uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL CALCULUS. Problems

Degree in Applied Mathematics and Computation

Chapter 1

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Open Course Ware, UC3M

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1 Real variable functions

1.1 The real line

Problem 1.1.1

a) Consider three real numbers $0 < a < b$, $k > 0$. Prove the following inequalities:

$$i)\quad a < \sqrt{ab} < \frac{a+b}{2} < b, \qquad ii)\quad \frac{a}{b} < \frac{a+k}{b+k}.$$

b) Prove that $|a+b| = |a| + |b| \iff ab \geq 0$.

c) Prove the inequality $|a-b| \geq \Big||a| - |b|\Big|$, for every $a, b \in \mathbb{R}$.

d) Prove that:

$$i)\ \max\{x,y\} = \frac{x+y+|x-y|}{2}, \qquad ii)\ \min\{x,y\} = \frac{x+y-|x-y|}{2}.$$

e) Express in one single formula the function: $f(x) = (x)_+ = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$.

Problem 1.1.2 Prove that if $n \in \mathbb{N}$ is not a perfect square, then $\sqrt{n} \notin \mathbb{Q}$.
Hint: write $n = z^2 r$, where r does not contain any square factor.

Problem 1.1.3 Find the sets of $x \in \mathbb{R}$ that verify:

$i)\quad A = \{\, |x-3| \leq 8 \,\}$, $\qquad ii)\quad B = \{\, 0 < |x-2| < 1/2 \,\}$,

$iii)\quad C = \{\, x^2 - 5x + 6 \geq 0 \,\}$, $\qquad iv)\quad D = \{\, x^3(x+3)(x-5) < 0 \,\}$,

$v)\quad E = \{\, \dfrac{2x+8}{x^2+8x+7} > 0 \,\}$, $\qquad vi)\quad F = \{\, \dfrac{4}{x} < x \,\}$,

$vii)\quad G = \{\, 4x < 2x+1 \leq 3x+2 \,\}$, $\qquad viii)\quad H = \{\, |x^2 - 2x| < 1 \,\}$,

$ix)\quad I = \{\, |x-1||x+2| = 10 \,\}$, $\qquad x)\quad J = \{\, |x-1| + |x-2| > 1 \,\}$.

Problem 1.1.4 Given two real numbers $a < b$, we define, for each $t \in \mathbb{R}$, the number $x(t) = (1-t)a + tb$. Describe the following sets of numbers:

$i)\quad A = \{\, x(t) \,:\, t = 0,\, 1,\, 1/2 \,\}$, $\qquad ii)\quad B = \{\, x(t) \,:\, t \in (0,1) \,\}$,

$iii)\quad C = \{\, x(t) \,:\, t < 0 \,\}$, $\qquad iv)\quad D = \{\, x(t) \,:\, t > 1 \,\}$.

Problem 1.1.5 Find the supremum, the infimum, the maximum and the minimum (if they exist) of the following sets of real numbers:

a) $A = \{-1\} \cup [2,3)$;

b) $B = \{3\} \cup \{2\} \cup \{-1\} \cup [0,1]$;

c) $C = \{ x = 2 + 1/n \, : \, n \in \mathbb{N} \}$;

d) $D = \{ x = (n^2+1)/n \, : \, n \in \mathbb{N} \}$;

e) $E = \{ x \in \mathbb{R} \, : \, 3x^2 - 10x + 3 \le 0 \}$;

f) $F = \{ x \in \mathbb{R} \, : \, (x-a)(x-b)(x-c)(x-d) < 0 \}$, with $a < b < c < d$ fixed;

g) $G = \{ x = 2^{-p} + 5^{-q} \, : \, p, q \in \mathbb{N} \}$;

h) $H = \{ x = (-1)^n + 1/m \, : \, n, m \in \mathbb{N} \}$.

Problem 1.1.6 Represent in $\mathbb{R}^2$ the following sets:

i) $A = \{ |x-y| < 1 \}$,

ii) $B = \{ x^2 < y < x \}$,

iii) $C = \{ x + y \in \mathbb{Z} \}$,

iv) $D = \{ |2x| + |y| = 1 \}$,

v) $E = \{ (x-1)^2 + (y+2)^2 < 4 \}$,

vi) $F = \{ |1-x| = |y-1| \}$,

vii) $G = \{ 4x^2 + y^2 \le 4, \, xy \ge 0 \}$,

viii) $H = \{ 1 \le x^2 + y^2 < 9, \, y \ge 0 \}$.

Problem 1.1.7 Prove that the straight lines $y = mx + b$, $y = nx + c$ are orthogonal if $mn = -1$.

Problem 1.1.8

a) Consider the parabola $G = \{y = x^2\}$, and the point $P = (0, 1/4)$. Find $\lambda \in \mathbb{R}$ such that the points of G are equidistant from P and the horizontal line $L = \{y = \lambda\}$.

b) Conversely, the set G such that its points are equidistant from a point $P = (a, b)$ and a straight line $L = \{y = \lambda\}$, is the parabola $y = \alpha x^2 + \beta x + \gamma$. Find α, β and γ.

Problem 1.1.9

a) Obtain the set of points in the plane such that the sum of distances to the points $F_1 = (c, 0)$ and $F_2 = (-c, 0)$ is $2a$, $(a > c)$.

b) Now, substitute sum by difference in the previous exercise (with $a < c$).

1.2 Elementary functions

Problem 1.2.1 Find the domain of the following functions:

i) $f(x) = \dfrac{1}{x^2 - 5x + 6}$,

ii) $f(x) = \sqrt{1 - x^2} + \sqrt{x^2 - 1}$,

iii) $f(x) = \dfrac{1}{x - \sqrt{1 - x^2}}$,

iv) $f(x) = \sqrt{1 - \sqrt{4 - x^2}}$,

v) $f(x) = \dfrac{1}{1 - \log x}$,

vi) $f(x) = \log(x - x^2)$,

vii) $f(x) = \dfrac{\sqrt{5 - x}}{\log x}$,

viii) $f(x) = \arcsin(\log x)$.

Problem 1.2.2

a) If both f and g are odd functions, how are $f+g$, $f\cdot g$ and $f\circ g$?

b) What happens if f is even and g is odd?

Problem 1.2.3 Study the symmetry of the following functions:

$$i)\quad f(x)=\frac{x}{x^2-1},\qquad ii)\quad f(x)=\frac{x^2-x}{x^2+1},$$

$$iii)\quad f(x)=\frac{\sin x}{x},\qquad iv)\quad f(x)=(\cos x^3)(\sin x^2)\mathrm{e}^{-x^4},$$

$$v)\quad f(x)=\frac{1}{\sqrt{x^2+1}-x},\qquad vi)\quad f(x)=\log(\sqrt{x^2+1}-x).$$

Hint: vi) is odd.

Problem 1.2.4 For what numbers $a,b,c,d\in\mathbb{R}$ does the function $f(x)=\dfrac{ax+b}{cx+d}$ satisfy $f\circ f=I$ (the identity) on the domain of f?

Problem 1.2.5 Check that the function $f(x)=\dfrac{x+3}{1+2x}$ is bijective defined from $\mathbb{R}\setminus\{-1/2\}$ to $\mathbb{R}\setminus\{1/2\}$, and find its inverse.

Problem 1.2.6

a) Study which ones of the following functions are injective, find their inverses when they are or an example of two points with the same image otherwise.

$$i)\quad f(x)=7x-4,\qquad ii)\quad f(x)=\sin(7x-4),$$

$$iii)\quad f(x)=(x+1)^3+2,\qquad iv)\quad f(x)=\frac{x+2}{x+1},$$

$$v)\quad f(x)=x^2-3x+2,\qquad vi)\quad f(x)=\frac{x}{x^2+1},$$

$$vii)\quad f(x)=\mathrm{e}^{-x},\qquad viii)\quad f(x)=\log(x+1).$$

b) Prove that the function $f(x)=x^2-3x+2$ is injective on $(3/2,\infty)$.

c) Prove that the function $f(x)=\dfrac{x}{x^2+1}$ is injective on $(1,\infty)$ and find $f^{-1}(\sqrt{2}/3)$.

d) Study whether the previous functions are surjective or bijective on their domain $D(f)$ in $\mathbb{R}$.

Problem 1.2.7 Starting with the formulas:

$$\sin(x+y)=\sin x\cos y+\sin y\cos x,\qquad \cos(x+y)=\cos x\cos y-\sin x\sin y,$$

prove the following relations:

$$i)\quad \sin^2 x=\frac{1-\cos(2x)}{2},\qquad ii)\quad \cos^2 x=\frac{1+\cos(2x)}{2},$$

$$iii)\quad \operatorname{tg}(\operatorname{arctg} x+\operatorname{arctg} y)=\frac{x+y}{1-xy},\quad iv)\quad \operatorname{tg}(2x)=\frac{2\operatorname{tg} x}{1-\operatorname{tg}^2 x}.$$

Problem 1.2.8 Prove that $a\sin x + b\cos x$ can be written as $A\sin(x+B)$, and obtain A and B.

Problem 1.2.9 Calculate

$$i)\quad \operatorname{arctg}\frac{1}{2} + \operatorname{arctg}\frac{1}{3},$$

$$ii)\quad \operatorname{arctg} 2 + \operatorname{arctg} 3,$$

$$iii)\quad \operatorname{arctg}\frac{1}{2} + \operatorname{arctg}\frac{1}{5} + \operatorname{arctg}\frac{1}{8}.$$

Hint: use the formula for the tangent of a sum and study the signs.

Problem 1.2.10 Simplify the following expressions

$$i)\quad f(x) = \sin(\arccos x), \qquad ii)\quad f(x) = \sin(2\arcsin x),$$

$$iii)\quad f(x) = \operatorname{tg}(\arccos x), \qquad iv)\quad f(x) = \sin(2\operatorname{arctg} x),$$

$$v)\quad f(x) = \cos(2\operatorname{arctg} x), \qquad vi)\quad f(x) = \mathrm{e}^{4\log x}.$$

Problem 1.2.11 Solve the following system of equations, for $x, y > 0$,

$$\begin{cases} x^y = y^x \\ y = 3x. \end{cases}$$

Problem 1.2.12 Describe the function g in terms of f in the following cases ($c \in \mathbb{R}$ is a constant). Plot them for $f(x) = x^2$ and $f(x) = \sin x$.

$$i)\quad g(x) = f(x) + c, \qquad ii)\quad g(x) = f(x+c),$$

$$iii)\quad g(x) = f(cx), \qquad iv)\quad g(x) = f(1/x),$$

$$v)\quad g(x) = f(|x|), \qquad vi)\quad g(x) = |f(x)|,$$

$$vii)\quad g(x) = 1/f(x), \qquad viii)\quad g(x) = (f(x))_+ = \max\{f(x), 0\}.$$

Problem 1.2.13 Sketch, with as few calculations as possible, the graph of the following functions:

$$i)\quad f(x) = (x+2)^2 - 1, \qquad ii)\quad f(x) = \sqrt{4-x},$$

$$iii)\quad f(x) = x^2 + 1/x, \qquad iv)\quad f(x) = 1/(1+x^2),$$

$$v)\quad f(x) = \min\{x, x^2\}, \qquad vi)\quad f(x) = |\mathrm{e}^x - 1|,$$

$$vii)\quad f(x) = \sqrt{x - [x]}, \qquad viii)\quad f(x) = 1/[1/x],$$

$$ix)\quad f(x) = |x^2 - 1|, \qquad x)\quad f(x) = 1 - \mathrm{e}^{-x},$$

$$xi)\quad f(x) = \log(x^2 - 1), \qquad xii)\quad f(x) = x\sin(1/x).$$

Hint: $[x] = n$ denotes the integer part of x, that is, the biggest integer $n \le x$.

Problem 1.2.14 We define the hyperbolic sine and cosine functions by:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

a) Study their symmetry.

b) Prove the formulas

$$i)\ \cosh^2 x - \sinh^2 x = 1, \qquad ii)\ \sinh 2x = 2\sinh x \cosh x.$$

c) Simplify the function $f(x) = \sinh^{-1} x$.

d) Sketch the graph of the functions $\sinh x$ and $\cosh x$.

Problem 1.2.15 Sketch the following curves given in polar coordinates:

i) $r = 1, \quad \theta \in [0, \pi]$,
ii) $\theta = 3\pi/4, \quad r \geq 2$,
iii) $r = 2\sin\theta, \quad \theta \in [0, \pi]$,
iv) $r = \theta, \quad \theta \in [0, 2\pi]$,
v) $r = e^\theta, \quad \theta \in [-2\pi, 2\pi]$,
vi) $r = \sec\theta, \quad \theta \in [0, \pi/2]$,
vii) $r = 1 - \sin\theta, \quad \theta \in [0, 2\pi]$,
viii) $r = (\cos 2\theta)_+, \quad \theta \in [0, 2\pi]$,
ix) $r = |\cos 2\theta|, \quad \theta \in [0, 2\pi]$,
x) $r = (\sin 3\theta)_+, \quad \theta \in [0, 2\pi/3]$.

Problem 1.2.16 Sketch the following sets in the plane given in polar coordinates:

i) $A = \{ 1 < r < 4 \}$,
ii) $B = \{ \pi/6 \leq \theta \leq \pi/3 \}$,
iii) $C = \{ r \leq \theta,\ 0 \leq \theta \leq 3\pi/2 \}$,
iv) $D = \{ r \leq \sec\theta,\ 0 \leq \theta \leq \pi/4 \}$.

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