uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL CALCULUS. Solutions

Degree in Applied Mathematics and Computation

Chapter 2

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2 Limits and continuity

2.1 Limits of functions

Problem 2.1.1 *i*) If $\delta = \min\{1, \varepsilon/5\}$ and $0 < |x-2| < \delta$, then $|x+2| < 5$, so

$$|x^2 - 4| = |x+2||x-2| < 5\delta \leq \varepsilon.$$

ii) Consider $\varepsilon = 1$. Given any $\delta > 0$ we take $x = 3 - \delta/2$ and obtain that $f(x) < 14$, so

$$|x-3| = \delta/2 < \delta \quad \text{and} \quad |f(x) - 16| > 2 > \varepsilon.$$

iii) It is enough to take $\delta = \varepsilon$ since $\left|\dfrac{x}{1+\sin^2 x}\right| \leq |x|$.
iv) If $\delta = \min\{9, 3\varepsilon\}$ and $0 < |x-9| < \delta$, then $x > 0$ (the root is defined) and $|\sqrt{x}+3| > 3$, so

$$|\sqrt{x} - 3| = \frac{|x-9|}{|\sqrt{x}+3|} < \delta/3 \leq \varepsilon.$$

Problem 2.1.2 *i*) Divide by Ruffini,

$$\lim_{x\to a} \frac{x^n - a^n}{x-a} = \lim_{x\to a} \sum_{k=1}^{n} a^{k-1}x^{n-k} = na^{n-1},$$

or, alternatively, use the Newton binomial,

$$\lim_{x\to a} \frac{x^n - a^n}{x-a} = \lim_{h\to 0} \frac{(a+h)^n - a^n}{h} = \lim_{h\to 0} \sum_{k=1}^{n} \binom{n}{k} a^{n-k}h^{k-1} = na^{n-1}.$$

The common factor is $(x-a)$. *ii*) Multiply by the conjugate $(\sqrt{x}+\sqrt{a})$, or write $y = \sqrt{x}$ and apply the previous limit, $L = \dfrac{1}{2\sqrt{a}}$. The common factor is $(\sqrt{x}-\sqrt{a})$. *iii*) Writing $z = x^{1/6}$ we have

$$L = \lim_{z\to 2} \frac{z^3 - 8}{z^2 - 4} = \lim_{z\to 2} \frac{z^2 + 2z + 4}{z+2} = 3.$$

The common factor is $(x^{1/6} - 2)$. *iv*) Multiply by the conjugate, $L = 1/2$. The common factor is x^2. *v*) By the Newton binomial,

$$L = \lim_{h\to 0} \frac{h^2 - 3h + 3}{(1-h)^3} = 3.$$

The common factor is h. *vi*) $L = \lim_{x\to 1} \dfrac{\sqrt{x}-1}{x-1} = 1/2$. The common factor is $(\sqrt{x}-1)$.

Problem 2.1.3 *i*) $L = 4\left(\lim_{x\to 0} \dfrac{\sin 2x^3}{2x^3}\right)^2 = 4$. *ii*) Multiply by the conjugate, $L = \lim_{x\to 0} \dfrac{\sin^2 x}{x^2(1+\cos x)} = 1/2$. *iii*) $L = \lim_{x\to 0} \dfrac{\frac{\sin x^2}{x\cos x^2} + 2}{1+x} = 2$. *iv*) Develop the sine of the sum, $L = \cos a$. *v*) Define $y = \log(1+x)$, $L = \lim_{y\to 0} \dfrac{y}{\mathrm{e}^y - 1} = 1$. *vi*) Use the exponential form and the

previous limit, $L = \mathrm{e}$. *vii*) Multiply and divide by $-2x$, $L = -2$. *viii*) Write it in exponential form and multiply and divide the exponent by $\sin x$, $L = \mathrm{e}^2$, or also $L = \exp(\lim_{x\to 0} \frac{2\sin x}{x}) = \mathrm{e}^2$.
ix) Take the common factor $\mathrm{e}^{\sin x}$ and define $y = x - \sin x$, $L = 1$. *x*) $L = 1/2$.
xi) $L = \exp(\lim_{x\to 0} \frac{\sin x}{\sin x - x}(\frac{x}{\sin x} - 1)) = \mathrm{e}^{-1}$. *xii*) $L = \exp(\lim_{x\to 0} \frac{\cos x - 1}{x^2}) = \mathrm{e}^{-1/2}$.
xiii) Define $y = x - \pi$, $L = 1/8$. *xiv*) $L = \lim_{x\to 0} \frac{\mathrm{e}^{x\log a} - 1 + 1 - \mathrm{e}^{x\log b}}{x} = \log(a/b)$.

Problem 2.1.4 *i*) Divide numerator and denominator by x^3, $L = -\frac{1}{\sqrt{2}}$. *ii*) Divide numerator and denominator by x, $L = 1/5$, since $\frac{\sin x^3}{x} \to 0$. *iii*) Divide numerator and denominator by $\sqrt{x}$, $L = 1$. *iv*) Multiply by the conjugate, $L = 2$. *v*) $L = 1$.
vi) $L = 0$. *vii*) $L = 1/2$. *viii*) Observe that $\sqrt{x^2} = |x| = -x$ if $x < 0$, $L = -1/2$.

Problem 2.1.5 *i*) $L = 1$ since $\left(\frac{1}{t}\right)^{[t]} = 1$ if $0 < t < 1$. *ii*) $L = 0$ since $\left(\frac{1}{t}\right)^{[t]} = t$ if $-1 < t < 0$. *iii*) $L = \infty$. *iv*) $L = 0$. *v*) $L = \lim_{x\to\infty} \frac{1-x}{1+x} = -1$. *vi*) $L = \lim_{x\to 0^+} \frac{1-x}{1+x} = 1$.
vii) $L = \exp\Big(\lim_{x\to\infty} \frac{13\sqrt{4x^2+x-3}}{2x-6}\Big) = \mathrm{e}^{13}$. *viii*) $L = \exp\Big(\lim_{x\to -\infty} \frac{13\sqrt{4x^2+x-3}}{2x-6}\Big) = \mathrm{e}^{-13}$.

Problem 2.1.6 *a*) $b = 2a$. *b*) *i*) $\lim_{x\to\infty} \log\left(\frac{\log x}{\log\alpha + \log x}\right) = 0$, *ii*) $\lim_{x\to\infty} \log\left(\frac{\log x}{\alpha\log x}\right) = -\log\alpha$.

Problem 2.1.7 *a*) By the pinching lemma, since $|f(x)\sin 1/x| \le |f(x)|$. *b*) $L = 0$ again by the pinching lemma, since $\left|\frac{x}{2+\sin 1/x}\right| \le |x|$.

2.2 Continuity

Problem 2.2.1 *a*) For any $\varepsilon > 0$, there exists a $\delta_1 > 0$ such that if $|x - f(a)| < \delta_1$ then $|g(x) - g(f(a))| < \varepsilon$, and also there exists a $\delta_2 > 0$ such that if $|x - a| < \delta_2$ then $|f(x) - f(a)| < \delta_1$. So, if $|x - a| < \delta_2$ we have $|g(f(x)) - g(f(a))| < \varepsilon$.
b) Apply the previous part to the function $h(x) = |x|$, that is continuous. the reciprocal is not true, for example, with the function $f(x) = \text{sign}(x) = \begin{cases} 1 & \text{si } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$.
c) If it is continuous ant takes two different values then it must take all the intermediate values. But between two rational numbers there are infinite irrational numbers. So the function must be constant.

Problem 2.2.2 If $\lambda = 0$ obviously the function is continuous since it is $b(x) = 1$. If $\lambda \neq 0$ the roots of the denominator are $1 \pm \sqrt{1 - 1/\lambda}$. *i*) To obtain a non null denominator in $\mathbb{R}$ the roots cannot be real, so the values of λ must be in $D = \{0 \leq \lambda < 1\}$. *ii*) Now simply the roots cannot belong to the interval $[0, 1]$, and so we need $\lambda < 0$, join this with the values for which there are no real roots and obtain $D = \{\lambda < 1\}$.

Problem 2.2.3 All of them are continuous in their domains. $D(f) = \mathbb{R} \setminus \{2, 6\}$, $D(g) = \mathbb{R} \setminus \{0\}$, $D(h) = \mathbb{R} \setminus \{3x + 2 = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\}$, $D(j) = (-\infty, 2] \cup [3, \infty)$, $D(k) = [-1, 1]$, $D(m) = (3/8, \infty)$.

Problem 2.2.4 *a*) Continuous in $\mathbb{R} \setminus \mathbb{Z}$, because at each point $a \in \mathbb{Z}$ there is a unity jump.
b) Continuous in $\mathbb{R}$, since the limit at $a = 0$ is zero and in the rest of the points it is a composition of continuous functions.
c) Continuous in $\mathbb{R} \setminus \{\frac{\pi}{2} + k\pi,\ k = 0, 1, \cdots\}$, since the one-side limits at zero are both zero, the value of the function there, then we only have to study the discontinuities of the tangent function.
d) Continuous only at $a = 0$: for $a \neq 0$ the function oscillates with amplitude a; only if $a = 0$ we can apply the pinching lemma and obtain that $-|x| \leq f(x) \leq |x|$.

Problem 2.2.5 *a*) Apply the Bolzano theorem to the function $g(x) = f(x) - x$ in $[0, 1]$: it is continuous and changes the sign.
ii) Apply the Bolzano theorem to the function $h(x) = f(x) - g(x)$ in $[a, b]$.

Problem 2.2.6 *a*) The function is continuous in $\mathbb{R}$, so it is uniformly continuous in any closed interval. For $a \in [1, \infty)$ if we fix $\varepsilon > 0$:

$$|x^2 - a^2| < \varepsilon \quad \Longrightarrow \quad |x - a| < \frac{\varepsilon}{|x + a|},$$

and this tends to zero if $a \to \infty$, so it is not uniformly continuous in this interval. *b*) If $a \in [1, \infty)$ consider $\delta < 1/2$ and compute:

$$\left| \frac{1}{x} - \frac{1}{a} \right| = \frac{|a - x|}{|xa|} < \varepsilon \quad \text{if} \quad |x - a| < \varepsilon xa,$$

and this is attained if $\delta = min\{1/2, \varepsilon/2\}$ since $xa \geq 1/2$. At the point $a = 0$ the function is not continuous, so it is not absolutely continuous in $[0, 1]$. *c*) Consider $\delta < 1$. For $a \in [0, \infty)$ and a fixed ε we have:

$$\left| \frac{1}{x^2 + 1} - \frac{1}{a^2 + 1} \right| = \frac{|a^2 - x^2|}{(x^2 + 1)(a^2 + 1)} < \varepsilon \quad \Longrightarrow \quad |x - a| < \frac{\varepsilon(x^2 + 1)(a^2 + 1)}{|x + a|},$$

for $a \in [0, 1]$ this is bigger than ε so we take $\delta < min\{1, \varepsilon\}$. For $a \geq 1$ we take $\delta = \varepsilon$ since:

$$|x - a| < \varepsilon < \frac{\varepsilon(x^2 + 1)(a^2 + 1)}{|x + a|}.$$

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