etc...) are true also when some or both of the limits ℓ and m are infinite, whenever the expressions are well defined or make sense.

Example: 2.5. $\lim\limits_{x\to \pi/4} \cos(x) = \dfrac{\sqrt{2}}{2}$ and $\lim\limits_{x\to \pi/4} \frac{1}{(x-\pi/4)^2} = \infty$, so

$$\lim_{x\to \pi/4} (\cos(x))^{1/(x-\pi/4)^2} = \left(\frac{\sqrt{2}}{2}\right)^{\infty} = 0.$$

2.1.2 Calculus of limits

There are some formal operations with infinite that are well defined:

$$
\begin{aligned}
a + \infty &= \infty\,, & a - \infty &= -\infty\,,\\
\infty + \infty &= \infty\,, & -\infty - \infty &= -\infty\,,\\
\infty \cdot \infty &= \infty\,, & -\infty \cdot \infty &= -\infty\,,\\
\frac{a}{\infty} &= 0\,, & a \cdot \infty &= \infty\,,\ \text{ if }\ a > 0\,,\\
\frac{a}{0} &= \infty\,,\ \text{ if }\ a > 0\,, & \frac{a}{0} &= -\infty\,,\ \text{ if }\ a < 0\,,\\
\infty^a &= \infty\,,\ \text{ if }\ a > 0\,, & \infty^a &= 0\,,\ \text{ if }\ a < 0\,,\\
\infty^\infty &= \infty\,, & \infty^{-\infty} &= 0\,,\\
a^\infty &= \infty\,,\ \text{ if }\ a > 1\,, & a^\infty &= 0\,,\ \text{ if }\ 0 \le a < 1\,.
\end{aligned}
$$

But we also can find **indeterminate forms**: expressions whose value cannot be determined in advance, and may be different in each case. The most typical are the following:

$$\infty - \infty\,,\ \ \frac{\infty}{\infty}\,,\ \ \frac{0}{0}\,,\ \ 0 \cdot \infty\,,\ \ \infty^0\,,\ \ 0^0\,,\ \ 1^\infty\,,\ \ 1^{-\infty}\,.$$

Some of these indeterminate forms can be simplified by looking for common factors, dividing by the greatest term or using the conjugate.

Example: 2.6. 1. $\displaystyle \lim_{x\to 3} \frac{x^2-9}{x^2-5x+6} = \lim_{x\to 3} \frac{x+3}{x-2} = 6.$

2. $\displaystyle \lim_{x\to\infty} \frac{x+3}{x-4} = \lim_{x\to\infty} \frac{1+3/x}{1-4/x} = 1.$

3. $\displaystyle \lim_{x\to\infty} \left(\sqrt{x+1} - \sqrt{x}\right) = \lim_{x\to\infty} \frac{1}{\sqrt{x+1}+\sqrt{x}} = 0.$

4. $\displaystyle \lim_{x\to\infty} x^{1/x} = \mathrm{e}^{\lim\limits_{x\to\infty} \frac{\log x}{x}} = 1.$

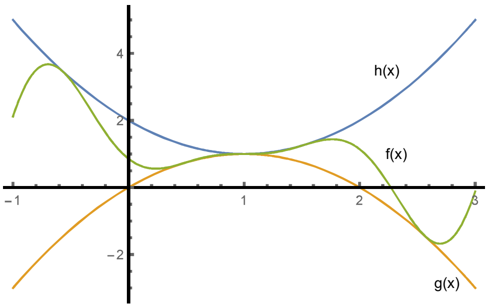
Some others can be solved using the following lemma:

Lemma 2.3. *(Sandwich or pinching lemma)*
If $g(x) \le f(x) \le h(x)$ for every $0 < |x - x_0| < \delta$ and

$$\lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x) = \ell$$

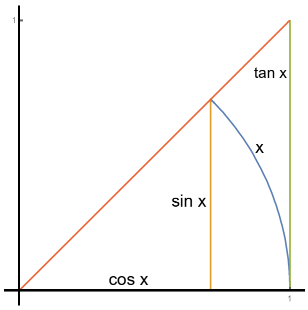
then

$$\lim_{x\to x_0} f(x) = \ell.$$



Example: 2.7. The limit $\lim_{x\to 0} \frac{\sin x}{x}$ can be obtained geometrically using this lemma: the areas of the three triangles involved are related as:

If $x > 0 \Rightarrow \frac{\sin x \cos x}{2} \leq \frac{x}{2} \leq \frac{\tan x}{2} \Rightarrow \cos x \leq \frac{x}{\sin x} \leq \frac{1}{\cos x} \Rightarrow$ $\lim_{x\to 0^+} \frac{x}{\sin x} = 1 \iff \lim_{x\to 0^+} \frac{\sin x}{x} = 1.$



By symmetry, $\lim_{x\to 0} \frac{\sin x}{x} = 1.$

The indeterminations related to the exponential can be solved using the following result.

Theorem 2.4. *(Exponentials)*
If $\lim_{x\to\alpha} f(x) = 1$ *and* $\lim_{x\to\alpha} g(x)$ *is* ∞ *or* $-\infty$, *then*

$$\lim_{x\to\alpha} \left(f(x)\right)^{g(x)} = e^{\lim_{x\to\alpha} (f(x)-1)g(x)},$$

if the last limit exists, where α *can be* x_0, x_0^+, x_0^-, ∞ *or* $-\infty$.

Example: 2.8. $\lim\limits_{x\to\infty}\left(\frac{2x+3}{2x-7}\right)^{x+5} = \mathrm{e}^{\lim\limits_{x\to\infty}(x+5)\,\frac{10}{2x-7}} = \mathrm{e}^5.$

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2.2 Continuity

2.2.1 Continuity at a point

A function f is **continuous at** x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$, that is, if:

$$\forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon, x_0) > 0 \text{ such that } |x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon.$$

An equivalent expression is $\lim_{h \to 0} \Big(f(x_0 + h) - f(x_0) \Big) = 0$.

Properties of continuity

If f and g are continuous at x_0:

1. $f + g$ is continuous at x_0,
2. $f \cdot g$ is continuous at x_0,
3. $\dfrac{f(x)}{g(x)}$ is continuous at x_0 if $g(x_0) \neq 0$,
4. If g is continuous at $f(x_0)$ then $\lim_{x \to x_0} g(f(x)) = g(\lim_{x \to x_0} f(x))$.

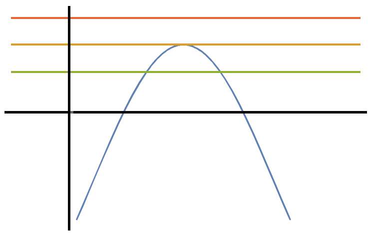
Example: 2.9. $\lim_{x \to 0} \mathrm{e}^{\frac{\sin x}{x}} = \mathrm{e}^{\lim_{x \to 0} \frac{\sin x}{x}} = \mathrm{e}.$

Theorem 2.5. *(Composition)*
If g is continuous at x_0 and f is continuous at $g(x_0)$, then $f \circ g$ is continuous at x_0.

Theorem 2.6. *(Sign)*

i) If f is continuous at x_0 and $f(x_0) > 0$ then there exists a $\delta > 0$ such that $f(x) > 0$ on $(x_0 - \delta, x_0 + \delta)$.

ii) If f is continuous at x_0 and $f(x_0) < 0$ then there exists a $\delta > 0$ such that $f(x) < 0$ on $(x_0 - \delta, x_0 + \delta)$.



2.2.2 Fundamental theorems

A function is **continuous on** (a, b) if it is continuous at any point of that interval. A function is **continuous on** $[a, b]$ if it is continuous on (a, b) and also:

$$\lim_{x \to a^+} = f(a), \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b).$$

Example: 2.10. Polynomials, sine and cosine, exponential, square root, logarithm are continuous in their domains.

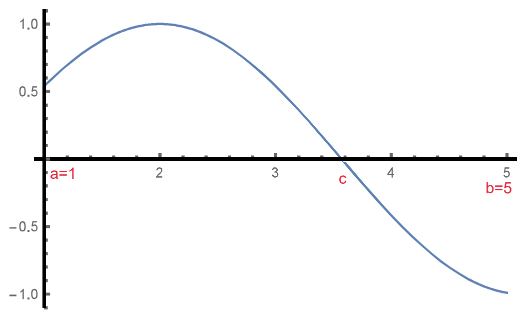
There is an **avoidable discontinuity** at x_0 if there exists the $\lim_{x \to x_0} f(x)$ and it is finite but it is not $f(x_0)$ or $f(x_0)$ does not exist.

Example: 2.11. $f(x) = \dfrac{\sin x}{x}$ is not continuous at $x_0 = 0$, but

$$g(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases} \qquad \text{is continuous on } \mathbb{R}.$$

The continuity on a closed interval implies some important theorems that we study now.

Theorem 2.7. *(Bolzano)*
If f is continuous on $[a, b]$ and $f(a)f(b) < 0$ then there exists a point $c \in (a, b)$ such that $f(c) = 0$.



Example: 2.12. The following functions do not satisfy this theorem:

$$f(x) = 1/x \quad \text{on } [-1, 1]; \qquad f(x) = 1/x \quad \text{on } (0, 1];$$
$$f(x) = x^2 \quad \text{on } [0, 1); \qquad f(x) = 1/x \quad \text{on } [1, \infty).$$

There are two important applications of this result, the first one is useful for computational calculation:

Newton's Bisection Method

To find a root of a continuous function $f(x)$ on $[a, b]$ when we know that $f(a)f(b) < 0$, we compute $f(\frac{a+b}{2})$:

1. If $f(\frac{a+b}{2}) = 0$ we have finished.
2. If it is not we have a smaller interval in which there is a root and we compute f in the middle point of it.
3. We continue until we arrive to the desired accuracy.

Another consequence of the theorem is:

Theorem 2.8. *(Intermediate values)*

1. *If f is continuous on $[a, b]$ and $f(a) < f(b)$ then for any $y \in (f(a), f(b))$ there exists a $c \in (a, b)$ such that $f(c) = y$.*
2. *Similarly, if $f(a) > f(b)$ then for any $y \in (f(b), f(a))$ there exists a $c \in (a, b)$ such that $f(c) = y$.*

If we want to find the biggest and smallest values of a function we need the following result:

Theorem 2.9. *(Boundedness)*
If f is continuous on $[a, b]$, it is bounded above and below on $[a, b]$.

We also define for a function $f : A \to \mathbb{R}$ (continuous or not):

1. if there exist a $c \in A$ such that $f(c) \geq f(x)$, $\forall x \in A$ we say that c is a **maximum point** of f on A and that $f(c)$ is the **maximum value** of f on A.
2. Similarly, if there exist a $c \in A$ such that $f(c) \leq f(x)$, $\forall x \in A$ we say that c is a **minimum point** of f on A and that $f(c)$ is the **minimum value** of f on A.

Using the continuity on a closed interval and the previous results we also prove the following theorems:

THEOREM 2.10. *(Max-Min)*
If f is continuous on $[a, b]$ then there exist maximum and minimum points of f on $[a, b]$. That is,

$$\exists c, d \in [a, b] \text{ such that } f(c) \leq f(x) \leq f(d), \ \forall x \in [a, b].$$

THEOREM 2.11. *(Odd degree polynomials)*
Any polynomial of odd degree has at least one root.

THEOREM 2.12. *(Even degree polynomials)*
Any polynomial of even degree with positive highest coefficient is bounded below.

2.2.3 Uniform continuity

We study now a stronger kind of continuity: A function is **uniformly continuous on an interval** A if

$$\forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon) > 0 \text{ such that } |x - y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon$$

wherever $x, y \in A$, or equivalently if:

$$\forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon) > 0 \text{ such that } |x - a| < \delta \Longrightarrow |f(x) - f(a)| < \varepsilon, \ \forall a \in A.$$

That is, now δ is independent of the point, the same δ is valid on the whole interval.

Example: 2.13. The function $f(x) = 2x - 1$ is uniformly continuous on $\mathbb{R}$.

Example: 2.14. The function $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$.

If f is uniformly continuous on $[a, b]$, then it is continuous on $[a, b]$, and besides:

THEOREM 2.13. *(Uniform continuity)*
If f is continuous on a closed interval then it is uniformly continuous on that interval.

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