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Departamento de Matemáticas

DIFFERENTIAL CALCULUS

Degree in Applied Mathematics and Computation

Chapter 5

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5

SEQUENCES AND SERIES OF REAL NUMBERS

We study in this chapter the sequences of numbers with their convergence and limits and also the series of numbers, that are sums of all the terms of a sequence. What we know about functions will be useful here.

Contents

5.1 Sequences of numbers

5.1.1 Definitions and properties

A list of infinite numbers in a precise order is called a **sequence**. It is the image of a function whose domain is the set of the natural numbers:

$$f : \mathbb{N} \to \mathbb{R}, \qquad f(n) = a_n.$$

The sequence is denoted by

$$\{a_n\}_{n=1}^{\infty} = \{a_1,\, a_2,\, a_3, \cdots\}$$

It also may start with a_0 or a_k. The general term is $a_n = f(n)$.

We are interested in the convergence, and we say that the sequence $\{a_n\}_{n=1}^{\infty}$ **converges to the limit** ℓ (ℓ finite) if:

$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N}$ such that $n > N \Longrightarrow |a_n - \ell| < \varepsilon$, we write $\lim\limits_{n\to\infty} a_n = \ell$.

We say that $\{a_n\}_{n=1}^{\infty}$ **converges** if it converges to some finite ℓ and that it **diverges** if it does not converge to any finite limit.

Example: 5.1. $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$, and the two sequences $a_n = (-1)^n,\ b_n = 2n + 3$ diverge both.

Theorem 5.1. *(Uniqueness of the limit)*
If the limit of a sequence exists, then it is unique.

We need some more definitions:

1. A sequence $\{a_n\}_{n=1}^{\infty}$ is **increasing** if $a_{n+1} \geq a_n \quad \forall\, n \in \mathbb{N}$, it is also called **non decreasing**. It is **strictly increasing** if $a_{n+1} > a_n$ $\forall\, n \in \mathbb{N}$.

2. $\{a_n\}_{n=1}^{\infty}$ is **decreasing** if $a_{n+1} \leq a_n \quad \forall\, n \in \mathbb{N}$, it is also called **non increasing**. It is **strictly decreasing** if $a_{n+1} < a_n \quad \forall\, n \in \mathbb{N}$.

3. $\{a_n\}_{n=1}^{\infty}$ is **bounded above** if $a_n \leq C \quad \forall\, n \in \mathbb{N}$ for some $C \in \mathbb{R}$. It is **bounded below** if $a_n \geq c \quad \forall\, n \in \mathbb{N}$ for some $c \in \mathbb{R}$.

4. $\{a_n\}_{n=1}^{\infty}$ is a **recurring sequence** if each term is given by a function of some previous terms, for example:

$$a_{n+1} = g(a_n), \quad n \geq 1, \quad a_1 = K.$$

> **THEOREM 5.2.** *(Monotonous sequences)*
>
> 1. *If $\{a_n\}_{n=1}^{\infty}$ is increasing and bounded above, then it converges.*
> 2. *If $\{a_n\}_{n=1}^{\infty}$ is decreasing and bounded below, then it converges.*

Example: 5.2. $a_n = \dfrac{n+1}{n+2}$ is increasing, since if $a_n = f(n)$ then $f'(x) = \dfrac{1}{(x+2)^2}$. Moreover $a_n \leq 1$ for every n. Thus the sequence is convergent. The limit is $\lim_{n\to\infty} a_n = \lim_{t\to 0} \dfrac{1+t}{1+2t} = 1$.

> **THEOREM 5.3.** *(Boundedness)*
> *Any convergent sequence is bounded.*

A **subsequence** is a sequence extracted from another given sequence, where the terms are in the same order of the original sequence.

> **THEOREM 5.4.** *(Subsequences)*
> *Any subsequence of a convergent sequence is convergent to the same limit of the original sequence. If a sequence has subsequences with different limits, the sequence diverges.*

Example: 5.3. $a_n = (-1)^n \arctan n$ diverges because $a_{2n} \to \dfrac{\pi}{2}$ while $a_{2n+1} \to \dfrac{-\pi}{2}$.

> **LEMMA 5.5.** *(Rising sun lemma)*
> *Any convergent sequence has a subsequence that is either non increasing or non decreasing.*

For the proof: The number n is a **peak point** of $\{a_n\}_{n=1}^{\infty}$ if $a_m < a_n$ for $m > n$ (i.e. a_n receives the rising sun light).

Theorem 5.6. *(Bolzano-Weierstrass)*
Any bounded sequence has a convergent subsequence. Any non-bounded sequence has a subsequence that tends to ∞ or to $-\infty$.

Example: 5.4. $a_n = (-1)^n \arctan n$ is bounded and divergent and the subsequence a_{2n} is convergent to $\pi/2$.

A number k is an **accumulation point** of a set $A \in \mathbb{R}$ if:

$$\forall \varepsilon > 0 \quad \exists a \in A, \quad a \neq k \;:\; |k - a| < \varepsilon.$$

Example: 5.5. $\dfrac{\pi}{2}$ and $-\dfrac{\pi}{2}$ are accumulation points of $a_n = (-1)^n \arctan n$.

Even more, a new kind of sequence is useful to study convergence: We say that $\{a_n\}_{n=1}^{\infty}$ is a **Cauchy sequence** if:

$$\lim_{m,n\to\infty} |a_m - a_n| = 0.$$

Theorem 5.7. *(Cauchy sequences)*
A sequence converges if and only if it is a Cauchy sequence.

Example: 5.6. $a_n = (-1)^n$ is not a Cauchy sequence because $|a_n - a_{n+1}| = 2 \quad \forall n \in \mathbb{N}$.

5.1.2 Limits

Properties

If $\lim_{n\to\infty} a_n = \ell$ and $\lim_{n\to\infty} b_n = m$, both finite, then:

1. $\lim_{n\to\infty} (a_n + b_n) = \ell + m$.
2. $\lim_{n\to\infty} a_n \cdot b_n = \ell \cdot m$.
3. $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\ell}{m}$, if $m \neq 0$ and $b_n \neq 0$ for n big enough.

4. If h is a continuous function at ℓ, then

$$\lim_{n\to\infty} h(a_n) = h(\lim_{n\to\infty} a_n) = h(\ell).$$

Techniques for limits

1. If $a_n = f(n)$, where f is a function defined on $\mathbb{R}$ (or at least on $[M, \infty)$ for some M), then

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$$

whenever this last limit exists. We can use the techniques we know to calculate the limit of f, such as the sandwich lemma, L'Hôpital's rule or Taylor's theorem.

Example: 5.7.

$$\lim_{n\to\infty} \big(\cos(3/n)\big)^{n^2} = \lim_{x\to 0} \big(\cos(3x)\big)^{1/x^2} = \mathrm{e}^{\lim_{x\to 0} \frac{\cos(3x)-1}{x^2}} = \mathrm{e}^{-9/2}.$$

2.

Theorem 5.8. *(Stolz criterion)*
If one of the following properties is true

a) $\{b_n\}_{n=1}^{\infty}$ *is strictly increasing with* $\lim_{n\to\infty} b_n = \infty$, *or*

b) $\{b_n\}_{n=1}^{\infty}$ *is strictly decreasing with* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$,

then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$$

whenever this last limit exists.

Example: 5.8.

$$\lim_{n\to\infty} \frac{1 + 2 + 4 + \cdots + 2^n}{2^n} = \lim_{n\to\infty} \frac{2^{n+1}}{2^{n+1} - 2^n} = 2.$$

3. **Stirling's formula**:

$$\lim_{n\to\infty} \frac{n!}{(n/\mathrm{e})^n \sqrt{2\pi n}} = 1\,.$$

It is useful when factorials are involved.

Example: 5.9.

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = \lim_{n\to\infty} \frac{\mathrm{e}n}{n(2\pi n)^{1/2n}} = \mathrm{e}.$$

5.1.3 Recurring sequences

Theorem 5.9. *(Recurring sequences)*
If $a_{n+1} = g(a_n)$, where g is a differentiable function defined on $\mathbb{R}$, then it is monotonous if $g' \geq 0$ and oscillating if $g' \leq 0$.

This is true because using the mean value theorem we obtain:

$$a_{n+1} - a_n = g(a_n) - g(a_{n-1}) = g'(c)(a_n - a_{n-1}), \quad \text{for some } c.$$

Even more, we have:

Theorem 5.10. *(Fixed point theorem)*
If $a_{n+1} = g(a_n)$, where g is a differentiable function defined on $\mathbb{R}$, with $|g'(x)| \leq \lambda < 1$ on some interval $I \subset \mathbb{R}$, and $a_k \in I$ if $k \geq M$ for some M, then $\{a_n\}_{n=1}^{\infty}$ is a convergent sequence and the limit ℓ is the only fixed point of g, that is $\ell = g(\ell)$, on I.

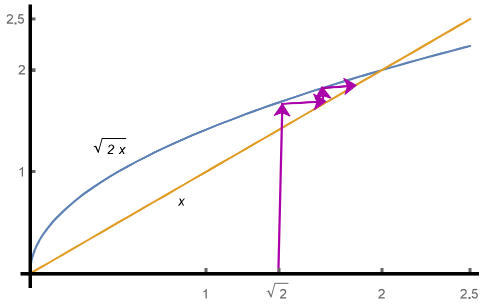
Example: 5.10. $a_{n+1} = \sqrt{2a_n}$ with $a_1 = 1$ is increasing, since $g'(x) = \dfrac{1}{\sqrt{2x}}$ and $a_2 - a_1 = \sqrt{2} - 1 > 0$. It is moreover bounded (by induction) $a_n \leq 2$. It is therefore convergent, and the limit satisfies $\ell = \sqrt{2\ell}$, that is $\ell = 0$ or $\ell = 2$. But $\ell > a_1 = 1$, so $\lim_{n\to\infty} a_n = 2$.

Example: 5.11. $a_{n+1} = 1 - \dfrac{a_n}{2}$, with $a_1 = 10$, is convergent since $|g'(x)| = \dfrac{1}{2}$ (oscillating since $g' < 0$) and the limit satisfies $\ell = 1 - \dfrac{\ell}{2}$, that is, $\lim_{n\to\infty} a_n = \dfrac{2}{3}$.

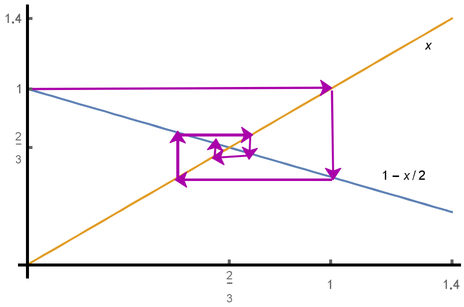
Cobweb diagram

The behaviour of a recurring sequence can be represented in a picture, known as cobweb diagram. Here we have two examples, one monotonic and the other oscillating.

Example: 5.12. $a_{n+1} = \sqrt{2a_n}, \quad a_1 = \sqrt{2}, \qquad \lim_{n\to\infty} a_n = 2.$



Example: 5.13. $\quad b_{n+1} = 1 - \dfrac{b_n}{2}, \quad b_1 = 0, \quad \lim_{n\to\infty} b_n = 2/3.$



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5.2 Series of numbers

5.2.1 Preliminaries

A **series** is the sum of all the terms of a sequence: $\sum_{n=1}^{\infty} a_n$. It is **convergent** if the sequence of the partial sums is convergent, that is, if:

$$\lim_{N\to\infty} \sum_{n=1}^{N} a_n = L,$$

for some finite L, this is called the **sum of the series**. In this case we say that the sequence $\{a_n\}_{n=1}^{\infty}$ is **summable**. If L is not finite or if it does not exist the series is **divergent**.

Example: 5.14. The series $\sum_{n=1}^{\infty} r^n$ is called **geometric series** of **ratio** r. We have:

$$\begin{cases} |r| < 1 \quad \Rightarrow \quad \displaystyle\sum_{n=1}^{\infty} r^n = \lim_{N\to\infty} \frac{r^{N+1} - r}{r - 1} = \frac{r}{1-r} \rightsquigarrow \text{ converges,} \\ |r| \geq 1 \quad \Rightarrow \quad \displaystyle\sum_{n=1}^{\infty} r^n \rightsquigarrow \text{ diverges.} \end{cases}$$

As a direct application of the Cauchy sequences we have the following:

Theorem 5.11. *(Cauchy criterion)*
The sequence $\{a_n\}_{n=1}^{\infty}$ is summable if and only if

$$\lim_{n\to\infty} (a_{n+1} + a_{n+2} + \cdots + a_m) = 0.$$

This is difficult to compute, but it has the following consequence:

Theorem 5.12. *(Remaind criterion)*
If a series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

This is a necessary condition, but it is not enough to obtain convergence:

Example: 5.15. The **harmonic series**, $\sum_{n=1}^{\infty} \frac{1}{n}$, satisfies this criterion, but it diverges.

In general, for a convergent series it is not possible to obtain the value of the sum, but it is essential to know if it converges or not if we want to apply it in any numerical approximation.

To determine the convergence we have **convergence tests**, that we study in the following sections.

5.2.2 Series of non-negative terms

If $a_n \geq 0 \;\; \forall n \in \mathbb{N}$ then the sequence $\left\{ \sum_{n=1}^{N} a_n \right\}_{N \in \mathbb{N}}$ is increasing, so $\sum_{n=1}^{\infty} a_n$ converges if the set of the partial sums is bounded.

Theorem 5.13. *(Comparison test)*
Assume $0 \leq a_n \leq b_n$ *for every* $n \geq k$ *and some* k. *Then*

$$\begin{cases} \sum^{\infty} b_n \text{ converges} & \Rightarrow \quad \sum^{\infty} a_n \text{ converges}, \\ \sum^{\infty} a_n \text{ diverges} & \Rightarrow \quad \sum^{\infty} b_n \text{ diverges}. \end{cases}$$

Theorem 5.14. *(Ratio test)*
When $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell$ *then*

$$\begin{cases} \ell < 1 \quad \Rightarrow \quad \sum^{\infty} a_n \text{ converges}, \\ \ell > 1 \quad \Rightarrow \quad \sum^{\infty} a_n \text{ diverges}, \\ \text{if } \ell = 1 \text{ the test is inconclusive}. \end{cases}$$

This is also known as **D'Alembert's criterion.**

Example: 5.16. $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ converges since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$.

THEOREM 5.15. *(Root test)*
If we have $\lim_{n\to\infty} \sqrt[n]{a_n} = \ell$, *then*

$$\begin{cases} \ell < 1 \quad \Rightarrow \quad \sum^{\infty} a_n \text{ converges,} \\ \ell > 1 \quad \Rightarrow \quad \sum^{\infty} a_n \text{ diverges,} \\ \text{if } \ell = 1 \text{ the test is inconclusive.} \end{cases}$$

This last one is known as **Cauchy's criterion**.

Example: 5.17. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+1}{n+5}\right)^n$ diverges since $\lim_{n\to\infty} \sqrt[n]{a_n} = 2 > 1$.

The root test is stronger than the ratio test: whenever the ratio test determines the convergence or divergence of a series, the root test does too, but not conversely. Moreover

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell \qquad \Longrightarrow \qquad \lim_{n\to\infty} \sqrt[n]{a_n} = \ell.$$

Thus if we try the ratio test and get $\ell = 1$ as limit, it is useless to computing the root test.

THEOREM 5.16. *(Limit comparison test)*
If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = c$, *with* $0 < c < \infty$, *then:*

1. *The series* $\sum^{\infty} a_n$ and $\sum^{\infty} b_n$ *both converge or both diverge.*
2. *If* $c = 0$ *and* $\sum^{\infty} b_n$ *converges, then* $\sum^{\infty} a_n$ *converges and if* $\sum^{\infty} a_n$ *diverges, then* $\sum^{\infty} b_n$ *diverges.*
3. *If* $c = \infty$ *and* $\sum^{\infty} b_n$ *diverges, then* $\sum^{\infty} a_n$ *diverges and if* $\sum^{\infty} a_n$ *converges, then* $\sum^{\infty} b_n$ *converges.*

THEOREM 5.17. *(Integral test)*
If f is a continuous positive decreasing function on $[0, \infty)$ and $a_n = f(n)$ $\forall n$, then the series $\sum^{\infty} a_n$ converges if and only if the following limit exists:

$$\int_1^{\infty} f(x)dx = \lim_{A\to\infty} \int_1^A f(x)dx.$$

Example: 5.18. The p-**harmonic series**, defined by $\sum^{\infty} \frac{1}{n^p}$ are a generalization of the harmonic series (where $p = 1$). Using the integral test we obtain that:

$$\begin{cases} p > 1 \quad \Rightarrow \quad \sum^{\infty} \frac{1}{n^p} \text{ converges,} \\ p \leq 1 \quad \Rightarrow \quad \sum^{\infty} \frac{1}{n^p} \text{ diverges.} \end{cases}$$

Example: 5.19. $\sum_{n=1}^{\infty} \arcsin(1/\sqrt{n})$ diverges since $\lim_{n\to\infty} \frac{a_n}{n^{-1/2}} = 1$, and the 1/2-series diverges.

5.2.3 Absolute convergence of series

If $a_n < 0$ $\forall n > K$ for some K, it can be studied like a series of positive terms, taking the sign outside the sum. The problem arises if there are infinite positive terms and infinite negative terms. To study it we define two kinds of convergence:

A series $\sum^{\infty} a_n$ with $a_n \in \mathbb{R}$ is **absolutely convergent** if the series of absolute values, $\sum^{\infty} |a_n|$, converges. The series is **conditionally convergent** if it converges but not absolutely.

THEOREM 5.18. *(Absolute convergence)*
If a series converges absolutely, then it converges. Besides, it is absolutely convergent if and only if the series of its positive terms and the series of its negative terms both converge.

Some sequences that are easy to study are the **alternating series**, that have the form $\sum_{n=1}^{\infty}(-1)^n a_n$, where $a_n \geq 0$. If the series $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty}(-1)^n a_n$ also converges, but the converse is false.

Theorem 5.19. *(Leibniz criterion)*
If the sequence $\{a_n\}$ is decreaasing with $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

Example: 5.20. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally.

Conditional convergence is very strange: If we have a **rearrangement** of a sequence, that is, a sequence with the same terms but in different order, then:

Theorem 5.20. *(Conditional convergence)*
If $\sum_{1}^{\infty} a_n$ converges but not absolutely, then for any $\alpha \in \mathbb{R}$ there is a rearrangement $\{b_n\}_{n=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$, such that $\sum_{1}^{\infty} b_n = \alpha$.

But when we have absolute convergence:

Theorem 5.21. *(Absolute convergence)*
If $\sum_{1}^{\infty} a_n$ converges absolutely, and $\{b_n\}_{n=1}^{\infty}$ is any rearrangement of $\{a_n\}_{n=1}^{\infty}$ then $\sum_{1}^{\infty} b_n$ also converges absolutely and $\sum_{1}^{\infty} a_n = \sum_{1}^{\infty} b_n$.

Some explicit sums

1. **Geometric series:**

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \qquad \sum_{n=k}^{\infty} r^n = \frac{r^k}{1-r}, \qquad \text{if } |r| < 1.$$

2. **Arithmetic-geometric series:** $\sum_{n=0}^{\infty}(an+b)r^n$. We use

$$\sum_{n=1}^{\infty} nr^n = \frac{r}{(1-r)^2}, \qquad \text{if } |r| < 1.$$

Thus

$$\sum_{n=0}^{\infty}(an+b)r^n = \frac{ar}{(1-r)^2} + \frac{b}{1-r}.$$

3. **Telescopic series:**

$$\sum_{n=1}^{\infty}(b_{n+1} - b_n) = \lim_{n\to\infty} b_n - b_1.$$

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