uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL CALCULUS

Degree in Applied Mathematics and Computation

Chapter 6

Elena Romera

Open Course Ware, UC3M

6

SEQUENCES AND SERIES OF FUNCTIONS

We study in this short chapter the sequences and series of functions, leading to the study of Taylor series.

Contents

6.1 Sequences of functions

A **sequence of functions** is a sequence whose elements are functions: $\{f_n\}_{n\in\mathbb{N}}$. For each value x, $\{f_n(x)\}_{n\in\mathbb{N}}$ is a sequence of numbers, so we have to study infinite sequences of numbers when we study a sequence of functions.

Some properties are surprising:

1. f_n continuous $\forall n \in \mathbb{N}$ does not imply $\lim\limits_{n\to\infty} f_n(x) = f(x)$ continuous.

   Example: 6.1.

$$f_n(x) = \begin{cases} x^n, & 0 \le x < 1, \\ 1, & 1 \le x, \end{cases} \longrightarrow \lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 0, & 0 \le x < 1, \\ 1, & 1 \le x, \end{cases}$$

2. f_n differentiable $\forall n \in \mathbb{N}$ does not imply $\lim\limits_{n\to\infty} f_n(x) = f(x)$ differentiable.

   Example: 6.2.

$$f_n(x) = \begin{cases} -1, & x \le \frac{-1}{n}, \\ \sin\frac{n\pi x}{2}, & \frac{-1}{n} \le x \le \frac{1}{n} \\ 1, & \frac{1}{n} \le x \end{cases} \longrightarrow \lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

3. f_n with the same area below $\forall n \in \mathbb{N}$ does not imply $\lim\limits_{n\to\infty} f_n(x) = f(x)$ with the same area.

   Example: 6.3.

$$f_n(x) = \begin{cases} 2n^2x, & 0 < x \le \frac{1}{2n}, \\ 2n - 2n^2x, & \frac{1}{2n} \le x \le \frac{1}{n} \\ 0, & \frac{1}{n} \le x \le 1, \end{cases} \longrightarrow \lim_{n\to\infty} f_n(x) = f(x) = 0$$

In all these examples we have that:

$$\forall\varepsilon > 0,\ \forall x \in \mathrm{Dom}(f), \quad \exists N(\varepsilon, x): \quad n > N \quad \Longrightarrow \quad |f_n(x) - f(x)| < \varepsilon.$$

So we have to study different kinds of convergence:

1. A sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ **converges punctually** to f on $\mathrm{Dom}(f)$ if

$$\lim_{n\to\infty} f_n(x) = f(x), \quad \forall x \in \mathrm{Dom}(f).$$

2. A sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ **converges uniformly** to f on Dom(f) if

$$\forall\varepsilon > 0 \quad \exists N(\varepsilon): \quad n > N \quad \Longrightarrow \quad |f_n(x) - f(x)| < \varepsilon \quad \forall x \in \mathrm{Dom}(f).$$

Uniform convergence $\Longrightarrow$ punctual convergence (the converse is false).

The convergence of the previous examples is not uniform.

Theorem 6.1. *(Uniform convergence of sequences)*

1. *If $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f on $[a,b]$ and f_n and f are all integrable on $[a,b]$, then:*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx.$$

2. *If $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f on $[a,b]$ and each f_n is continuous on $[a,b]$, then f is continuous on $[a,b]$.*

3. *If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of differentiable functions on $[a,b]$ and converges punctually to f and $\{f'_n\}_{n\in\mathbb{N}}$ converges uniformly on $[a,b]$ to some continuous function g, then f is differentiable on $[a,b]$ and*

$$f'(x) = g(x) = \lim_{n\to\infty} f'_n(x).$$

Observe that the differentiability needs more conditions.

Example: 6.4. $f_n(x) = \frac{1}{n}\sin(n^2x)$ converges uniformly to $f(x) = 0$ and all the functions are differentiable but: $f'_n(x) = n\cos(n^2x)$ does not tend to zero.

6.2 Series of functions

A **series of functions** is the sum of the terms of a sequence of functions, $\sum_{n=1}^{\infty} f_n$.

The series $\sum_{n=1}^{\infty} f_n$ **converges uniformly** to f on A if the sequence of the partial sums converges uniformly to f on A.

Theorem 6.2. *(Uniform convergence of series)*

1. *If* $\sum_{n=1}^{\infty} f_n$ *converges uniformly to* f *on* $[a,b]$ *and* f_n *and* f *are all integrable on* $[a,b]$*, then:*

$$\int_a^b f(x)dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)dx.$$

2. *If* $\sum_{n=1}^{\infty} f_n$ *converges uniformly to* f *on* $[a,b]$ *and each* f_n *is continuous on* $[a,b]$*, then* f *is continuous on* $[a,b]$*.*

3. *If* $\sum_{n=1}^{\infty} f_n$ *converges punctually to* f *on* $[a,b]$ *and* $\sum_{n=1}^{\infty} f'_n$ *converges uniformly on* $[a,b]$ *to some continuous function, then* f *is differentiable on* $[a,b]$ *and*

$$f'(x) = \sum_{n_1}^{\infty} f'_n(x), \quad \forall x \in [a,b].$$

Theorem 6.3. *(Weierstrass M-test)*
Consider $\{f_n\}_{n=1}^{\infty}$*, a sequence of functions defined on* A *and* $\{M_n\}_{n=1}^{\infty}$ *a sequence of numbers with* $|f_n(x)| \leq M_n$*,* $\forall x \in A$*. If* $\sum_{n=1}^{\infty} M_n$ *converges then* $\sum_{n=1}^{\infty} f_n$ *converges uniformly on* A *and* $\sum_{n=1}^{\infty} f_n(x)$ *converges absolutely to the function:*

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

Example: 6.5. The series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ converges uniformly because:

$$\frac{\sin nx}{n^2} \leq \frac{1}{n^2},$$

and the 2-harmonic series converges.

6.3 Taylor series

A **power series** is a sum of the form

$$S(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Also, we define for a function f that admits infinite derivatives its **Taylor series** centered at x_0 as the power series:

$$\sum_{n=0}^{\infty} \frac{f^{n)}(x_0)}{n!}(x - x_0)^n.$$

This is a generalization of the Taylor polynomial, so if the Taylor remainder $R_{n,x_0} f(x)$ tends to zero when $n \to \infty$ this coincides with f and we have:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n)}(x_0)}{n!}(x - x_0)^n.$$

A power series does not converge usually on $\mathbb{R}$, the set of points where it converges is called **convergence set**; it is always a symmetric interval centered at x_0: $(x_0 - R, x_0 + R)$, including or not the end-points. The number $R \in [0, \infty]$ is called **radius of convergence**.

Example: 6.6. $\log(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots,$ $-1 < x \leq 1$. The radius in this case is $R = 1$.

Theorem 6.4. *(Taylor series)*
If the numeric series $\sum_{n=0}^{\infty} a_n (a - x_0)^n$ *converges and* $0 < r < |a - x_0|$, *then, the series* $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ *converges uniformly and absolutely on the interval* $[x_0 - r, x_0 + r]$. *Besides, the same happens with the series* $\sum_{n=1}^{\infty} a_n n (x - x_0)^{n-1}$ *and* f *is differentiable with derivative:*

$$f'(x) = \sum_{n=1}^{\infty} a_n n (x - x_0)^{n-1}, \qquad \forall x : |x - x_0| < |a - x_0|.$$

This means that any convergent power series is a Taylor series. To obtain the radius of convergence we have the following result:

Theorem 6.5. *(Cauchy-Hadamard formula)*
If one of the following limits exists:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|, \qquad \lim_{n\to\infty}\sqrt[n]{|a_n|},$$

(if both exist they are equal) and L is the value, then the radius of convergence R of the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is $R = \frac{1}{L}$, with the convention $1/0 = \infty$, $1/\infty = 0$.

In fact, the limits that appear in this formula are the **limits superior**. The limit superior exists always and is equal to the usual limit if the limit exists. It is defined by (sup means supremum):

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_n, a_{n+1}, a_{n+2}, \dots\}.$$

Example: 6.7. For $a_n = (-1)^n$ we have: $\limsup_{n\to\infty} a_n = 1$.

Some Taylor series

- $\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \qquad x \in \mathbb{R}.$
- $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots, \qquad x \in \mathbb{R}.$
- $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots, \qquad x \in \mathbb{R}.$
- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \qquad -1 < x < 1.$
- $\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots, \qquad -1 < x \leq 1.$

Taylor series are the most important of the power series because:

THEOREM 6.6. *(Convergent power series)*
If a power series centered at a point x_0 converges, then it is the Taylor series of some function $f(x)$ centered at x_0.

– *ERC*–