uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL CALCULUS. Problems

Degree in Applied Mathematics and Computation

Chapter 3

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3 Derivatives and their applications

3.1 Differentiability

Problem 3.1.1 Let f, g be differentiable functions in all $\mathbb{R}$. Write down the derivative of the following functions on their domains:

$$i)\quad h(x) = \sqrt{f^2(x) + g^2(x)}, \qquad ii)\quad h(x) = \operatorname{arctg}\left(\frac{f(x)}{g(x)}\right),$$

$$iii)\quad h(x) = f(g(x))\mathrm{e}^{f(x)}, \qquad iv)\quad h(x) = \log(g(x)\sin(f(x))),$$

$$v)\quad h(x) = (f(x))^{g(x)}, \qquad vi)\quad h(x) = \frac{1}{\log(f(x) + g^2(x))}.$$

Problem 3.1.2

a) Build a continuous function on $\mathbb{R}$ that vanishes for $|x| \geq 2$ and takes the value one for $|x| \leq 1$.

b) Build another function that is also differentiable.

Problem 3.1.3 From the hyperbolic functions $\sinh x$ and $\cosh x$, we define the hyperbolic tangent and secant by $\operatorname{tgh} x = \dfrac{\sinh x}{\cosh x}$ and $\operatorname{sech} x = \dfrac{1}{\cosh x}$. Prove the formulas:

$$i)\quad (\sinh x)' = \cosh x, \qquad ii)\quad (\cosh x)' = \sinh x,$$

$$iii)\quad (\operatorname{tgh} x)' = \operatorname{sech}^2 x, \qquad iv)\quad (\operatorname{sech} x)' = -\operatorname{sech} x \operatorname{tgh} x.$$

Problem 3.1.4 Check that the following functions satisfy the specified differential equations, where c, c_1 and c_2 are constants.

$$i)\quad f(x) = \frac{c}{x}, \qquad xf' + f = 0;$$

$$ii)\quad f(x) = x \operatorname{tg} x, \qquad xf' - f - f^2 = x^2;$$

$$iii)\quad f(x) = c_1 \sin 3x + c_2 \cos 3x, \qquad f'' + 9f = 0;$$

$$iv)\quad f(x) = c_1\mathrm{e}^{3x} + c_2\mathrm{e}^{-3x}, \qquad f'' - 9f = 0;$$

$$v)\quad f(x) = c_1\mathrm{e}^{2x} + c_2\mathrm{e}^{5x}, \qquad f'' - 7f' + 10f = 0;$$

$$vi)\quad f(x) = \log(c_1\mathrm{e}^{x} + \mathrm{e}^{-x}) + c_2, \qquad f'' + (f')^2 = 1.$$

Problem 3.1.5 Prove the identities

$$i)\quad \operatorname{arctg} x + \operatorname{arctg}\frac{1}{x} = \frac{\pi}{2}, \qquad x > 0;$$

$$ii)\quad \operatorname{arctg}\frac{1+x}{1-x} - \operatorname{arctg} x = \frac{\pi}{4}, \qquad x < 1;$$

$$iii)\quad 2\operatorname{arctg} x + \arcsin\frac{2x}{1+x^2} = \pi, \qquad x \geq 1.$$

Hint: differentiate and substitute at some point of the interval. The result is *not* true outside the specified intervals.

Problem 3.1.6 Find the value of $a \in \mathbb{R}$ for which the parabola $f(x) = ax^2$ is tangent to the curve $g(x) = \log x$ and write the equation of the common tangent.

Problem 3.1.7 Find the points at which the graph of the function $f(x) = x + (\sin x)^{1/3}$ has a vertical tangent.

Problem 3.1.8 Obtain the angle spanned by the left and right tangents at the origin to the graph of the function

$$f(x) = \begin{cases} \dfrac{x}{1 + \mathrm{e}^{1/x}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Problem 3.1.9 Given the function

$$f(x) = \begin{cases} (3 - x^2)/2 & \text{if } x < 1, \\ 1/x & \text{if } x \geq 1, \end{cases}$$

a) study its continuity and differentiability;

b) can we apply the mean value theorem on [0,2]? If we can, find the point (or points) for which the thesis of the theorem is satisfied.

Problem 3.1.10 Study the continuity and differentiability of the function

$$f(x) = \sqrt{x + 2}\, \arccos(x + 2).$$

Problem 3.1.11 Find the minimum value of α for which the function $f(x) = |\alpha x^2 - x + 3|$ is differentiable on all $\mathbb{R}$.

Problem 3.1.12 The function $f(x) = 1 - x^{2/3}$ vanishes at -1 and at 1 and, nevertheless, $f'(x) \neq 0$ in $(-1, 1)$. Explain this apparent contradiction with Rolle's theorem.

Problem 3.1.13

a) Compute the derivative, for $x \neq 0$, of the following functions:

$$i)\ f(x) = |x|^k, \qquad ii)\ g(x) = |x|^{k-1}x.$$

b) Prove that if $k > 1$, both are differentiable at the origin and obtain $f'(0)$.

c) Prove that if a function h satisfies $|h(x)| \leq |x|^k$, $k > 1$, for every x on some neighbourhood of the origin, then h is differentiable at the origin and obtain $h'(0)$.

d) Prove that the function

$$f(x) = \begin{cases} x^2(1 - x)^2, & \text{if} x \notin \mathbb{Q}, \\ 0, & \text{if} x \in \mathbb{Q}, \end{cases}$$

is differentiable only at two points.

Problem 3.1.14 Given the function

$$f(x) = \begin{cases} a + bx^2, & |x| \le c, \\ |x|^{-1}, & |x| > c, \end{cases}$$

with $c > 0$, find a and b so that it is continuous and differentiable on the whole real line, $\mathbb{R}$.

Problem 3.1.15 Using the mean value theorem, approximate $26^{2/3}$ and $\log(3/2)$.

Problem 3.1.16

a) If f is a differentiable function, obtain

$$\lim_{h\to 0} \frac{f(x+bh) - f(x-ah)}{h}.$$

b) If f is a differentiable and even function, find $f'(0)$.

c) If f is a twice differentiable function, obtain

$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}.$$

Problem 3.1.17 Obtain the limits of the problems 2.1.2 and 2.1.3 using L'Hôpital's rule, writing them previously in the appropriate form.

Problem 3.1.18 Calculate the following limits:

i) $\displaystyle\lim_{x\to 0} \frac{e^x - \sin x - 1}{x^2}$, *ii*) $\displaystyle\lim_{x\to 0} \frac{\log|\sin 7x|}{\log|\sin x|}$,

iii) $\displaystyle\lim_{x\to 1} \log x \cdot \log(x-1)$, *iv*) $\displaystyle\lim_{x\to\infty} x^{1/x}$,

v) $\displaystyle\lim_{x\to 0} \frac{(1+x)^{1+x} - 1 - x - x^2}{x^3}$, *vi*) $\displaystyle\lim_{x\to\infty} x\Big(\operatorname{tg}(2/x) - \operatorname{tg}(1/x)\Big)$.

Problem 3.1.19 Compute the following limits:

i) $\displaystyle\lim_{x\to\infty} \frac{x^{x-1}}{(x-1)^x}$, *ii*) $\displaystyle\lim_{x\to 0} \frac{1 + \sin x - e^x}{\operatorname{arctg} x}$,

iii) $\displaystyle\lim_{x\to 0} \frac{\operatorname{tg} x - \sin x}{x^3}$, *iv*) $\displaystyle\lim_{x\to 0} (1+x^2)^{3/(2\arcsin x)}$,

v) $\displaystyle\lim_{x\to 1/2} (2x^2 + 3x - 2)\operatorname{tg}(\pi x)$, *vi*) $\displaystyle\lim_{x\to 0} \frac{2x\sin x}{\sec x - 1}$,

vii) $\displaystyle\lim_{x\to -\infty} (\sqrt{x^2+1} - \sqrt{x^2 - 4x})$, *viii*) $\displaystyle\lim_{x\to 0^+} x^{1/\log x}$.

Problem 3.1.20 Let h be a twice differentiable function, and consider

$$f(x) = \begin{cases} h(x)/x^2, & x \ne 0, \\ 1, & x = 0. \end{cases}$$

Assuming that f is continuous, find $h(0)$, $h'(0)$ and $h''(0)$.

Problem 3.1.21 Find a so that $\lim_{x\to 0} \dfrac{e^{ax} - e^x - x}{x^2}$ is finite and obtain the value of the limit.

Problem 3.1.22 Calculate the following limits:

$$i)\ \lim_{x\to\infty} x\Big((1+1/x)^x - e\Big), \qquad ii)\ \lim_{x\to\infty} \frac{(1+1/x)^{x^2}}{e^x},$$

$$iii)\ \lim_{x\to\infty} \Big(\frac{2^{1/x}+18^{1/x}}{2}\Big)^x, \qquad iv)\ \lim_{x\to\infty} \Big(\frac{1}{p}\sum_{i=1}^{p} a_i^{1/x}\Big)^x,\ p\in\mathbb{N},\ a_i>0.$$

Problem 3.1.23 Given a differentiable function, f, that satisfies $\lim_{x\to 0} \dfrac{f(2x^3)}{5x^3} = 1$,

a) justify that $f(0) = 0$;

b) prove that $f'(0) = 5/2$;

c) compute $\lim_{x\to 0} \dfrac{(f\circ f)(2x)}{f^{-1}(3x)}$.

Problem 3.1.24 Use the mean value theorem to obtain the limit:

$$\lim_{x\to\infty} \Big[(1+x)^{1+\frac{1}{1+x}} - x^{1+\frac{1}{x}}\Big].$$

Problem 3.1.25

a) Consider $f(x) = \sin x$. Calculate the values of x for which we have $(f^{-1})'(x) = 5/4$.

b) Consider now $g(x) = \log(x+\sqrt{x^2+1})$ and obtain x such that $(g^{-1})'(x) = 2$.

Problem 3.1.26 The equation

$$\begin{cases} e^{-f} f' = 2 + \operatorname{tg} x, \\ f(0) = 1, \end{cases}$$

defines a differentiable one-to-one (bijective) function f on the interval $[-\pi/4, \pi/4]$. We define the function $g(x) = f^{-1}(x+1)$. Obtain the limit

$$\lim_{x\to 0} \frac{e^x - e^{-\sin x}}{g(x)}.$$

Problem 3.1.27

a) Let $f : [a,b] \longrightarrow \mathbb{R}$ be differentiable. Prove that if f has $k \geq 2$ roots in $[a,b]$, then f' has at least $k-1$ roots in $[a,b]$.

b) If f is n times differentiable on $[a,b]$ and vanishes in $n+1$ different points in $[a,b]$, prove that $f^{(n)}$ vanishes at least once on $[a,b]$.

Problem 3.1.28 Calculate how many different solutions the following equations have on the given intervals:

$i)$	$x^7 + 4x = 3,$	in $\mathbb{R}$;	$ii)$	$x^5 = 5x - 6,$	in $\mathbb{R}$;
$iii)$	$x^4 - 4x^3 = 1,$	in $\mathbb{R}$;	$iv)$	$\sin x = 2x - 1,$	in $\mathbb{R}$;
$v)$	$x^x = 2,$	in $[1,\infty)$;	$vi)$	$x^2 = \log(1/x),$	in $(1,\infty)$.

Problem 3.1.29 For $0 < \alpha \leq 1$, we define $\Lambda_\alpha([a,b])$ as the set of functions $f : [a,b] \to \mathbb{R}$ such that there is a constant $k > 0$ for which

$$|f(x) - f(y)| \leq k|x - y|^\alpha, \qquad \forall x, y \in [a,b].$$

Prove that:

a) if $f \in \Lambda_\alpha([a,b])$, then f is continuous;

b) $f(x) = \sqrt{x}$ belongs to $\Lambda_{1/2}([0,1])$;

c) if f has a continuous derivative on $[a,b]$, then $f \in \Lambda_1([a,b])$.

d) if f is differentiable on $[a,b]$ and belongs to $\Lambda_1([a,b])$ with constant k, then $|f'(x)| \leq k$ for all $x \in [a,b]$.

e) $f(x) = |x|$ is not differentiable at the origin, but belongs to $\Lambda_1([a,b])$ for every pair $a, b \in \mathbb{R}$, $a < b$.

f) if f is in $\Lambda_\alpha([a,b])$ with $\alpha > 1$, then f is constant.

(The functions of $\Lambda_\alpha([a,b])$ are known as *Hölder continuous* if $0 < \alpha < 1$, and as *Lipschitz continuous* if $\alpha = 1$).
Hint: *b*) if $0 \leq y \leq x/2$, $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x-y}} \leq \sqrt{2}$, and if $x/2 \leq y \leq x$, by the mean value theorem, $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x-y}} \leq \frac{1}{2}$; *c*) if $|f'|$ has a maximum value k in $[a.b]$, by the mean value theorem this is the (Lipschitz) constant; *f*) $\left|\frac{f(x)-f(y)}{x-y}\right| \leq k|x-y|^{\alpha-1}$.

3.2 Extrema

Problem 3.2.1 Consider the function $f(x) = |x^3(x-4)| - 1$.

a) Study its continuity and differentiability.

b) Find its relative extrema.

c) Prove that the equation $f(x) = 0$ has a single solution on the interval $[0,1]$.

Problem 3.2.2 A tomato sauce company wants to manufacture cylindrical cans of fixed volume V. What is the relation between the radius r of the basis and its height, h, so that the minimum amount of material is required?

Problem 3.2.3 Find the area of the rectangle with sides parallel to the coordinate axes and inscribed in the ellipse $(x/a)^2 + (y/b)^2 = 1$ with maximum area.

Problem 3.2.4 Obtain the area of the triangle formed by the tangent to the parabola $y = 6 - x^2$ and the two positive semiaxes that has the minimum area.

Problem 3.2.5 A right triangle ABC has the vertex A at the origin, B on the circumference $(x-1)^2 + y^2 = 1$, and the side AC in the horizontal axis. Calculate C such that the area of the triangle is maximum.

Problem 3.2.6 Consider a point $P = (x_0, y_0)$ on the first quadrant. Draw a straight line which passes through P and cuts the axes at $A = (x_0 + \alpha, 0)$ and $B = (0, y_0 + \beta)$ respectively. Calculate $\alpha, \beta > 0$ such that the following magnitudes are minimized:

a) the length of AB;

b) the sum of the lengths of OA and OB;

c) the area of the triangle OAB.

Hint: $\beta = x_0 y_0/\alpha$.

Problem 3.2.7

a) Prove the Bernoulli inequality for $x > -1$:

$$\begin{array}{ll} (1+x)^a \geq 1 + ax, & \text{if} \quad a \geq 1, \\ (1+x)^a \leq 1 + ax, & \text{if} \quad 0 < a \leq 1. \end{array}$$

b) Prove that $\mathrm{e}^x \geq 1 + x$ for all $x \in \mathbb{R}$.

c) Prove that $\dfrac{x}{1+x} \leq \log(1+x) \leq x$ for all $x > -1$.

Hint: minimize the appropriate functions.

Problem 3.2.8

a) Prove that $\dfrac{\log x}{x} < \dfrac{1}{\mathrm{e}}$ for all $x > 0$, $x \neq \mathrm{e}$.

b) As a conclusion, obtain that $\mathrm{e}^x > x^{\mathrm{e}}$ for all $x > 0$, $x \neq \mathrm{e}$.

Problem 3.2.9

a) Find the absolute maxima and minima of the function $f(x) = 2x^{5/3} + 5x^{2/3}$ on the interval $[-2, 1]$.

b) The same question with $f(x) = |x/\sqrt{2}| + \cos x$ in the interval $[-\pi, \pi]$.

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