uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL CALCULUS. Solutions

Degree in Applied Mathematics and Computation

Chapter 3

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Open Course Ware, UC3M

3 Derivatives and their applications

3.1 Differentiability

Problem 3.1.1 *i)* $h'(x) = \dfrac{f(x)f'(x) + g(x)g'(x)}{\sqrt{f^2(x) + g^2(x)}}$.

ii) $h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{f^2(x) + g^2(x)}$.

iii) $h'(x) = \Big[f'(g(x))g'(x) + f(g(x))f'(x)\Big]\mathrm{e}^{f(x)}$.

iv) $h'(x) = \dfrac{g'(x)\sin(f(x)) + g(x)f'(x)\cos(f(x))}{g(x)\sin(f(x))}$.

v) $h'(x) = (f(x))^{g(x)}[g'(x)\log(f(x)) + \dfrac{g(x)f'(x)}{f(x)}]$.

vi) $h'(x) = \dfrac{-[f'(x) + 2g(x)g'(x)]}{[\log(f(x) + g^2(x))]^2\,[f(x) + g^2(x)]}$.

Problem 3.1.2 *a)* For example, for $1 \leq |x| \leq 2$ we can define $f(x) = 2 - |x|$.
b) For example, for $1 \leq x \leq 2$ we can take $f(x) = 2x^3 - 9x^2 + 12x - 4$, and its even symmetric function in $-2 \leq x \leq -1$. Another easier function is $f(x) = \sin^2(\pi x/2)$ in $1 \leq |x| \leq 2$.

Problem 3.1.3 It is direct. For example

$$(\sinh x)' = \left(\frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}\right)' = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} = \cosh x.$$

Problem 3.1.4 It is direct. For example
ii) $xf' - f - f^2 - x^2 = x\Big(\operatorname{tg} x + x(1 + \operatorname{tg}^2 x)\Big) - x\operatorname{tg} x - (x\operatorname{tg} x)^2 - x^2 = 0$.

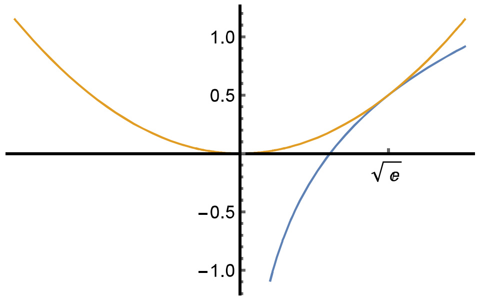
Problem 3.1.5 The derivatives of the three functions are zero in their domains, so they are constant on each interval of the domain. For example:

$$(\operatorname{arctg} x + \operatorname{arctg}\frac{1}{x})' = \frac{1}{1+x^2} + \frac{-1/x^2}{1+(1/x)^2} = \frac{1}{1+x^2} - \frac{1}{x^2+1} = 0.$$

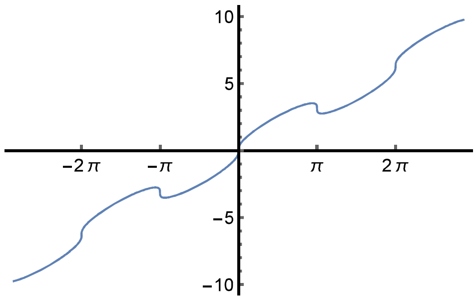
i) At $x = 1$ we have $\operatorname{arctg} 1 + \operatorname{arctg} 1 = \pi/2$ (also we can compute the limit for $x \to 0^+$, or for $x \to \infty$).
ii) At $x = 0$ we have $\operatorname{arctg} 1 + \operatorname{arctg} 0 = \pi/4$.
iii) At $x = 1$ we have $2\operatorname{arctg} 1 + \arcsin 1 = \pi$.
Observe, for example, that the first function is odd and has the value $-\dfrac{\pi}{2}$ for $x < 0$, and it is not continuous at $x = 0$.

Problem 3.1.6

The system $\begin{cases} ax^2 = \log x \\ 2ax = 1/x \end{cases}$ gives $x = \sqrt{\mathrm{e}}$, $a = 1/2\mathrm{e}$, and the tangent line is $y = \dfrac{x}{\sqrt{\mathrm{e}}} - \dfrac{1}{2}$.



Problem 3.1.7

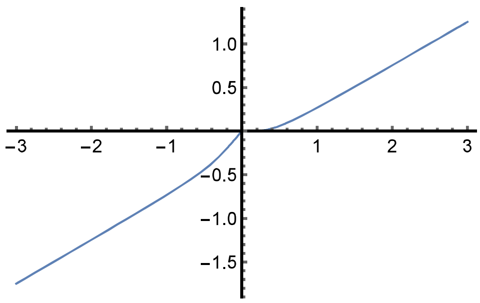


At the points where the sine function vanishes: $x = k\pi$, $k \in \mathbb{Z}$.

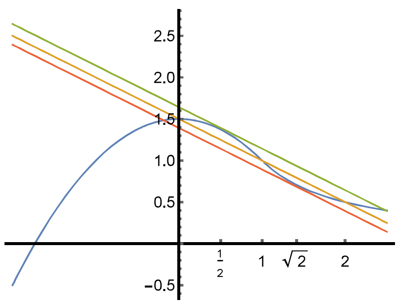
Problem 3.1.8

$$\lim_{h\to 0^+} \frac{1}{1+\mathrm{e}^{1/h}} = 0, \quad \lim_{h\to 0^-} \frac{1}{1+\mathrm{e}^{1/h}} = 1.$$

The angle that the tangent lines form is then $\operatorname{arctg} 1 = \pi/4$.



Problem 3.1.9 *a*) It is continuous and differentiable in all $\mathbb{R}$.



b) Yes, we can apply the mean value theorem in $[0, 2]$:

$$\frac{f(2) - f(0)}{2 - 0} = -1/2 \quad = f'(c) = \begin{cases} -c & \text{if } c < 1, \\ -1/c^2 & \text{if } c > 1. \end{cases}$$

We obtain the two values $c = 1/2 < 1$ and $c = \sqrt{2} > 1$.

Problem 3.1.10 It is continuous in $\{x+2 \geq 0\} \bigcap \{-1 \leq x+2 \leq 1\} = [-2,-1]$, ant it is differentiable in $(-2,-1)$ since $f'(x) = \dfrac{\arccos(x+2)}{2\sqrt{x+2}} - \dfrac{\sqrt{x+2}}{\sqrt{1-(x+2)^2}}$, and the denominators are null respectively at $x=-2$ and at $x=-1$.

Problem 3.1.11 f is differentiable in $\mathbb{R}$ if and only if the equation $\alpha x^2 - x + 3 = 0$ does not have two different roots; using the discriminant we obtain the condition $1 - 12\alpha \leq 0$, that is $\alpha \geq \dfrac{1}{12}$.

Problem 3.1.12 f is not differentiable at $x = 0$.

Problem 3.1.13 *a) i)* $f'(x) = kx|x|^{k-2}$, *ii)* $g'(x) = k|x|^{k-1}$, *b)* They are both continuous at zero and, for $k > 1$, in f and g the side derivatives at the origin have limit zero, so they are both differentiable at zero and $f'(0) = 0$. *c)* The function h is continuous at zero by the pinching lemma. For the derivative at zero:

$$\left|\lim_{\alpha\to 0} \frac{h(\alpha)-h(0)}{\alpha}\right| = \left|\lim_{\alpha\to 0} \frac{h(\alpha)}{\alpha}\right| \leq \lim_{\alpha\to 0} \frac{|\alpha|^k}{|\alpha|} = \lim_{\alpha\to 0} |\alpha|^{k-1} = 0,$$

hence, h is differentiable at zero with $h'(0) = 0$. *d)* By part *c)* applied to the points $x = 0$ and $x = 1$ it is differentiable at those points. For $x \neq 0, x \neq 1$ the function is not continuous, so it is not differentiable.

Problem 3.1.14 If $c \leq 0$ the function is simply $f(x) = |x|^{-1}$, that is not continuous at $x = 0$. For $c > 0$, since f is even, it is enough to study $x = c$, where we obtain $a + bc^2 = \dfrac{1}{c}$, $2bc = -\dfrac{1}{c^2}$, which implies $a = \dfrac{3}{2c}$, $b = -\dfrac{1}{2c^3}$.

Problem 3.1.15 Consider $f(x) = x^{2/3}$. Applying the mean value theorem in $[26, 27]$ we have

$$\frac{27^{2/3} - 26^{2/3}}{27-26} = \frac{2}{3}x^{-1/3}, \quad \text{for some } x \in (26,27) \subset (8,27),$$

which implies $9 - 26^{2/3} \in \left(\dfrac{2}{9}, \dfrac{1}{3}\right)$ and finally $\dfrac{78}{9} < 26^{2/3} < \dfrac{79}{9}$.

If we define now $g(x) = \log x$ and apply the mean value theorem in $[1, 3/2]$ we obtain $\dfrac{1}{3} < \log(3/2) < \dfrac{1}{2}$.

Problem 3.1.16 *a)*

$$\lim_{h\to 0}\frac{f(x+bh)-f(x-ah)}{h}=\lim_{h\to 0}\left(\frac{f(x+bh)-f(x)}{bh}b+\frac{f(x-ah)-f(x)}{-ah}a\right)=(b+a)f'(x).$$

b) Using the previous part:

$$2f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0-h)}{h}=0 \quad\Longrightarrow\quad f'(0)=0.$$

c) We can use L'Hôpital:

$$\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}=\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}=\frac{2}{2}f''(x)=f''(x).$$

Problem 3.1.17 The limits of 2.1.2 are immediate using L'Hôpital's rule; for example

$$iii)\quad L=\lim_{x\to 64}\frac{1/(2x^{1/2})}{1/(3x^{2/3})}=3.$$

In some limits of 2.1.3 it is necessary first to work a little bit to obtain a quotient. For example in *vi)*, *viii)*, *xi)* and *xii)* write them in exponential form. If it is necessary to use L'Hôpital's rule more than once, simplify as much as you can before differentiating again. For example

$$i)\quad L=\lim_{x\to 0}\frac{12x^2\sin 2x^3\cos 2x^3}{6x^5}=2\lim_{x\to 0}\frac{\sin 2x^3}{x^3}=2\lim_{x\to 0}\frac{6x^2\cos 2x^3}{3x^2}=4.$$

Problem 3.1.18 *i)* Use L'Hôpital twice, $L=1/2$. *ii)* $L=1$.

$$iii)\quad L=\lim_{x\to 1^+}\frac{\log(x-1)}{1/\log x}=\lim_{x\to 1}\frac{1/(x-1)}{-1/(x\log^2 x)}=\lim_{x\to 1}\frac{\log^2 x}{1-x}=-\lim_{x\to 1}\frac{2\log x}{x}=0.$$

iv) $L=\exp(\lim_{x\to\infty}\dfrac{\log x}{x})=1$. *v)* define $x+1=t$ to simplify the derivatives:

$$\begin{aligned}L&=\lim_{t\to 1}\frac{t^t-t^2+t-1}{(t-1)^3}=\lim_{t\to 1}\frac{t^t(\log t+1)-2t+1}{3(t-1)^2}=\lim_{t\to 1}\frac{t^t(\log t+1)^2+t^{t-1}-2}{6(t-1)}\\&=\frac{1}{6}\lim_{t\to 1}\left(t^2(\log t+1)^3+2t^{t-1}(\log t+1)+t^{t-1}(\log t+(t-1)/t)\right)=1/2.\end{aligned}$$

vi) Define $t=1/x$, $L=1$.

Problem 3.1.19 *i)* $L=\lim_{x\to\infty}\dfrac{1}{x}\left(\dfrac{x}{x-1}\right)^x=0$. *ii)* $L=0$. *iii)* $L=1/2$.

iv) $L=\exp(\lim_{x\to 0}\frac{3x^2}{2\arcsin x})=1$. *v)* $L=\lim_{x\to 1/2}\dfrac{2x^2+3x-2}{\cos\pi x}=-\dfrac{5}{\pi}$. *vi)* $L=\lim_{x\to 0}\dfrac{2x\sin x}{1-\cos x}=4$.

vii) $L=\lim_{x\to\infty}(\sqrt{x^2+1}-\sqrt{x^2+4x})=\lim_{x\to\infty}\dfrac{1-4x}{\sqrt{x^2+1}+\sqrt{x^2+4x}}=-2$. *viii)* $L=\mathrm{e}$.

Problem 3.1.20 Since $\lim_{x\to 0}\frac{h(x)}{x^2} = 1$, we have $h(0) = \lim_{x\to 0} h(x) = \lim_{x\to 0}\frac{h(x)}{x} = 0$. Then, $h'(0) = \lim_{x\to 0}\frac{h(x)}{x} = 0$. Finally, $h''(0) = \lim_{x\to 0}\frac{h'(x)}{x}$. But using L'Hôpital's rule we obtain: $1 = \lim_{x\to 0}\frac{h(x)}{x^2} = \lim_{x\to 0}\frac{h'(x)}{2x}$, so $h''(0) = 2$.

Problem 3.1.21

$$L = \lim_{x\to 0}\frac{a\mathrm{e}^{ax} - \mathrm{e}^x - 1}{2x} = \lim_{x\to 0}\frac{a^2\mathrm{e}^{ax} - \mathrm{e}^x}{2} = \frac{a^2-1}{2},$$

But, in order to make sense in the second application of L'Hôpital it is necessary to have $\lim_{x\to 0}(a\mathrm{e}^{ax} - \mathrm{e}^x - 1) = 0$ (in other case the limit does not exist) which implies $a = 2$ and so $L = 3/2$.

Problem 3.1.22

$$\begin{aligned} i)\quad L &= \lim_{t\to 0}\frac{(1+t)^{1/t} - \mathrm{e}}{t} = \lim_{t\to 0}(1+t)^{1/t}\Big(\frac{1}{t(t+1)} - \frac{\log(1+t)}{t^2}\Big) = \\ &= \mathrm{e}\lim_{t\to 0}\frac{t-(1+t)\log(1+t)}{t^2} = \mathrm{e}\lim_{t\to 0}\frac{-\log(1+t)}{2t} = -\frac{\mathrm{e}}{2}. \end{aligned}$$

$$\begin{aligned} ii)\quad L &= \lim_{x\to\infty}\Big[\frac{(1+1/x)^x}{\mathrm{e}}\Big]^x = \exp\Big[\lim_{x\to\infty} x\Big(\frac{(1+1/x)^x}{\mathrm{e}} - 1\Big)\Big] = \\ &= \exp\Big[\frac{1}{\mathrm{e}}\lim_{x\to\infty} x\Big((1+1/x)^x - \mathrm{e}\Big)\Big] = \exp\Big[\frac{1}{\mathrm{e}}(-\frac{\mathrm{e}}{2})\Big] = \mathrm{e}^{-1/2}, \end{aligned}$$

using the previous limit.

$$\begin{aligned} iii)\quad L &= \exp\Big[\lim_{x\to\infty} x\Big(\frac{2^{1/x}+18^{1/x}}{2} - 1\Big)\Big] = \exp\Big[\lim_{t\to 0}\frac{2^t+18^t-2}{2t}\Big] \\ &= \exp\Big[\frac{\log 2 + \log 18}{2}\Big] = 6\,. \end{aligned}$$

In fact, this is a particular case of the following limit, $L = \sqrt{2\cdot 18} = 6$.

$$\begin{aligned} iv)\quad L &= \exp\Big[\lim_{x\to\infty} x\Big(\frac{1}{p}\sum_{i=1}^{p} a_i^{1/x} - 1\Big)\Big] = \exp\left[\lim_{t\to 0}\frac{\sum_{i=1}^{p} a_i^t - p}{pt}\right] \\ &= \exp\Big[\lim_{t\to 0}\frac{1}{p}\sum_{i=1}^{p} a_i^t\log a_i\Big] = \exp\Big[\frac{1}{p}\sum_{i=1}^{p}\log a_i\Big] = \Big(\prod_{i=1}^{p} a_i\Big)^{1/p}, \end{aligned}$$

that is, the geometric mean of the a_i terms.

Problem 3.1.23 *a)* $f(0) = 0$ so that the limit exists.
b) $f'(0) = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{f(2x^3)}{2x^3} = \frac{5}{2}$.
c) $\lim_{x\to 0} \frac{(f\circ f)(2x)}{f^{-1}(3x)} = \lim_{x\to 0} \frac{(f(f(2x))}{f(2x)} \frac{f(2x)}{2x} \frac{3x}{f^{-1}(3x)} \frac{2}{3} = \frac{125}{12}$.

Problem 3.1.24 The function $f(x) = x^{1+\frac{1}{x}}$ is continuous and differentiable in $x > 0$. For every $x > 0$, applying the mean value theorem to the interval $[x, x+1]$ there is an $\alpha \in (x, x+1)$ such that:

$$(1+x)^{1+\frac{1}{1+x}} - x^{1+\frac{1}{x}} = \frac{f(x+1) - f(x)}{1} = f'(\alpha),$$

so, the limit we want is equal to

$$\lim_{\alpha\to\infty} f'(\alpha) = \lim_{\alpha\to\infty} \alpha^{\frac{1}{\alpha}} \left(-\frac{\log\alpha}{\alpha} + 1 + \frac{1}{\alpha} \right) = 1.$$

Problem 3.1.25 *a)* $f^{-1}(x) = \arcsin x$, for $x \in [-1, 1]$, and we can differentiate directly, also we can use the inverse function theorem to obtain:

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\cos(f^{-1}(x))} = \frac{5}{4} \implies \cos(f^{-1}(x)) = \frac{4}{5} \implies f^{-1}(x) = \arccos\frac{4}{5} \in [0, \pi],$$

then $x = \sin\left(\arccos\frac{4}{5}\right) = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \frac{3}{5}$. *b)* call $g^{-1}(x) = y$

$$\left(g^{-1}\right)'(x) = \frac{1}{g'(g^{-1}(x))} = \frac{1}{g'(y)} = \frac{y + \sqrt{1+y^2}}{1 + \frac{y}{\sqrt{1+y^2}}} = \sqrt{y^2+1} = \sqrt{\left(g^{-1}(x)\right)^2 + 1} = 2,$$

this means that $g^{-1}(x) = \pm\sqrt{3}$, and so $x = g(\pm\sqrt{3}) = \log(\pm\sqrt{3} + 2)$.

Problem 3.1.26 Observe first that if f is continuous then so is f', since $f' = \mathrm{e}^f(2 + \operatorname{tg} x)$. Also the inverse g is continuous. We have

$$f(0) = 1, \quad g(0) = f^{-1}(1) = 0, \quad f'(0) = \lim_{x\to 0} f'(x) = 2\mathrm{e}^{f(0)} = 2e.$$

Besides, since $f'(0) \neq 0$, the following limit exists

$$\lim_{x\to 0} g'(x) = \lim_{x\to 0} \frac{1}{f'(g(x))} = \frac{1}{2\mathrm{e}}.$$

Now, we use L'Hôpital:

$$L = \lim_{x\to 0} \frac{\mathrm{e}^x - \mathrm{e}^{-\sin x}}{g(x)} = \lim_{x\to 0} \frac{\mathrm{e}^x + \cos x\, \mathrm{e}^{-\sin x}}{g'(x)} = 4\mathrm{e}.$$

Also we can decompose the limit instead:

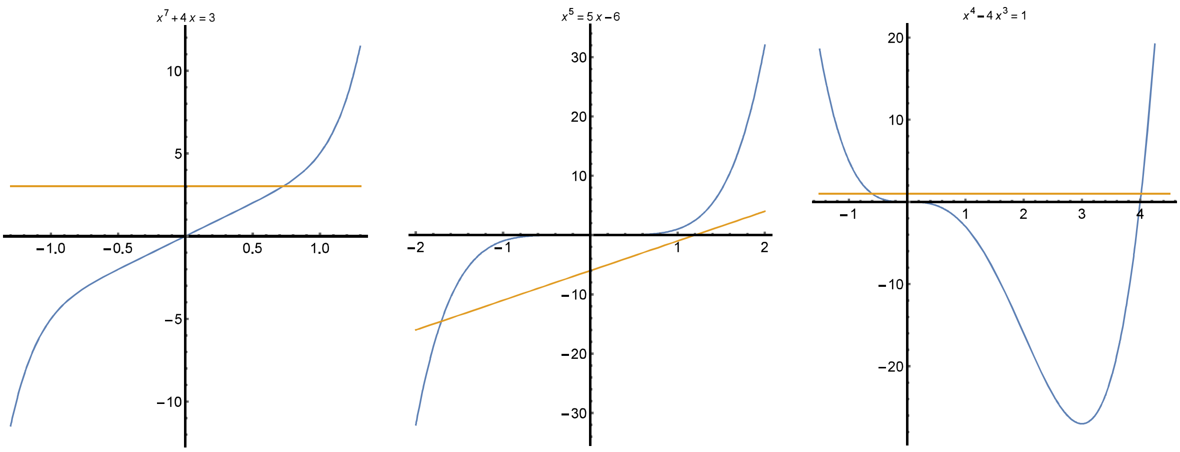
$$L = \lim_{x\to 0} \frac{\mathrm{e}^x - \mathrm{e}^{-\sin x}}{g(x)} = \lim_{x\to 0} \frac{\mathrm{e}^x - \mathrm{e}^{-\sin x}}{x} \lim_{x\to 0} \frac{x}{g(x)}.$$

The first limit is 2; for the second:

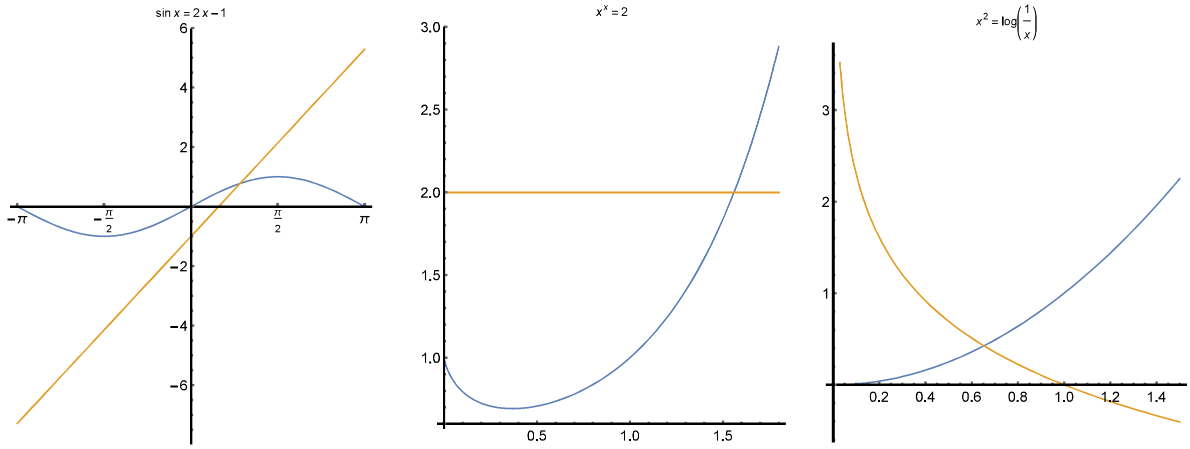
$$\lim_{x\to 0} \frac{x}{g(x)} = \lim_{z\to 0} \frac{f(z) - 1}{z} = f'(0) = 2\mathrm{e}.$$

Problem 3.1.27 *a*) Apply Rolle's theorem in the intervals determined by each pair of consecutive roots of f, we obtain that f' must have at least the amount of roots of f minus one.
b) Apply the previous result to the successive derivatives of f.

Problem 3.1.28 Using Bolzano's and Rolle's theorems we obtain that the equations *i*), *ii*), *iv*) and *v*) have one root, while the equation *iii*) has two and the equation *vi*) has none. Once we know how to plot a function the problem becomes much easier to solve.
i) $f(x) = x^7 + 4x - 3$ verifies $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = \infty$, so it has at least one root; in case of more than one root the derivative should be zero at some intermediate point, but $f'(x) = 7x^4 + 4 \neq 0$ for all $x \in \mathbb{R}$.
ii) $f(x) = x^5 - 5x + 6$ verifies $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = \infty$, $f(-1) > 0$, $f(1) > 0$, and $f'(x) = 5(x^4 - 1) = 0 \Leftrightarrow x = \pm 1$; so there is a root in $(-\infty, -1)$, none in $[1, \infty)$, and also no roots in $[-1, 1]$ since this would imply $f'(x) = 0$ for some intermediate point.
iii) $f(x) = x^4 - 4x^3 - 1$ verifies $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = \infty$, $f(0) < 0$, $f(3) < 0$, and $f'(x) = 4x^2(x - 3) = 0 \Leftrightarrow x = 0$ or $x = 3$; so there is a root in $(-\infty, 0)$, another in $(3, \infty)$ and none in $[0, 3]$.



iv) $f(x) = \sin x - 2x + 1$ verifies $f(0) > 0$, $f(\pi) < 0$ and $f'(x) = \cos x - 2 \neq 0$ for all $x \in \mathbb{R}$; so there is a unique root, and it is in $(0, \pi)$.
v) $f(x) = x^x - 2$ verifies $f(1) < 0$, $f(2) > 0$ and $f'(x) = x^x(\log x + 1) \neq 0$ for all $x \geq 1$; so there is a unique root, and it is in $(1, 2)$.
vi) $f(x) = x^2 + \log x$ verifies $f(1) > 0$, and $f'(x) = 2x + 1/x > 0$ for all $x \geq 1$; so there are no roots.



Problem 3.1.29 *a*) For a given ε, we have $|f(x)-f(\alpha)| \leq k|x-a|^{\alpha} < \varepsilon$ if $|x-a| < \left(\frac{\varepsilon}{k}\right)^{1/\alpha}$, this is the δ we need. *b*) With the hint we obtain $k=\sqrt{2}$. Another solution: If $x \geq y \geq 0$ then $\sqrt{x}-\sqrt{y} \leq \sqrt{x-y}$ (compute the squares and arrive to $y \leq \sqrt{xy}$, that is true). By the same reason, if $y \geq x \geq 0$ then $\sqrt{y}-\sqrt{x} \leq \sqrt{y-x}$, so

$$\left|\sqrt{x}-\sqrt{y}\right| \leq \sqrt{|x-y|},$$

so $f(x)=\sqrt{x}$ belongs to $\Lambda_{1/2}([0,1])$ with constant $k=1$. *c*) Use the mean value theorem for any pair of points $x, y \in [a,b]$, there is a $\xi \in (a,b)$ such that:

$$\frac{f(x)-f(y)}{x-y} = f'(\xi) \quad \Longrightarrow \quad |f(x)-f(y)| = |f'(\xi)||x-y|.$$

Since f' is continuous in $[a,b]$, then it is also bounded there, for example by K, hence: $|f(x)-f(y)| = K|x-y|$. *d*) For any $x, y \in [a,b]$ there is a k such that: $|f(x)-f(y)| \leq k|x-y|$, use now the mean value theorem and obtain:

$$|f'(x)| = \lim_{y \to x}\left|\frac{f(y)-f(x)}{y-x}\right| \leq k.$$

e) Directly, $|f(x)-f(y)| = \Big||x|-|y|\Big| \leq |x-y|$. *f*) We compute the derivative:

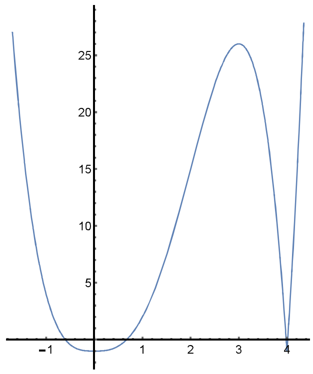
$$|f'(x)| = \lim_{y\to x}\left|\frac{f(y)-f(x)}{y-x}\right| = \lim_{y\to x}\frac{|f(y)-f(x)|}{|y-x|^{\alpha}}|y-x|^{\alpha-1} \leq k\lim_{y\to x}|y-x|^{\alpha-1} = 0.$$

So, the function is constant.

3.2 Extrema

Problem 3.2.1

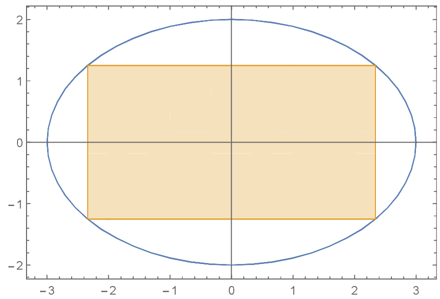
i) f is continuous in $\mathbb{R}$ and differentiable in $\mathbb{R} \setminus \{4\}$.
ii) Local maximum at $x = 3$, local and absolute minima
at $x = 0$, $x = 4$.
iii) f is increasing in $[0, 1]$, with $f(0) = -1$, $f(1) = 2$.



Problem 3.2.2 Minimize the area of the surface, $A(r, h) = 2\pi r^2 + 2\pi rh$, with the restriction of fixed volume $V = \pi r^2 h$. That is, minimize $f(r) = 2\pi\big(r^2 + \dfrac{V}{\pi r}\big)$ for $r \in (0, \infty)$. We obtain $r = (V/2\pi)^{1/3}$, $h = 2r$.

Problem 3.2.3

For a given point in the first quadrant $P = (x, y)$, maximize the area $A(x, y) = 4xy$ with the restriction that the point belongs to the ellipse $(x/a)^2 + (y/b)^2 = 1$. That is, maximize $f(x) = \dfrac{4bx}{a}\sqrt{a^2 - x^2}$ for $x \in [0, a]$. We obtain $x = \dfrac{a}{\sqrt{2}}$, $y = \dfrac{b}{\sqrt{2}}$, and the area $A = 2ab$.

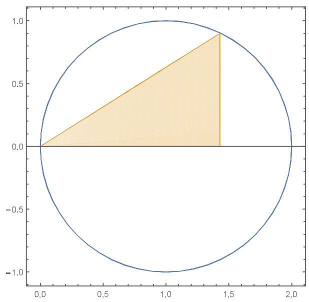


Problem 3.2.4 Given a point in the first quadrant $P = (x_0, y_0)$ on the parabola $y = 6 - x^2$, the tangent line to the curve through P is $y = 6 + x_0^2 - 2x_0 x$. The area of the triangle determined by this line and the coordinate axes is $A(x_0) = \dfrac{1}{4x_0}(x_0^2 + 6)^2$. Minimize this function for $x_0 \in (0, \sqrt{6}]$ and we obtain $x_0 = \sqrt{2}$, with area $A = 8\sqrt{2}$.

Problem 3.2.5

Given a point in the first quadrant $P = (x, y)$ on the circumference $(x - 1)^2 + y^2 = 1$, the area we want to minimize is $A(x, y) = \dfrac{1}{2}xy = \dfrac{1}{2}x\sqrt{2x - x^2}$, for $x \in [0, 2]$.

We obtain $x = 3/2$.



Problem 3.2.6 By symmetry of triangles we have that $\dfrac{x_0+\alpha}{x_0} = \dfrac{y_0+\beta}{\beta}$.

a) Minimizing the length $f(\alpha) = \sqrt{(x_0+\alpha)^2 + (y_0 + \frac{x_0 y_0}{\alpha})^2}$, for $\alpha > 0$, gives us $\alpha = (x_0 y_0^2)^{1/3}$.

b) Minimizing the sum of lengths $g(\alpha) = x_0 + \alpha + y_0 + \dfrac{x_0 y_0}{\alpha}$, for $\alpha > 0$, gives us $\alpha = (x_0 y_0)^{1/2}$.

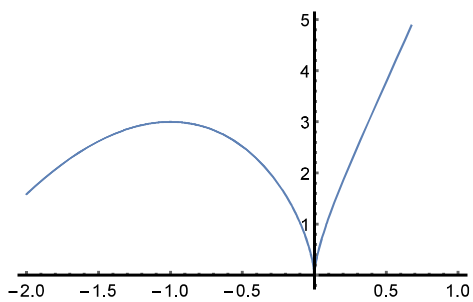
c) Minimizing the area $h(\alpha) = \dfrac{1}{2}(x_0+\alpha)(y_0 + \dfrac{x_0 y_0}{\alpha})$ for $\alpha > 0$ gives us $\alpha = x_0$.

Problem 3.2.7 *a*) Consider a fixed $a \geq 1$ and define the function, for $x \geq -1$, $f(x) = (1+x)^a - 1 - ax$. The point $x = 0$ is an absolute minimum of f in $[-1, \infty)$, so $f(x) \geq f(0) = 0$ for all $x \geq -1$.
Now consider a fixed $0 < a \leq 1$ and the function, for $x \geq -1$, $f(x) = (1+x)^a - 1 - ax$. The point $x = 0$ is a global maximum of f in $[-1, \infty)$, so $f(x) \leq f(0) = 0$ for all $x \geq -1$.
b) Define $g(x) = \mathrm{e}^x - 1 - x$. The point $x = 0$ is a global minimum of g, so $g(x) \geq g(0) = 0$ for all $x \in \mathbb{R}$.
c) Consider $h(x) = \log(1+x) - \dfrac{x}{1+x}$. The point $x = 0$ is a global minimum of h, so $h(x) \geq h(0) = 0$ for all $x > -1$. For the second inequality use part *b*) and obtain directly $\mathrm{e}^x \geq x + 1$, which implies $x \geq \log(x+1)$.

Problem 3.2.8 *a*) Define $f(x) = \dfrac{\log x}{x}$. Since $x = \mathrm{e}$ is a global maximum we have $f(x) < f(\mathrm{e}) = \dfrac{1}{\mathrm{e}}$, for all $x > 0$, $x \neq \mathrm{e}$;
b) We have $\dfrac{\log x}{x} < \dfrac{1}{\mathrm{e}}$ for all $x > 0$, $x \neq \mathrm{e}$, so $(\mathrm{e} \log x) < x$, that is, $x^{\mathrm{e}} < \mathrm{e}^x$.

Problem 3.2.9

a) Consider $f(x) = 2x^{5/3} + 5x^{2/3}$, then:
$f'(x) = \dfrac{10}{3}(x+1)x^{-1/3}$.
Thus $f'(0)$ does not exist, while $f'(-1) = 0$. Comparing the values $f(0) = 0$, $f(-1) = 3$, $f(-2) = 2^{2/3}$, $f(1) = 7$, we find a maximum at $x = 1$, and a minimum at $x = 0$.



b) f is continuous on $[-\pi, \pi]$, so there exist the absolute maximum and the absolute minimum. $f'(0)$ does not exist. Comparing the values $f(0) = 1$, $f(\pi) = f(-\pi) = -1 + \frac{\pi}{\sqrt{2}}$, $f(\frac{\pi}{4}) = f(-\frac{\pi}{4}) = \frac{\pi+4}{4\sqrt{2}}$, the absolute maxima are at $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$ and the absolute minimum is at $x = 0$.

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