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DIFFERENTIAL CALCULUS

Degree in Applied Mathematics and Computation

Chapter 4

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4

LOCAL STUDY OF A FUNCTION

We study now the graphic representation of a function and some other applications of the derivatives, arriving to the local approximation using a Taylor polynomial.

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4.1 Graphic representation

4.1.1 Convexity

A set in $\mathbb{R}^2$ is said **convex** if given any two points in the set, the segment joining those points is inside the set.

A function f is said to be **convex** on the interval $[a, b]$ if the set $\{(x, y) \in \mathbb{R}^2 : a \leq x \leq b,\ y \geq f(x)\}$ is convex.

A function f is **concave** if $-f$ is convex.

Alternative definitions

1. f is convex on the interval $[a, b]$ if $\forall x \in [a, b]$ we have:

$$f(x) < f(a) + \frac{f(b) - f(a)}{b - a}(x - a),$$

2. f is convex on the interval $[a, b]$ if $\forall x \in [a, b]$

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a},$$

3. f is convex on the interval $[a, b]$ if $\forall x, y \in (a, b)$ and $0 < \lambda < 1$:

$$f\Big(\lambda x + (1 - \lambda)y\Big) \leq \lambda f(x) + (1 - \lambda)f(y).$$

Theorem 4.1. *(Convexity)*
If f is convex and differentiable at a, then the graph of f is above the tangent line at the point $(a, f(a))$ on a neighbourhood of a, except for the tangent point. In particular, if f is differentiable at a and b, then:

$$f'(a) < f'(b) \qquad \forall\ a < b.$$

The first and second derivatives help us to study convexity:

Theorem 4.2.

1. *If f is differentiable and f' is increasing on $[a, b]$, then f is convex on $[a, b]$.*
2. *If f is differentiable and f' is decreasing on $[a, b]$, then f is concave on $[a, b]$.*

Corollary 4.3.

1. *If f'' exists and $f'' > 0$ on $[a, b]$, then f is convex on $[a, b]$.*
2. *If f'' exists and $f'' < 0$ on $[a, b]$, then f is concave on $[a, b]$.*

A point x_0 is an **inflection point** of a function f if the convexity changes at that point, i.e. (id est), f is convex on one side (near the point) and concave on the other side.

If x_0 is an inflection point of f and the function is twice differentiable at that point, then $f''(x_0) = 0$.

Example: 4.1. x^3 has an inflection point at 0, but x^4 does not.

In order to find the intervals of convexity of a function, as well as the inflection points, we determine the sign of the second derivative on the different intervals.

4.1.2 Asymptotes

An **asymptote** of a function f as $x \to \infty$ is another function g such that

$$\lim_{x\to\infty} (f(x) - g(x)) = 0,$$

and g is asymptote as $x \to -\infty$ if

$$\lim_{x\to-\infty} (f(x) - g(x)) = 0.$$

We are going to study only asymptotes that are straight lines:

1. f has a **vertical asymptote** at $x = x_0$ if some of the side limits at that point is infinite:

$$\lim_{x\to x_0^{\pm}} f(x) = \pm\infty.$$

2. f has the **horizontal asymptote** $y = a$ as $x \to \infty$ if $\lim_{x\to\infty} f(x) = a$.

   The asymptote is for $x \to -\infty$ if $\lim_{x\to-\infty} f(x) = a$.

3. f has a **slant (oblique) asymptote** $y = mx + b$ as $x \to \infty$ if the following limits are finite:

$$m = \lim_{x\to\infty} \frac{f(x)}{x}, \qquad b = \lim_{x\to\infty} \Big(f(x) - mx\Big).$$

   The asymptote is as $x \to -\infty$ if this happens when the limits are taken for $x \to -\infty$.

Observation 4.1. 1. The vertical asymptotes can be at the end points of the domain of f, if they are finite, and never at points where f is continuous.

2. The graph of f can cut a horizontal or slant asymptote, but never a vertical asymptote.

3. The polynomials that are not a straight line (degrees 0 or 1) never have straight asymptotes.

Example: 4.2. $f(x) = \dfrac{x^2}{x^2 - 1}$ has the horizontal asymptote $y = 1$ as $x \to \pm\infty$ and the vertical asymptotes $x = 1$, $x = -1$.

Now we can plot graphs, studying f in the following steps:

Construction of graphs

1. Domain and symmetries.
2. Asymptotes.
3. Critical points and points without derivative.
4. Increasing and decreasing intervals.
5. Local extrema.
6. Convexity and concavity intervals.
7. Inflection points.

4.2 Other applications of the derivative

4.2.1 Implicit derivative

Some curves are defined easily using an implicit expression, like $x^2+y^2=1$, but we need two functions to describe it explicitly.

If we are sure that the expression defines one or more functions in some domain we can differentiate directly in that formula, that is, **implicitly** and then clear y':

Example: 4.3. In the formula $2x^2+y^2=4$ (an ellipse) we differentiate with respect to x and obtain:

$$4x+2yy'=0, \qquad y'=-\frac{2x}{y}.$$

4.2.2 Logarithmic derivative

Remember first that $\log'|x| = \dfrac{1}{x}$ and that $(\log f(x))' = \dfrac{f'(x)}{f(x)}$. We can apply this to simplify the calculation of some derivatives of products and quotients:

1. **Derivatives of products:** If $f(x)=g_1(x)\cdot g_2(x)\cdots g_n(x)$ then

$$\log|f(x)| = \sum_{j=1}^{n}\log|g_j(x)| \quad\Longrightarrow\quad \frac{f'(x)}{f(x)} = \sum_{j=1}^{n}\frac{g_j'(x)}{g_j(x)}$$

$$\Longrightarrow \quad f'(x) = f(x)\sum_{j=1}^{n}\frac{g_j'(x)}{g_j(x)}.$$

2. **Derivatives of quotients:** If $f(x)=\dfrac{g_1(x)\cdot g_2(x)\cdots g_n(x)}{h_1(x)\cdot h_2(x)\cdots h_m(x)}$, then

$$\log|f(x)| = \sum_{j=1}^{n}\log|g_j(x)| - \sum_{k=1}^{m}\log|h_k(x)| \Longrightarrow \frac{f'(x)}{f(x)} = \sum_{j=1}^{n}\frac{g_j'(x)}{g_j(x)} - \sum_{k=1}^{m}\frac{h_k'(x)}{h_k(x)}$$

$$\Longrightarrow \quad f'(x) = f(x)\left(\sum_{j=1}^{n}\frac{g_j'(x)}{g_j(x)} - \sum_{k=1}^{m}\frac{h_k'(x)}{h_k(x)}\right).$$

4.3 Taylor polynomial

4.3.1 Construction

We want to approximate a function near a given point by a polynomial. The idea is to impose that the successive derivatives of the polynomial and the function coincide at the point.

For the point $x_0 = 0$ and the polynomial of degree n:

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = \sum_{k=0}^{n} a_k x^k,$$

the values of the derivatives at $x_0 = 0$ are easy:

$$P_n^{k)}(0) = k!\, a_k \quad \Longrightarrow \quad a_k = \frac{P_n^{k)}(0)}{k!}\,.$$

If now we consider a general point x_0, writing the polynomial as

$$\begin{aligned} P_n(x) &= a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n \\ &= \sum_{k=0}^{n} a_k (x-x_0)^k. \end{aligned}$$

The value of the derivatives at x_0 is:

$$P_n^{k)}(x_0) = k!\, a_k \quad \Longrightarrow \quad a_k = \frac{P_n^{k)}(x_0)}{k!}.$$

This suggests to approximate a function (that can be hard to manage) near a point x_0 by a polynomial with the same derivatives up to the n-th order. This means that:

$$a_k = \frac{f^{k)}(x_0)}{k!}.$$

So we define the **Taylor polynomial** of degree n of f near the point x_0, as

$$P_{n,x_0} f(x) = \sum_{k=0}^{n} \frac{f^{k)}(x_0)}{k!}(x-x_0)^k.$$

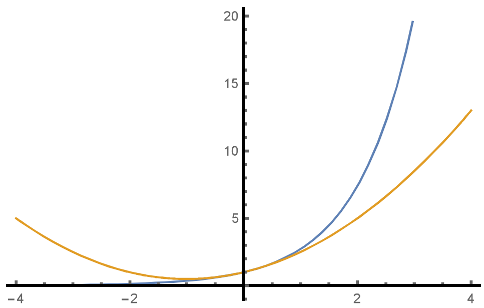
When the point is $x_0 = 0$ it is also known as **McLaurin polynomial**.

A polynomial of degree n is determined with $n+1$ data, so the Taylor polynomial is the unique polynomial of degree less than or equal to n with that property.

Example: 4.4. Let $f(x) = \mathrm{e}^x$. Since $f^{k)}(0) = 1 \;\forall\, k \geq 0$, the Taylor polynomial is

$$P_{n,0}f(x) = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

In the picture we see $f(x) = \mathrm{e}^x$ and $P_{2,0}f(x) = 1 + x + \dfrac{x^2}{2}$.



Main Taylor polynomials

In the following examples we take $x_0 = 0$.

$$\sin x \quad \rightsquigarrow \quad x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x \quad \rightsquigarrow \quad 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\frac{1}{1-x} \quad \rightsquigarrow \quad 1 + x + x^2 + x^3 + \cdots + x^n$$

$$\log(1+x) \quad \rightsquigarrow \quad x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{(-1)^{n+1} x^n}{n}$$

An example with $x_0 = 1$ is:

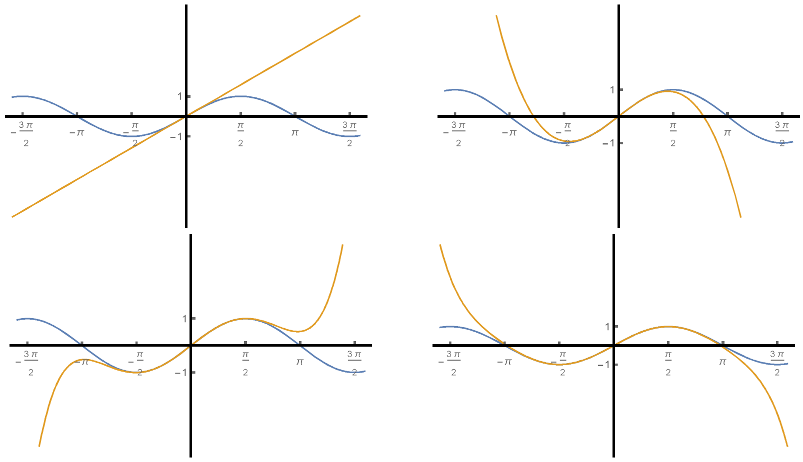
$$\log x \quad \rightsquigarrow \quad (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots + \frac{(-1)^{n+1}(x-1)^n}{n}$$

Theorem 4.4. *(Taylor)*
If f and its derivatives up to order n exist at x_0, then

$$\lim_{x \to x_0} \frac{f(x) - P_{n,x_0}f(x)}{(x-x_0)^n} = 0.$$

That is, the polynomial approaches the function near the point, and this approximation is better as the degree of the polynomial is bigger.

Example: 4.5. In the following picture we see the function $f(x) = \sin x$ compared with its Taylor polynomials at the origin of degrees 1, 3, 5 and 7.



We say that a function is **small-o** of $g(x)$ as $x \to x_0$ and write $f(x) = o\left(g(x)\right)$ if

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0.$$

This is Landau's notation.

We can write then,

$$f(x) = P_{n,x_0} f(x) + o(|x - x_0|^n) \qquad \text{for } x \to x_0.$$

Example: 4.6.

$$\cos x - 1 = o(\sin x) \qquad \text{for } x \to 0$$

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2} + o(x^2) \qquad \text{for } x \to 0$$

$$\log x = o(x) \qquad \text{for } x \to \infty.$$

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4.3.2 Properties

Theorem 4.5. *(Polynomials)*
If P and Q are two polynomials in powers of $(x - x_0)$ of degree less than or equal to n and are equal up to order n at x_0, then $P = Q$.

The expression equal up to order n means that:

$$P(x_0) = Q(x_0),\ P'(x_0) = Q'(x_0),\ P''(x_0) = Q''(x_0), \dots, P^{n)}(x_0) = Q^{n)}(x_0).$$

And observe that with $n+1$ data we determine in an unique way a polynomial of degree n.

Corollary 4.6. *()*
If P is a polynomial of degree at most n in powers of $(x - x_0)$ that is equal up to order n to an n-times differentiable function f at x_0, then it is the Taylor polynomial of f at x_0 of order n.

This means that we can obtain a Taylor polynomial using different methods. For example, the Taylor polynomial of degree n at $x = x_0$ of a polynomial of degree less than or equal to n is the same polynomial, but written in powers of $(x - x_0)$.

Example: 4.7. Taylor polynomial for $\arctan x$: We start with the derivative:

$$\arctan'(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \frac{(-1)^n x^{2n+2}}{1+x^2}.$$

Then:

$$\arctan(x) = \int_0^x \frac{dt}{1+t^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \int_0^x \frac{(-1)^n t^{2n+2} dt}{1+t^2}.$$

Observe that

$$\left| \int_0^x \frac{t^{2n+2} dt}{1+t^2} \right| \le \left| \int_0^x t^{2n+2} dt \right| = \frac{|x|^{2n+3}}{2n+3}.$$

Then:

$$\lim_{x \to 0} \frac{\displaystyle\int_0^x \frac{(-1)^n t^{2n+2} dt}{1+t^2}}{x^{2n+1}} = 0.$$

So, the Taylor polynomial of $\arctan x$ is:

$$P_{2n+1,0}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1}.$$

We call **Taylor remainder** of order n at $x = x_0$ of the function f to the difference:

$$R_{n,x_0}f(x) = f(x) - P_{n,x_0}f(x),$$

and then $R_{n,x_0}f(x) = o\left((x - x_0)^n\right)$.

THEOREM 4.7. *(Taylor's theorem)*
Suppose that $f, f', f'', \ldots, f^{n+1)}$ are all defined on $[x_0, x]$ (if $x > x_0$, and on $[x, x_0]$ if $x < x_0$) and consider the Taylor remainder $R_{n,x_0}f(x)$. Then:

$$R_{n,x_0}f(x) = \frac{f^{n+1)}(t)}{(n+1)!}(x - x_0)^{n+1},$$

for some $t \in (x_0, x)$ (or $t \in (x, x_0)$ if $x < x_0$). This is the ***Lagrange form of the remainder****.*

Example: 4.8. Consider $f(x) = \sin x$. In order to estimate $\sin 1$ using the Taylor polynomial of degree 5, we see that

$$|R_{5,0}f(1)| \leq \frac{1}{6!}$$

and then

$$\sin 1 = 1 - \frac{1}{6} + \frac{1}{120} + \varepsilon, \qquad |\varepsilon| \leq \frac{1}{720}$$

We obtain the value $\sin 1 \approx 0.8416$, with an error of 0.0014. The exact value with 6 significative figures is 0.841471.

4.3.3 Applications

1. **Calculus of limits**: We can approximate functions that appear in limits, using the Taylor polynomial of the proper degree.

Example: 4.9.

$$\begin{aligned}\lim_{x\to 0}\frac{\cos x - \mathrm{e}^x + x}{x^2} &= \lim_{x\to 0}\frac{1 - x^2/2 + o(x^2) - (1 + x + x^2/2 + o(x^2)) + x}{x^2}\\ &= \lim_{x\to 0}\frac{-x^2 + o(x^2)}{x^2} = -1.\end{aligned}$$

2. **Characterization of extrema:** Using Taylor's theorem we can characterize the critical points according to the first nonzero derivative.

> **THEOREM 4.8.** *(Extrema with Taylor)*
> *Assume that* $f'(x_0) = f''(x_0) = \cdots = f^{k-1)}(x_0) = 0$, $f^{k)}(x_0) \neq 0$, $k \geq 1$. *Then*
>
> a) *If* k *is even and* $f^{k)}(x_0) > 0$, *the point* $x = x_0$ *is a local minimum.*
>
> b) *If* k *is even and* $f^{k)}(x_0) < 0$, *the point* $x = x_0$ *is a local maximum.*
>
> c) *If* k *is odd, the point* $x = x_0$ *is not a maximum nor a minimum, it is an inflection point.*

The idea is that near the point $x = x_0$:

$$f(x) \approx f(x_0) + \frac{f^{k)}(x_0)}{k!}(x - x_0)^k.$$

If k is even, $(x - x_0)^k$ is positive and $f(x)$ is bigger or smaller than $f(x_0)$ depending on the sign of $f^{k)}(x_0)$. If k is odd then the last term changes sign at both sides of $x = x_0$.

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