uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL CALCULUS

Degree in Applied Mathematics and Computation

Chapter 3

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3

DERIVATIVES AND THEIR APPLICATIONS

This chapter is dedicated to the fundamental tool of Differential Calculus, the derivative. Some important theorems are proved and many interesting applications are studied.

Contents

3.1 Differentiability 34
3.1.1 Definition and basic derivatives 34
3.1.2 Basic properties 35
3.2 Meaning of the derivative 38
3.2.1 First and second derivatives 38
3.2.2 Strong theorems 40
3.2.3 Extrema 41

3.1 Differentiability

3.1.1 Definition and basic derivatives

Example: 3.1. The mean velocity in a time interval $[t_0, t_0+h]$, if $u(t)$ is the position is:

$$v_m = \frac{u(t_0+h) - u(t)}{h}.$$

Example: 3.2. The slope of a segment joining two points of a graph $y = f(x)$ of coordinates x_0 and $x_0 + h$ has the form:

$$m = \frac{f(x_0+h) - f(x_0)}{h}.$$

Considering the limits of those concepts we say that a function f is **differentiable** at a point a if there exists the limit

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

and it is finite. In that case it is called the **derivative** of f at a and denoted by $f'(a)$ (Newton's notation) or $\frac{df}{dx}(a)$ (Leibniz's notation). An alternative definition is:

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

Example: 3.3. If $f(x) = x^2$ then

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h} = 2x.$$

A function f is **differentiable on an interval** (a, b) if it is differentiable at all the points of (a, b). The function f' has as domain the points where f is differentiable, and it also can be differentiable.

Notation: The higher order derivatives of a function are denoted by: $f''(x)$, $f'''(x), \dots f^{(n}(x)$ (Newton's notation) or by

$$\frac{d^2 f}{dx^2}(x),\ \frac{d^3 f}{dx^3}(x),\ \dots,\ \frac{d^n f}{dx^n}(x).$$

With the definition, the **tangent line** to the graph of a function $y = f(x)$ at the point $(a, f(a))$ is

$$y = f(a) + f'(a)(x - a).$$

When the limit of the definition does not exist, but the following limits

$$\lim_{h\to 0^+} \frac{f(x)-f(a)}{h}, \qquad \lim_{h\to 0^-} \frac{f(x)-f(a)}{h}.$$

exist and are finite, they are called respectively **right hand derivative** and **left hand derivative**, that is, the **side derivatives** of f at a:

Example: 3.4. The derivatives of the most common functions are obtained easily:

$$\begin{array}{lll} f(x) = x^n \quad (n \in \mathbb{N}) & \rightsquigarrow & f'(x) = nx^{n-1} \\ f(x) = \sin x & \rightsquigarrow & f'(x) = \cos x \\ f(x) = \cos x & \rightsquigarrow & f'(x) = -\sin x \\ f(x) = \mathrm{e}^x & \rightsquigarrow & f'(x) = \mathrm{e}^x \\ f(x) = \log x & \rightsquigarrow & f'(x) = 1/x \end{array}$$

And, of course, we have that:

Theorem 3.1. *(Continuity)*
If f is differentiable at a then it is continuous at a.

That is: if f is not continuous at a point it cannot be differentiable at that point.

Example: 3.5. The converse is false: $f(x) = |x|$ is continuous but not differentiable at $x = 0$.

3.1.2 Basic properties

Usually we do not use the definition to obtain a derivative, we use instead the following properties:

Properties of derivatives

For f and g differentiable functions:

1. Differentiation is a linear operation: $(cf)' = cf'$, $(f+g)' = f' + g'$
2. Derivative of a product, **Leibniz's rule**: $(fg)' = f'g + fg'$.
3. Derivative of $\frac{1}{g}$: $\left(\frac{1}{g}\right)' = \frac{-g'}{g^2}$ (when $g \neq 0$).

4. Derivative of a quotient: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ (when $g \neq 0$).

Example: 3.6. If $f(x) = \tan x = \frac{\sin x}{\cos x}$, then

$$f'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x = 1 + \tan^2 x.$$

Theorem 3.2. *(Chain rule)*
If g is differentiable at a and f is differentiable at $g(a)$ then

$$(f \circ g)'(a) = f'(g(a))\, g'(a).$$

Example: 3.7. If $f(x) = \sin(\log x)$ then $f'(x) = \cos(\log x) \cdot \frac{1}{x}$.

Theorem 3.3. *(Continuity of inverse)*
If f is continuous and bijective on an interval, then f^{-1} is also continuous in the corresponding interval.

Observe that if f is bijective on an interval, then it is increasing or decreasing.

Theorem 3.4. *(Inverse function)*
If f is a bijective continuous function defined on an interval and it is differentiable on $f^{-1}(b)$ with $f'\left(f^{-1}(b)\right) \neq 0$, then f^{-1} is differentiable at b and

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}.$$

Observe, that if $f\left(f^{-1}(x)\right) = x$ and f^{-1} is differentiable, by the chain rule

$$f'\left(f^{-1}(x)\right)(f^{-1})'(x) = 1.$$

As an application of the theorem of the inverse function, now we can obtain the derivatives of many other functions, and add them to our list:

Applications

$$
\begin{array}{lll}
f(x) = \sec x & \rightsquigarrow & f'(x) = \sec x \tan x \\
f(x) = \operatorname{cosec} x & \rightsquigarrow & f'(x) = -\operatorname{cosec} x \operatorname{cotg} x \\
f(x) = a^x \quad (a > 0) & \rightsquigarrow & f'(x) = a^x \log a \\
f(x) = \log_a x \quad (a > 0) & \rightsquigarrow & f'(x) = \dfrac{1}{x \log a} \\
f(x) = x^\alpha \quad (\alpha \in \mathbb{R}) & \rightsquigarrow & f'(x) = \alpha x^{\alpha - 1} \\
f(x) = \arctan x & \rightsquigarrow & f'(x) = \dfrac{1}{1 + x^2} \\
f(x) = \arcsin x & \rightsquigarrow & f'(x) = \frac{1}{\sqrt{1-x^2}} \\
f(x) = \arccos x & \rightsquigarrow & f'(x) = \frac{-1}{\sqrt{1-x^2}}
\end{array}
$$

Example: 3.8. If $f(x) = a^x = \mathrm{e}^{x \log a}$ then

$$f'(x) = \mathrm{e}^{x \log a} \log a = a^x \log a.$$

Example: 3.9. If $f(x) = \arcsin x$ then

$$f'(x) = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1 - x^2}}.$$

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3.2 Meaning of the derivative

3.2.1 First and second derivatives

We need a definition: A point $x_0 \in A$ is a **local maximum [or minimum] point** of f in A if $\exists \delta > 0$ such that x_0 is a maximum [or minimum] of f in $A \cap (x_0 - \delta, x_0 + \delta)$.

> **THEOREM 3.5.** *(Zero Derivative)*
> *If x_0 is a local maximum or a local minimum of f and f is differentiable at x_0, then $f'(x_0) = 0$.*

PROOF: For a local minimum, if $f(x_0) \leq f(y)$ for every $y \in (x_0 - \delta, x_0 + \delta)$, then for $|h| < \delta$:

$$\frac{f(x_0+h) - f(x_0)}{h} \geq 0 \qquad \text{if} \quad h > 0,$$
$$\frac{f(x_0+h) - f(x_0)}{h} \leq 0 \qquad \text{if} \quad h < 0.$$

Therefore, since the limit exists, it must be zero.

Example: 3.10. The converse is not true: $f'(0) = 0$ for $f(x) = x^3$ and zero is not a maximum nor a minimum.

One of the most famous and useful theorems about differentiability is:

> **THEOREM 3.6.** *(Rolle)*
> *If f is continuous on $[a, b]$ and differentiable on (a, b), and satisfies $f(a) = f(b)$, then there is some $c \in (a, b)$ such that $f'(c) = 0$.*

PROOF: Since f is continuous on $[a, b]$, it attains its maximum and its minimum. If both are attained in a and b, then f is constant. On the contrary, the maximum or the minimum lies in the interior, and by the previous result the derivative vanishes there.

THEOREM 3.7. *(Mean Value Theorem)*
If f is continuous on $[a, b]$ and differentiable on (a, b), then there is some $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

PROOF: The function $g(x) = f(x) - \dfrac{f(b) - f(a)}{b - a}(x - a)$ satisfies all the hypotheses of Rolle's theorem.

COROLLARY 3.8.

1. *If f is continuous on (a, b) and $f'(x) = 0$ for every $x \in (a, b)$, then f is constant on (a, b).*
2. *If $f'(x) > 0$ [respectively $f'(x) < 0$] for every $x \in (a, b)$, then f is (strictly) increasing [respectively decreasing] on (a, b), that is*

$$f(x) < f(y) \quad \forall\, a < x < y < b.$$

3. *If $f'(x) > 0$ for every $x \in (a, b)$ and $f(a) < 0 < f(b)$ then the equation $f(x) = 0$ has a unique solution on that interval.*

APPLICATIONS

1. If for some $\delta > 0$ we have $f'(x) > 0$ for $x \in (a - \delta, a)$ and $f'(x) < 0$ for $x \in (a, a + \delta)$, then a is a local maximum.
2. If for some $\delta > 0$ we have $f'(x) < 0$ for $x \in (a - \delta, a)$ and $f'(x) > 0$ for $x \in (a, a + \delta)$, then a is a local minimum.
3. If f' has the same sign at both sides of a, there is no maximum or minimum at that point, even when $f'(a) = 0$.

THEOREM 3.9. *(Extrema with second derivative)*

1. *If $f'(a) = 0$ and $f''(a) > 0$ then f has a local minimum at a.*
2. *If $f'(a) = 0$ and $f''(a) < 0$ then f has a local maximum at a.*

In the other direction it is not exactly the same:

Theorem 3.10.

1. *If $f''(a)$ exists and f has a local minimum at a then $f''(a) \geq 0$.*

2. *If $f''(a)$ exists and f has a local maximum at a then $f''(a) \leq 0$.*

Example: 3.11. $f(x) = x^4$ at the origin.

3.2.2 Strong theorems

In this subsection we prove the famous L'Hôpital rule together with two other important results.

Theorem 3.11. *(Continuity of the derivative)*
If f is continuous at a and $\lim_{x\to a} f'(x)$ exists and is finite, then f' is continuous at a, that is:

$$f'(a) = \lim_{x\to a} f'(x).$$

That is: a derivative never has an avoidable discontinuity.

Example: 3.12. $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$ has $f'(0) = 0$.

Theorem 3.12. *(Cauchy Mean Value)*
If f and g are continuous on $[a,b]$ and differentiable on (a,b), then $\exists c \in (a,b)$ such that

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

And if also $g(b) - g(a) \neq 0$ and $g'(c) \neq 0$ then:

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

This is exactly the mean value theorem if $g(x) = x$.

Proof: Apply Rolle's theorem to

$$h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)].$$

THEOREM 3.13. *(L'Hôpital-Bernoulli rule)*
If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ *and* $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ *exists, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

This can be extended to side limits or limits at infinity.

Observation 3.1. L'Hôpital's rule also applies when

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty.$$

PROOF: We define $f(a) = g(a) = 0$ if necessary to obtain continuous functions. By Rolle's theorem, $g(x) \neq 0\ \forall x \in (a, a+\delta)$ for some δ. Apply now the Cauchy mean value theorem to $[a, x]$ with $x \in (a, a+\delta)$ and obtain $\exists \alpha_x \in (a, x)$ such that:

$$\frac{f(x)}{g(x)} = \frac{f'(\alpha_x)}{g'(\alpha_x)} \quad \Longrightarrow \quad \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{\alpha_x\to a} \frac{f'(\alpha_x)}{g'(\alpha_x)} = \lim_{y\to a} \frac{f'(y)}{g'(y)}.$$

Example: 3.13.

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}\,.$$

3.2.3 Extrema

These are some of the most desired things we usually want to know from a function. We need some definitions to study them:

If $f'(x_0) = 0$ then x_0 is called **critical point** of f. The number $f(x_0)$ is called **critical value**.

Remember that a continuous function on a closed interval always attains its maximum and minimum values, that is, there exist the maximum and minimum points.

To locate the maximum and minimum of a continuous function f on a closed interval $[a, b]$ we consider the three kinds of points:

1. The critical points of f.
2. The end points a and b.
3. The points $x \in [a, b]$ without derivative.

Finally we compare the value of the function at those points.

Example: 3.14. $f(x) = |x^2 - 1|$ on $[-2.1]$, has the critical point 0, $f(0) = 1$, at the end-points: $f(-2) = 3$, $f(1) = 0$; a point without derivative is: -1, $f(-1) = 0$, so the maximum point is -2, with maximum value 3, and the minimum point is -1, with minimum value 0.

If the set is not closed, or not bounded, or the function is not continuous at some point, the existence of maximum and minimum is not guaranteed.

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