uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL CALCULUS. Solutions

Degree in Applied Mathematics and Computation

Chapter 1

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Open Course Ware, UC3M

Index

1 Real variable functions

1.1 The real line

Problem 1.1.1 *a*) Direct calculation, for example

$$i)\ a < \sqrt{ab} \iff a^2 < ab \iff a < b.$$

$$\sqrt{ab} < \frac{a+b}{2} \iff 4ab < a^2 + b^2 + 2ab \iff 0 < a^2 + b^2 - 2ab \iff 0 < (a-b)^2 \iff a \neq b.$$

$$ii)\ \frac{a}{b} < \frac{a+k}{b+k} \iff a(b+k) < b(a+k) \iff ak < bk \iff a < b.$$

We have used the hypotheses about the signs of a, b and k; some of the operations are false with different sign.
b)

$$|a+b| = |a| + |b| \iff a^2 + b^2 + 2ab = a^2 + b^2 + 2|a|\,|b| \iff ab = |ab| \iff ab \geq 0.$$

c) By the triangular inequality, $|a| = |a-b+b| \leq |a-b| + |b|$, so $|a-b| \geq |a| - |b|$; by the same reason, $|a-b| \geq |b| - |a|$.

d) $\dfrac{x+y+|x-y|}{2} = \begin{cases} \dfrac{x+y+x-y}{2} & \text{if } x \geq y \\ \dfrac{x+y-x+y}{2} & \text{if } x \leq y. \end{cases}$ *e*) $f(x) = \max\{x, 0\} = \dfrac{x+|x|}{2}.$

Problem 1.1.2 Write $n = z^2 r$, where r contains no square factor. If there exist $p,\ q \in \mathbb{Z}$, with g.c.d.$(p,q) = 1$, such that $p^2/q^2 = n$, we have $p^2 = z^2 q^2 r$, which implies $p = kr$ for some $k \in \mathbb{Z}$. Then, $k^2 r^2 = z^2 q^2 r$, which implies $q = mr$ for some $m \in \mathbb{Z}$. Finally g.c.d.$(p,q) \geq r$ that is a contradiction.

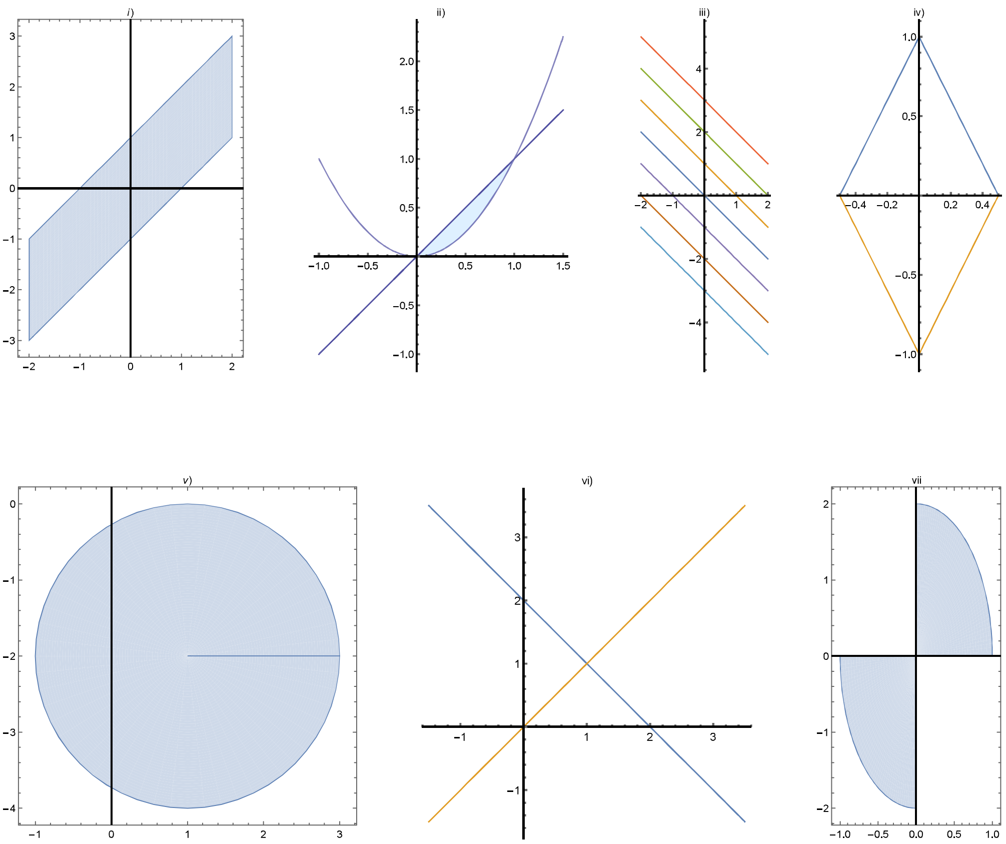
Problem 1.1.3 *i*) $A = \{-8 \leq x-3 \leq 8\} = [-5, 11]$. *ii*) $B = (3/2, 2) \cup (2, 5/2)$.
iii) $C = \{\,(x-2)(x-3) \geq 0\,\} = (-\infty, 2] \cup [3, \infty)$. *iv*) $D = (-\infty, -3) \cup (0, 5)$.
v) $E = \{\,\dfrac{x+4}{(x+1)(x+7)} > 0\,\} = (-7, -4) \cup (-1, \infty)$.
vi) $F = \{\,x^2 > 4,\ x > 0\,\} \cup \{\,x^2 < 4,\ x < 0\,\} = (-2, 0) \cup (2, \infty)$.
vii) $G = [-1, 1/2)$. *viii*) $H = (1-\sqrt{2}, 1) \cup (1, 1+\sqrt{2})$. *ix*) $I = \{\,3,\ -4\,\}$.
x)

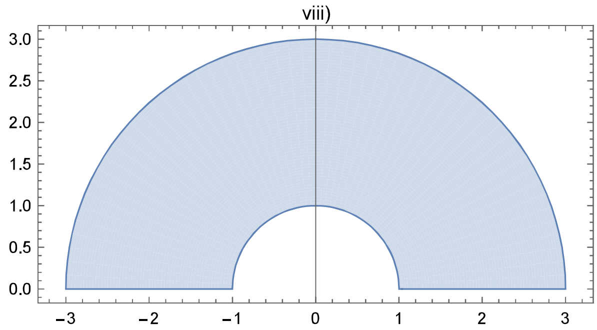
$$\begin{aligned} J &= \{x-1+x-2 > 1,\ x \geq 1,\ x \geq 2\} \cup \{x-1+2-x > 1,\ x \geq 1,\ x \leq 2\} \\ &\cup \{1-x+x-2 > 1,\ x \leq 1,\ x \geq 2\} \cup \{1-x+2-x > 1,\ x \leq 1,\ x \leq 2\} \\ &= \{x > 2\} \cup \emptyset \cup \emptyset \cup \{x < 1\} = (-\infty, 1) \cup (2, \infty). \end{aligned}$$

Problem 1.1.4
i) $A = \{\,a,\ \dfrac{a+b}{2},\ b\}$. *ii*) $B = (a, b)$. *iii*) $C = (-\infty, a)$. *iv*) $D = (b, \infty)$.

Problem 1.1.5 *a*) $\sup A = 3$, $\inf A = \min A = -1$, there is no maximum.
b) $\sup B = \max B = 3$, $\inf B = \min B = -1$.
c) $\sup C = \max C = 3$, $\inf C = 2$, there is no minimum.
d) $\inf D = \min D = 2$, there is no maximum nor supremum.
e) $\sup E = \max E = 3$, $\inf E = \min E = 1/3$.
f) $\sup F = d$, $\inf F = a$, there is no maximum nor minimum.
g) $\sup G = \max G = 7/10$, $\inf G = 0$, there is no maximum.
h) $\sup H = \max H = 2$, $\inf H = -1$, there is no minimum.

Problem 1.1.6 *i*) The interior of a band. *ii*) The region bounded by a parabola and a straight line. *iii*) Infinite parallel lines. *iv*) A diamond (rhombus). *v*) The interior of a circle. *vi*) Two secant lines. *vii*) The interior of two quarters of an ellipse, including the boundary. *viii*) The interior of a half annulus, with the lower boundary and the smaller half circumference.





Problem 1.1.7 The directing vectors are $(1, m)$ and $(1, n)$. They are orthogonal if their scalar product is null: $1 + mn = 0$.

Problem 1.1.8 *a)* $x^2 + (x^2 - 1/4)^2 = (x^2 - \lambda)^2 \Rightarrow \lambda = -1/4$.
b) $(x - a)^2 + (y - b)^2 = (y - \lambda)^2 \Rightarrow y = \alpha x^2 + \beta x + \gamma$ with $\alpha = \dfrac{1}{2(b - \lambda)}$, $\beta = \dfrac{a}{\lambda - b}$, $\gamma = \dfrac{a^2 + b^2 - \lambda^2}{2(b - \lambda)}$.

Problem 1.1.9 *a)* $\sqrt{(x - c)^2 + y^2} + \sqrt{(x + c)^2 + y^2} = 2a \Rightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - c^2} = 1$, an ellipse.
b) $\sqrt{(x - c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = 2a \Rightarrow \dfrac{x^2}{a^2} - \dfrac{y^2}{c^2 - a^2} = 1$, a hyperbola.

1.2 Elementary functions

Problem 1.2.1 *i)* $\mathbb{R} - \{2, 3\}$. *ii)* $\{-1, 1\}$. *iii)* $[-1, 1/\sqrt{2}) \cup (1/\sqrt{2}, 1]$.
iv) $\{\sqrt{3} \leq |x| \leq 2\}$. *v)* $(0, \mathrm{e}) \cup (\mathrm{e}, \infty)$. *vi)* $(0, 1)$. *vii)* $(0, 1) \cup (1, 5]$. *viii)* $[1/\mathrm{e}, \mathrm{e}]$.

Problem 1.2.2 *a)* $f + g$ is odd, fg is even and $f \circ g$ is odd. As an example, for the composition we have:

$$f \circ g(-x) = f(g(-x)) = f(-g(x)) = -f(g(x)) = -f \circ g(x).$$

b) $f + g$ is not even nor odd, fg is odd and $f \circ g$ is even. For example, for the product

$$f(-x)g(-x) = f(x)(-g(x)) = -f(x)g(x).$$

Problem 1.2.3 Even *iii)* and *iv)*; odd *i)* and *vi)*. For the last one observe that:

$$\log(\sqrt{x^2 + 1} + x) = \log\left(\frac{1}{\sqrt{x^2 + 1} - x}\right) = -\log(\sqrt{x^2 + 1} - x).$$

Problem 1.2.4 $\dfrac{a\,\frac{ax+b}{cx+d}+b}{c\,\frac{ax+b}{cx+d}+d} = x \Rightarrow a = -d$ with $a^2+bc \neq 0$ (in other case the denominator is null, or, what is the same, the function is constant and the conclusion is impossible). or $a = d$, $b = c = 0$.

Problem 1.2.5 It is injective because $\dfrac{x_1+3}{1+2x_1} = \dfrac{x_2+3}{1+2x_2}$ implies $x_1 + 2x_1x_2 + 3 + 6x_2 = x_2 + 2x_1x_2 + 3 + 6x_1$, that is, $x_1 = x_2$. The image is $\mathbb{R}\setminus\{1/2\}$ because $f^{-1}(y) = \dfrac{y-3}{1-2y}$ is defined for all $y \neq 1/2$.

Problem 1.2.6 *a*) *i*) Injective, since $7x_1 - 4 = 7x_2 - 4 \Rightarrow x_1 = x_2$; the inverse is $f^{-1}(y) = \dfrac{y+4}{7}$. *ii*) Non injective, $f(x) = f(x + 2\pi/7)$. *iii*) Injective, $f^{-1}(y) = (y-2)^{1/3} - 1$. *iv*) Injective, $f^{-1}(y) = \dfrac{y-2}{1-y}$. *v*) Non injective, $f(0) = f(3)$. *vi*) Non injective, $f(3) = f(1/3)$. *vii*) Injective, $f^{-1}(y) = -\log y$. *viii*) Injective, $f^{-1}(y) = \mathrm{e}^y - 1$.
b) $f(x_1) = f(x_2)$ implies $x_1 + x_2 = 3$, that is impossible if $x_1, x_2 > 3/2$.
c) $f(x_1) = f(x_2)$ implies $x_1x_2 = 1$, that is impossible if $x_1, x_2 > 1$. $f^{-1}(\sqrt{2}/3) = \sqrt{2}$. *d*) *i*) Bijective. *ii*) Non surjective: $Img(f) = [-1,1]$. *iii*) Bijective. *iv*) Non surjective: $Img(f) = \mathbb{R} - \{1\}$. *v*) Non surjective: $Img(f) = [-1/4, \infty)$. *vi*) Non surjective: $Img(f) = [-1/2, 1/2]$. *vii*) Non surjective: $Img(f) = (0,\infty)$. *viii*) Bijective.

Problem 1.2.7 For *i*) and *ii*) obtain first $\cos(2x)$. For *iii*) calculate first $\operatorname{tg}(x+y)$ and for *iv*) use $\cos(2x)$ and $\sin(2x)$.

Problem 1.2.8 $A(\sin x \cos B + \sin B \cos x) = a \sin x + b \cos x \Rightarrow A\cos B = a$, $A \sin B = b$ $\Rightarrow A = \sqrt{a^2+b^2}$, $B = \operatorname{arctg}(b/a)$.

Problem 1.2.9 *i*) $\operatorname{tg} x = \dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}} = 1$, so $x = \dfrac{\pi}{4} + k\pi$. Since $\operatorname{arctg}\dfrac{1}{2}$ and $\operatorname{arctg}\dfrac{1}{3}$ both belong to $(0, \dfrac{\pi}{4})$ we have that $x \in (0, \dfrac{\pi}{2})$, so $x = \dfrac{\pi}{4}$.

ii) $\operatorname{tg} x = \dfrac{2+3}{1-2\cdot 3} = -1 \Rightarrow x = -\dfrac{\pi}{4} + k\pi$;

$$\operatorname{arctg} 2, \operatorname{arctg} 3 \in (\frac{\pi}{4}, \frac{\pi}{2}) \Rightarrow x \in (\frac{\pi}{2}, \pi) \Rightarrow x = \frac{3\pi}{4}.$$

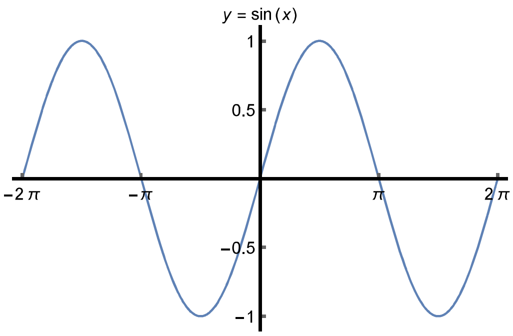
iii) $\operatorname{tg} x = \dfrac{\frac{1}{2} + \frac{1/5+1/8}{1-1/40}}{1-\frac{1}{2}\cdot\frac{1/5+1/8}{1-1/40}} = 1 \Rightarrow x = \dfrac{\pi}{4} + k\pi$;

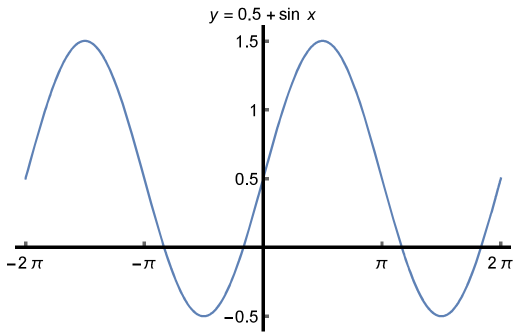
$$\operatorname{arctg}\frac{1}{2}, \operatorname{arctg}\frac{1}{5}, \operatorname{arctg}\frac{1}{8} \in (0, \frac{\pi}{4}) \Rightarrow x \in (0, \frac{3\pi}{4}) \Rightarrow x = \frac{\pi}{4}.$$

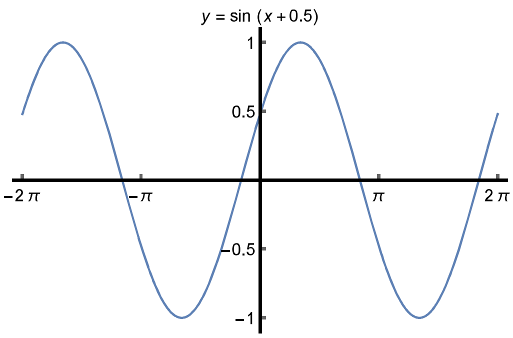
Problem 1.2.10 Be careful with the sign.
i) $\arccos x \in [0,\pi] \Rightarrow \sin(\arccos x) \geq 0 \Rightarrow \sin(\arccos x) = \sqrt{1-x^2}$.
ii) $\arcsin x \in [-\pi/2,\pi/2] \Rightarrow \cos(\arcsin x) \geq 0 \Rightarrow$
$\sin(2\arcsin x) = 2\sin(\arcsin x)\cos(\arcsin x) = 2x\sqrt{1-x^2}$.
iii) By the first part, $\operatorname{tg}(\arccos x) = \dfrac{\sin(\arccos x)}{\cos(\arccos x)} = \dfrac{\sqrt{1-x^2}}{x}$.
iv) $\sin(2\operatorname{arctg} x) = 2\operatorname{tg}(\operatorname{arctg} x)\cos^2(\operatorname{arctg} x) = \dfrac{2x}{x^2+1}$.
v) $\cos^2(2\operatorname{arctg} x) = 1 - \left(\dfrac{2x}{x^2+1}\right)^2 = \left(\dfrac{x^2-1}{1+x^2}\right)^2$; in order to know what is the sign in the root, observe that $|x| \leq 1 \Rightarrow 2\operatorname{arctg} x \in [-\pi/2,\pi/2] \Rightarrow \cos(2\operatorname{arctg} x) \geq 0 \Rightarrow \cos(2\operatorname{arctg} x) = \dfrac{1-x^2}{1+x^2}$; analogously, $\cos(2\operatorname{arctg} x) \leq 0$ for $|x| \geq 1$, that is, $\cos(2\operatorname{arctg} x) = \dfrac{1-x^2}{1+x^2}$ in all cases. *vi*) $e^{4\log x} = x^4$.

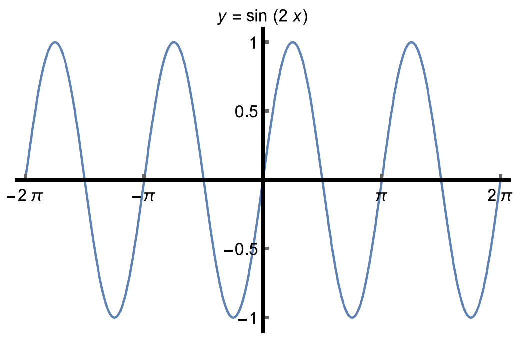
Problem 1.2.11 $x^{3x} = (3x)^x \Rightarrow 3x\log x = x\log(3x) \Rightarrow 2\log x = \log 3 \Rightarrow x = \sqrt{3}$, $y = 3\sqrt{3}$.

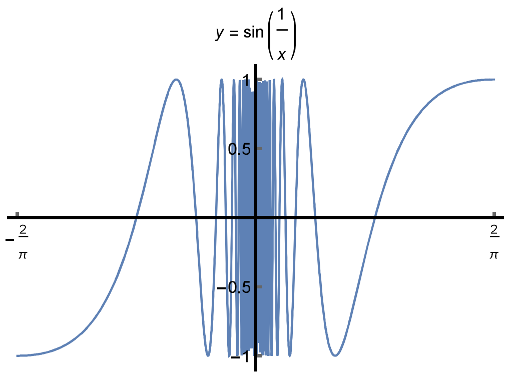
Problem 1.2.12 *i*) Vertical shift. *ii*) Horizontal shift. *iii*) Horizontal dilation or contraction. *iv*) Inversion in the horizontal axis. *v*) Replaces the part $\{x<0\}$ by its symmetric with respect to the vertical axis of $\{x>0\}$. *vi*) Symmetry with respect to the horizontal axis of the negative part of the function. *vii*) Inversion in the vertical axis. *viii*) Positive part.

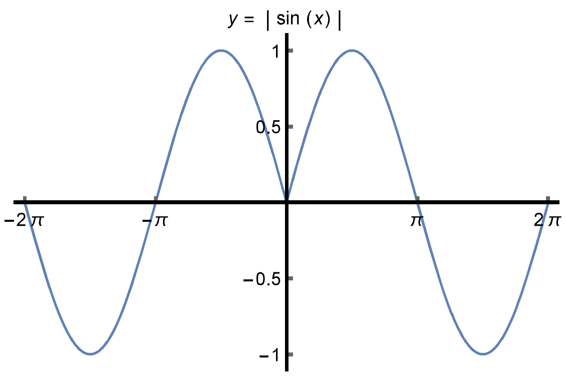


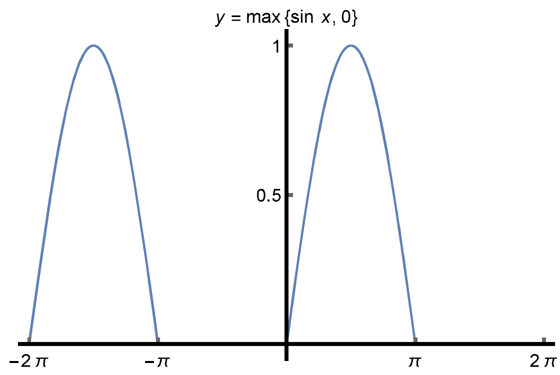


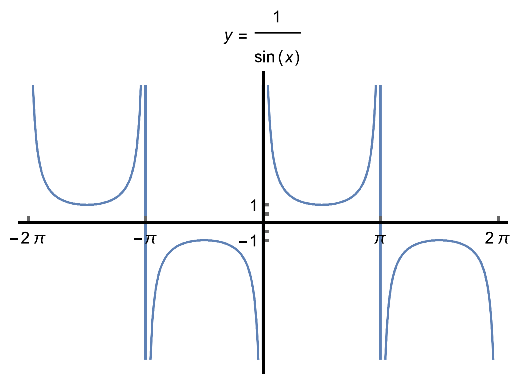


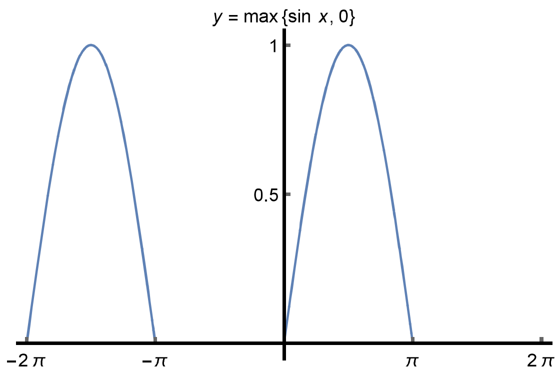



y = sin(1/x)
1
0.5
−0.5
−1
−2/π
2/π

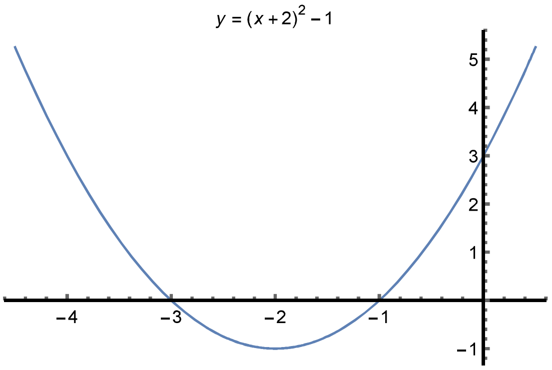

y = | sin (x) |
1
0.5
−0.5
−1
−2 π
−π
π
2 π

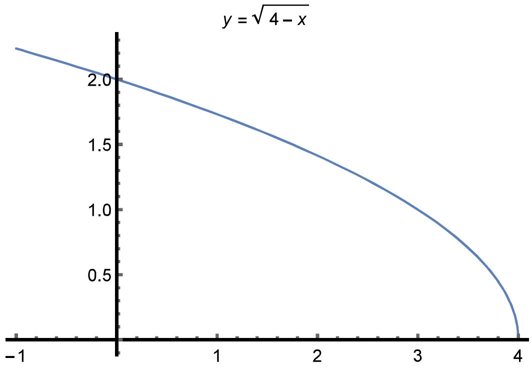

y = max {sin x, 0}
1
0.5
−2 π
−π
π
2 π

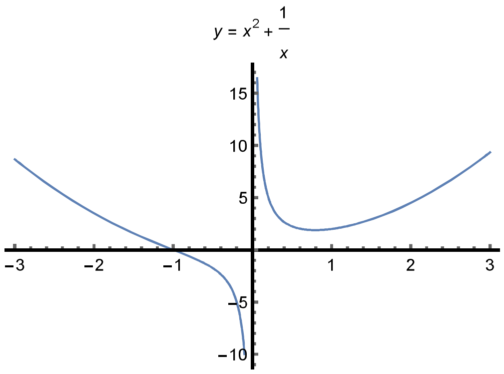

y = 1/sin(x)
1
−1
−2 π
−π
π
2 π

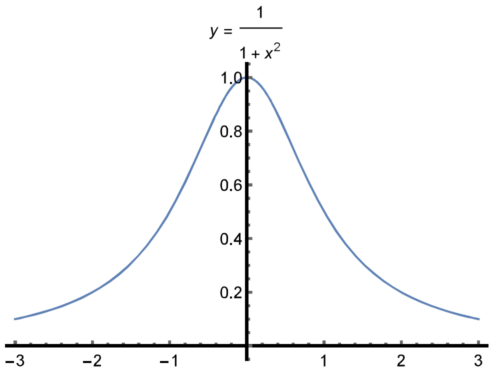

y = max {sin x, 0}
1
0.5
−2 π
−π
π
2 π

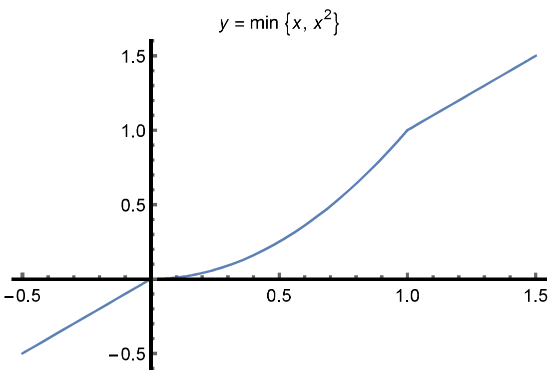
Problem 1.2.13

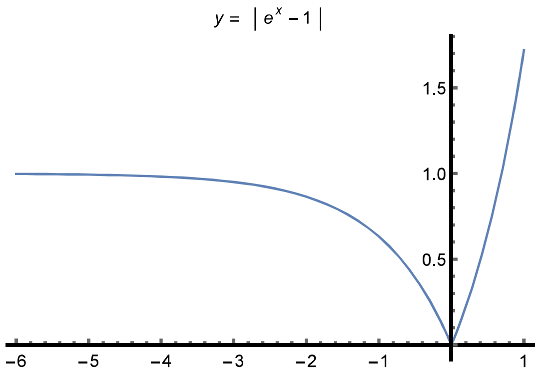


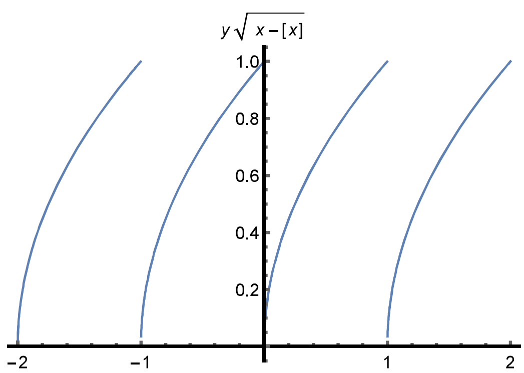


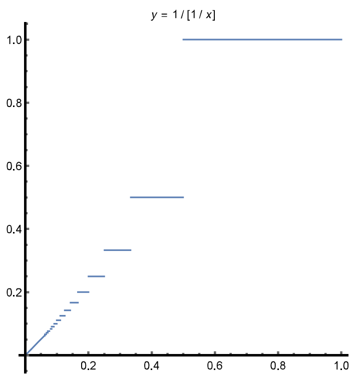


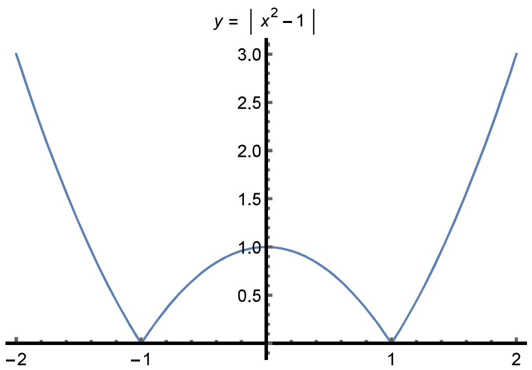


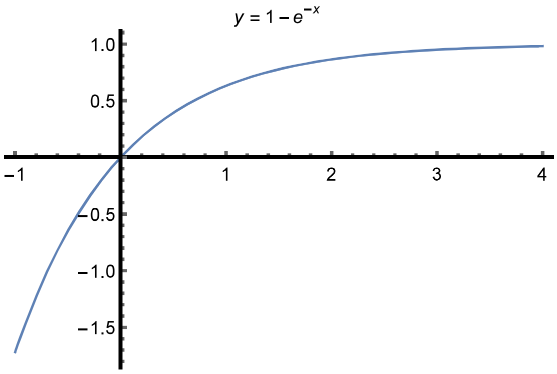


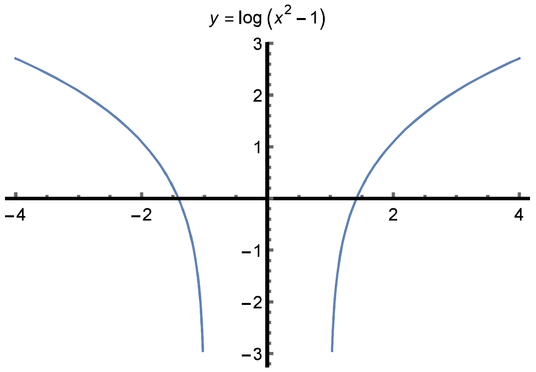


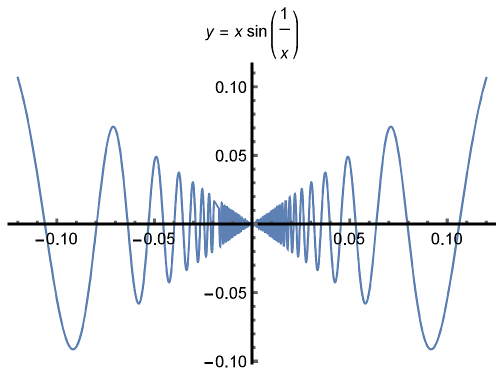












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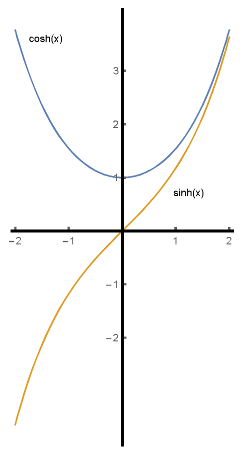


a) $\sinh x$ is odd and $\cosh x$ is even; for example

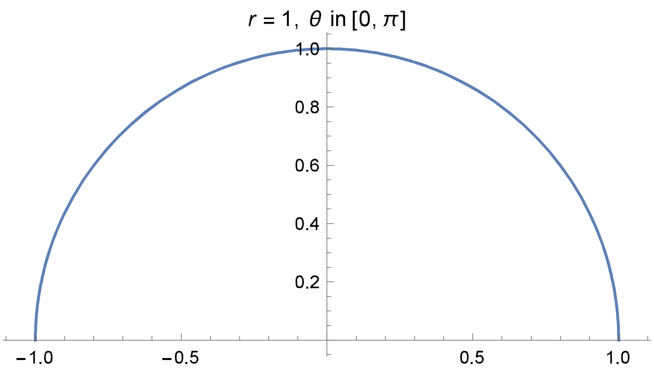
$$\sinh(-x) = \frac{e^{-x} - e^{x}}{2} = -\sinh(x).$$

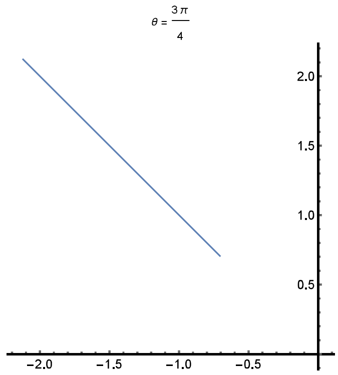
b) *i*) $\quad \dfrac{1}{4}(e^{2x} + e^{-2x} + 2) - \frac{1}{4}(e^{2x} + e^{-2x} - 2) = 1.$

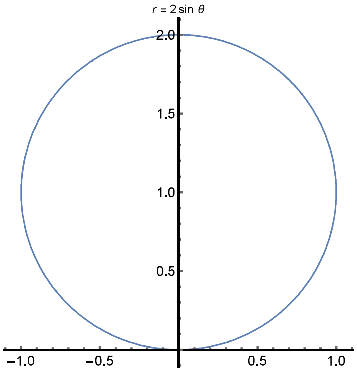
ii) $\quad 2 \cdot \dfrac{1}{2}(e^{x} - e^{-x}) \cdot \frac{1}{2}(e^{x} + e^{-x}) = \frac{1}{2}(e^{2x} - e^{-2x}).$

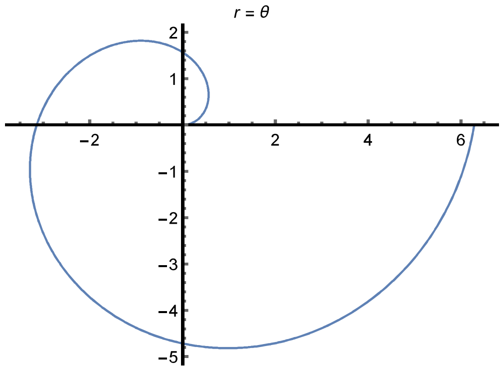
c) $x = \sinh(y) = \dfrac{1}{2}(e^{y} - e^{-y})$; writing $t = e^{y} > 0$ we have $t - 1/t = 2x \Rightarrow t = x \pm \sqrt{x^2+1} \Rightarrow t = x + \sqrt{x^2+1} \Rightarrow \sinh^{-1} x = \log(x + \sqrt{x^2+1})$.

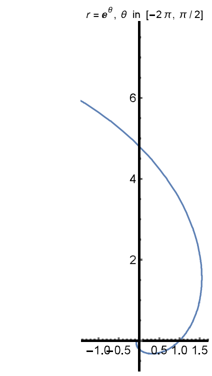
Problem 1.2.15 *i*) Half circumference. *ii*) A ray (half straight line). *iii*) A circumference. *iv*) A spiral. *v*) A spiral. *vi*) A ray. *vii*) A cardioid. *viii*) A lemniscata. *ix*) A rose of four petals. *x*) A rose of three petals.

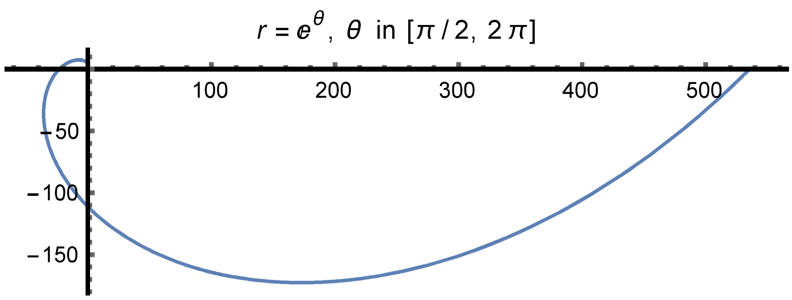

r = 1, θ in [0, π]
1.0
0.8
0.6
0.4
0.2
−1.0
−0.5
0.5
1.0

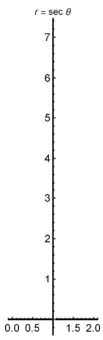

θ = 3π/4
2.0
1.5
1.0
0.5
−2.0
−1.5
−1.0
−0.5

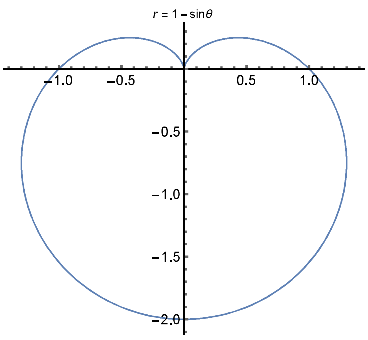

r = 2 sin θ
2.0
1.5
1.0
0.5
−1.0
−0.5
0.5
1.0

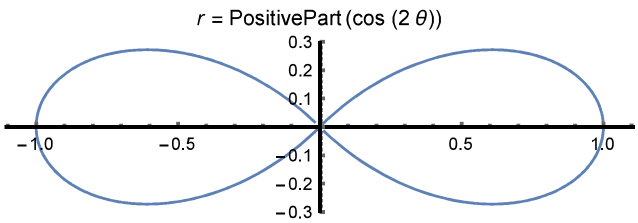

r = θ
2
1
−2
2
4
6
−1
−2
−3
−4
−5

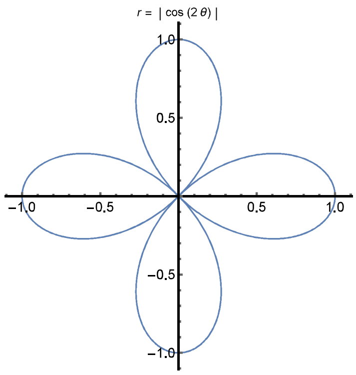

r = e^θ, θ in [−2π, π/2]
6
4
2

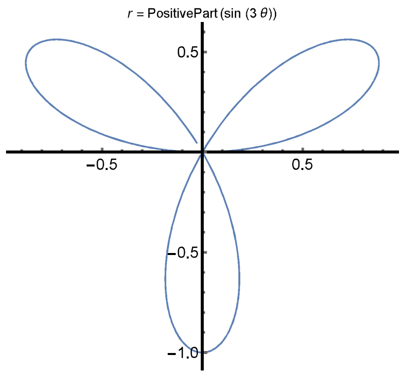

r = e^θ, θ in [π/2, 2π]
100
200
300
400
500
−50
−100
−150


r = sec θ
7
6
5
4
3
2
1
0.0 0.5
1.5 2.0

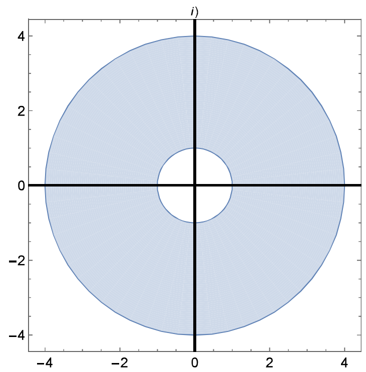

r = 1 − sinθ
−1.0
−0.5
0.5
1.0
−0.5
−1.0
−1.5
−2.0

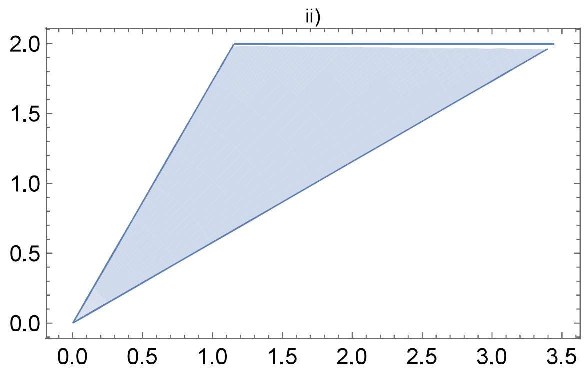

r = PositivePart (cos (2 θ))
0.3
0.2
0.1
−1.0
−0.5
0.5
1.0
−0.1
−0.2
−0.3

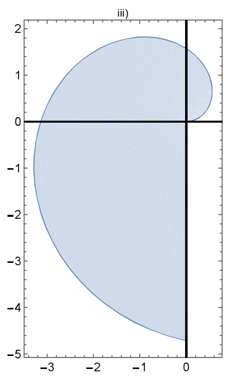


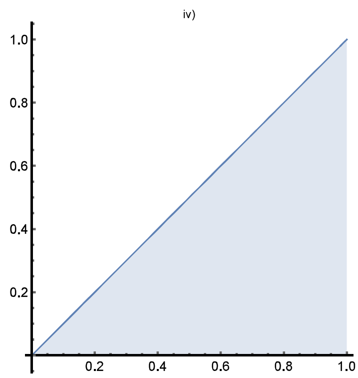


Problem 1.2.16 *i*) The interior of an annulus. *ii*) An angular sector. *iii*) The interior of an spiral arc. *iv*) A triangle.









– $\mathcal{ERC}$–
– $\mathcal{A}_\delta$P–