uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL CALCULUS. Problems

Degree in Applied Mathematics and Computation

Chapter 2

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2 Limits and continuity

2.1 Limits of functions

Problem 2.1.1 Using the ε–δ definition of limit, prove that:

i) $\lim_{x\to 2} x^2 = 4$, *ii)* $\lim_{x\to 3}(5x-1) \neq 16$,

iii) $\lim_{x\to 0} \dfrac{x}{1+\sin^2 x} = 0$, *iv)* $\lim_{x\to 9} \sqrt{x} = 3$.

Problem 2.1.2 Compute the following limits simplifying the common factors that may appear:

i) $\lim_{x\to a} \dfrac{x^n - a^n}{x-a}$, *ii)* $\lim_{x\to a} \dfrac{\sqrt{x}-\sqrt{a}}{x-a}$,

iii) $\lim_{x\to 64} \dfrac{\sqrt{x}-8}{\sqrt[3]{x}-4}$, *iv)* $\lim_{x\to 0} \dfrac{1-\sqrt{1-x^2}}{x^2}$,

v) $\lim_{h\to 0} \dfrac{\frac{1}{(1-h)^3}-1}{h}$, *vi)* $\lim_{x\to 1} \left(\dfrac{1}{\sqrt{x}-1} - \dfrac{2}{x-1}\right)$.

Problem 2.1.3 Using that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$ and $\lim_{x\to 0} \dfrac{\mathrm{e}^x - 1}{x} = 1$, compute the following limits:

i) $\lim_{x\to 0} \dfrac{(\sin 2x^3)^2}{x^6}$, *ii)* $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$,

iii) $\lim_{x\to 0} \dfrac{\operatorname{tg} x^2 + 2x}{x+x^2}$, *iv)* $\lim_{x\to 0} \dfrac{\sin(x+a)-\sin a}{x}$,

v) $\lim_{x\to 0} \dfrac{\log(1+x)}{x}$, *vi)* $\lim_{x\to 0}(1+x)^{1/x}$,

vii) $\lim_{x\to 0} \dfrac{\log(1-2x)}{\sin x}$, *viii)* $\lim_{x\to 0}(1+\sin x)^{2/x}$,

ix) $\lim_{x\to 0} \dfrac{\mathrm{e}^x - \mathrm{e}^{\sin x}}{x-\sin x}$, *x)* $\lim_{x\to 0} \dfrac{\operatorname{tg} x - \sin x}{x^3}$,

xi) $\lim_{x\to 0} \left(\dfrac{x}{\sin x}\right)^{\frac{\sin x}{\sin x - x}}$, *xii)* $\lim_{x\to 0}(\cos x)^{1/x^2}$,

xiii) $\lim_{x\to \pi} \dfrac{1-\sin(x/2)}{(x-\pi)^2}$, *xiv)* $\lim_{x\to 0} \dfrac{a^x - b^x}{x}$.

Hint: it may be necessary to use some change of variables and the limit of the composite function.

Problem 2.1.4 Calculate the following limits:

$$i)\quad \lim_{x\to\infty} \frac{x^3+4x-7}{7x^2-\sqrt{2x^6+x^5}}, \qquad ii)\quad \lim_{x\to\infty} \frac{x+\sin x^3}{5x+6},$$

$$iii)\quad \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}, \qquad iv)\quad \lim_{x\to\infty} \left(\sqrt{x^2+4x}-x\right),$$

$$v)\quad \lim_{x\to\infty} \frac{\mathrm{e}^x}{\mathrm{e}^x-1}, \qquad vi)\quad \lim_{x\to-\infty} \frac{\mathrm{e}^x}{\mathrm{e}^x-1},$$

$$vii)\quad \lim_{x\to\infty} \frac{x-2}{\sqrt{4x^2+1}}, \qquad viii)\quad \lim_{x\to-\infty} \frac{x-2}{\sqrt{4x^2+1}}.$$

Problem 2.1.5 Obtain the one-sided limits:

$$i)\quad \lim_{t\to 0^+} \left(\frac{1}{t}\right)^{[t]}, \qquad ii)\quad \lim_{t\to 0^-} \left(\frac{1}{t}\right)^{[t]},$$

$$iii)\quad \lim_{t\to 0^+} \mathrm{e}^{1/t}, \qquad iv)\quad \lim_{t\to 0^-} \mathrm{e}^{1/t},$$

$$v)\quad \lim_{t\to 0^+} \frac{1-\mathrm{e}^{1/t}}{1+\mathrm{e}^{1/t}}, \qquad vi)\quad \lim_{t\to 0^-} \frac{1-\mathrm{e}^{1/t}}{1+\mathrm{e}^{1/t}},$$

$$vii)\quad \lim_{x\to+\infty} \left(\frac{2x+7}{2x-6}\right)^{\sqrt{4x^2+x-3}}, \qquad viii)\quad \lim_{x\to-\infty} \left(\frac{2x+7}{2x-6}\right)^{\sqrt{4x^2+x-3}}.$$

Problem 2.1.6

a) Establish the relation between a and b so that

$$\lim_{x\to 1} x^{a/(1-x)} = \lim_{x\to 0} (\cos x)^{b/x^2}.$$

b) If $f(x) = \log(\log x)$ and $\alpha > 0$, find the limits:

$$i)\ \lim_{x\to\infty} (f(x) - f(\alpha x)), \qquad ii)\ \lim_{x\to\infty} (f(x) - f(x^\alpha)).$$

Problem 2.1.7

a) Prove that if $\lim_{x\to 0} f(x) = 0$ then $\lim_{x\to 0} f(x) \sin 1/x = 0$.

b) Calculate $\lim_{x\to 0} \dfrac{x}{2+\sin 1/x}$.

2.2 Continuity

Problem 2.2.1

a) Prove that if f is continuous at a point a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.

b) Prove that if f is continuous, then $|f|$ is continuous too. Is the reciprocal true?

c) What can we say of a continuous function that only takes values on $\mathbb{Q}$?

Problem 2.2.2 Find $\lambda \in \mathbb{R}$ so that the function $b(x) = \dfrac{1}{\lambda x^2 - 2\lambda x + 1}$ is continuous on:

$$i)\ \mathbb{R}, \qquad ii)\ [0,1].$$

Problem 2.2.3 Study the continuity of the following functions:

a) $f(x) = \dfrac{e^{-5x} + \cos x}{x^2 - 8x + 12}$;

b) $g(x) = e^{3/x} + x^3 - 9$;

c) $h(x) = x^3 \operatorname{tg}(3x + 2)$;

d) $j(x) = \sqrt{x^2 - 5x + 6}$;

e) $k(x) = (\arcsin x)^3$;

f) $m(x) = (x - 5)\log(8x - 3)$.

Problem 2.2.4 Study the continuity of the following functions defined in pieces:

a) $f(x) = x - [x]$;

b) $f(x) = \begin{cases} x^2 \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0; \end{cases}$

c) $f(x) = \begin{cases} \dfrac{\operatorname{tg} x}{\sqrt{x}}, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ e^{1/x}, & \text{if } x < 0; \end{cases}$

d) $f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q}, \\ -x, & \text{if } x \notin \mathbb{Q}. \end{cases}$

Problem 2.2.5 Prove the following fixed point theorems:

a) Let $f : [0,1] \longrightarrow [0,1]$ be a continuous function. Then there exists $c \in [0,1]$ such that $f(c) = c$.

b) Let $f, g : [a,b] \longrightarrow \mathbb{R}$ be two continuous functions such that $f(a) > g(a)$, $f(b) < g(b)$. Then there exists $c \in (a,b)$ such that $f(c) = g(c)$.

Problem 2.2.6 Prove that

a) $f(x) = x^2$ is uniformly continuous on $[0, 1]$ but it is not on $[1, \infty)$.

b) $f(x) = 1/x$ is uniformly continuous on $[1, \infty)$ but it is not on $[0, 1]$.

c) $f(x) = 1/(x^2 + 1)$ is uniformly continuous on $[0, \infty)$.

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