uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL CALCULUS. Solutions

Degree in Applied Mathematics and Computation

Chapter 4

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Open Course Ware, UC3M

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4 Local study of a function

4.1 Graphic representation

Problem 4.1.1 *i)* f is concave in $(-\infty, -\frac{2}{5})$, and convex in $(-\frac{2}{5}, 0) \cup (0, \infty)$. *ii)* f is convex in its domain, $(2, \infty)$. *iii)* f is convex in $\mathbb{R}$. *iv)* f is concave in its domain, $(-\infty, 2) \cup (4, \infty)$.

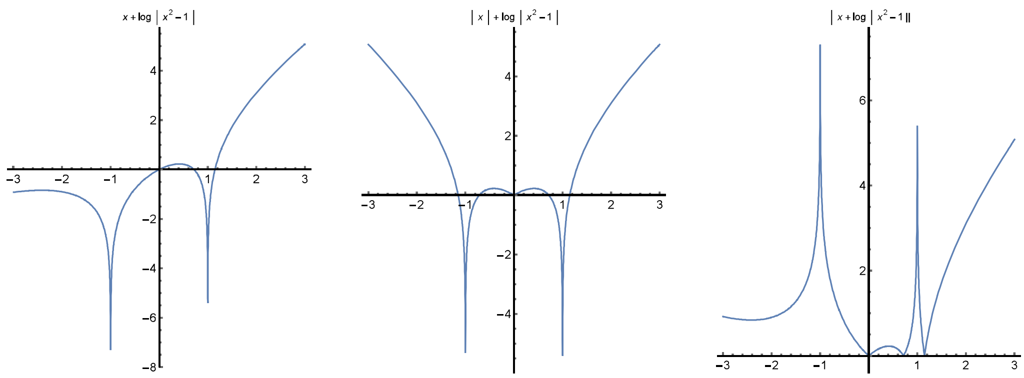
Problem 4.1.2 *a)* If $h = f \circ g$ and $f' \geq 0$, $f'' \geq 0$, $g'' \geq 0$, then we have:

$$h''(x) = f''(g(x))\,(g'(x))^2 + f'((g(x))\,g''(x) \geq 0.$$

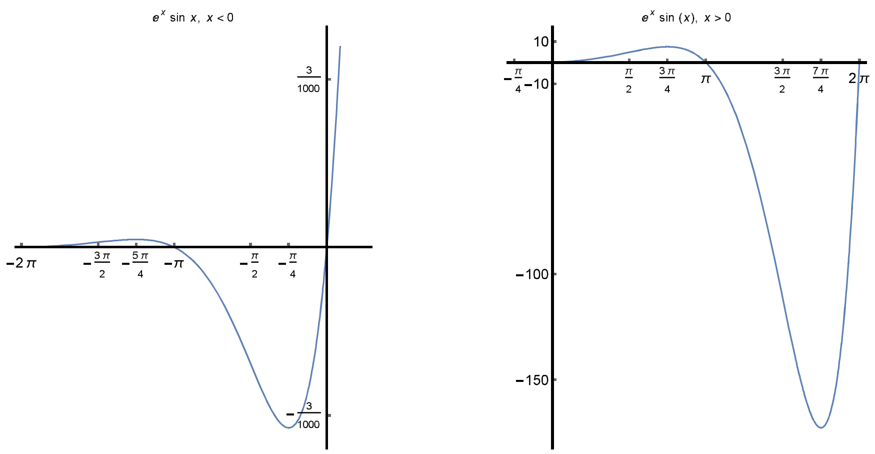
b) Since g is convex : $g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y)$, since f is increasing and convex:

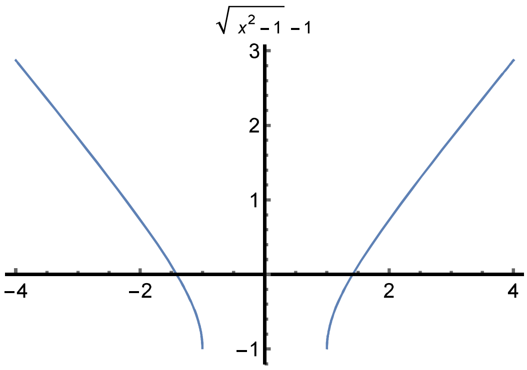
$$\begin{aligned} h(\lambda x + (1-\lambda)y) &= f(g(\lambda x + (1-\lambda)y)) \leq f(\lambda g(x) + (1-\lambda)g(y)) \\ &\leq \lambda f(g(x)) + (1-\lambda)f(g(y)) = \lambda h(x) + (1-\lambda)h(y). \end{aligned}$$

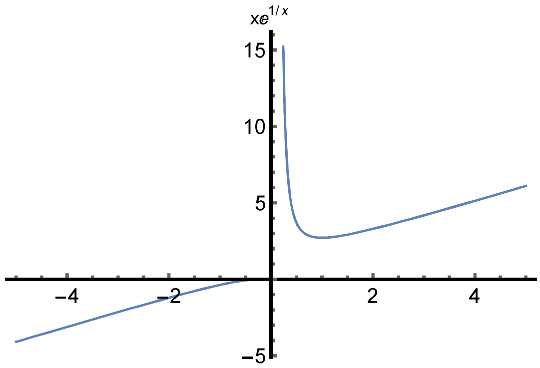
Problem 4.1.3

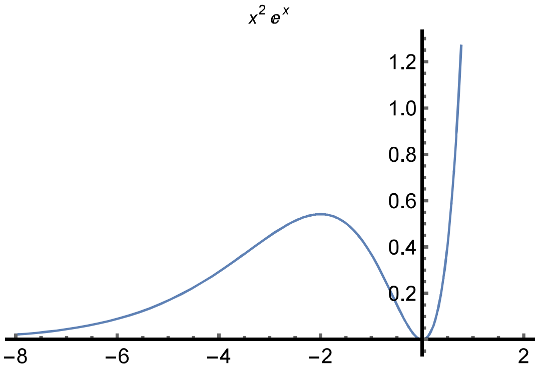


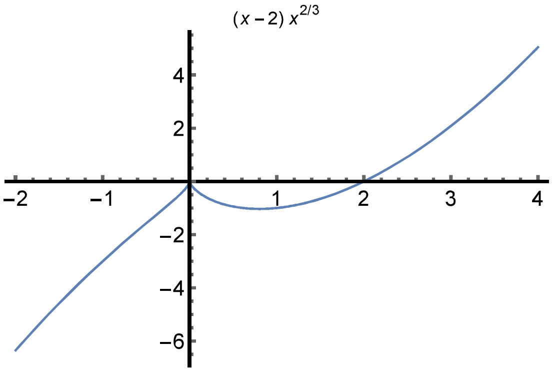
Problem 4.1.4

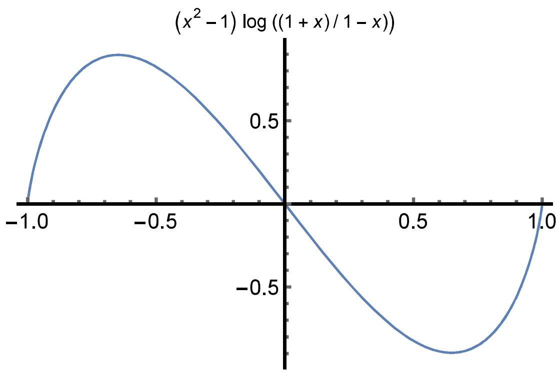


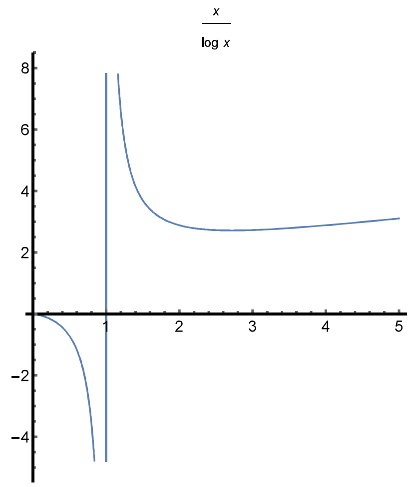

√(x² − 1) − 1
3
2
1
−1
−4
−2
2
4

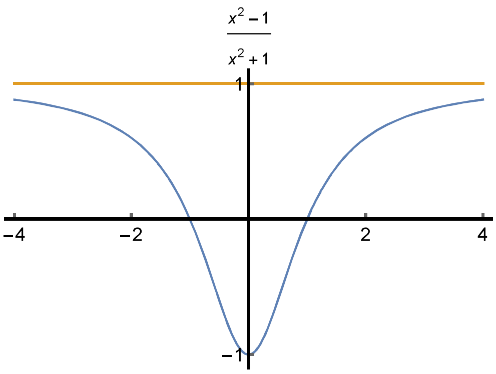

xe^{1/x}
15
10
5
−5
−4
−2
2
4


x² e^x
1.2
1.0
0.8
0.6
0.4
0.2
−8
−6
−4
−2
2

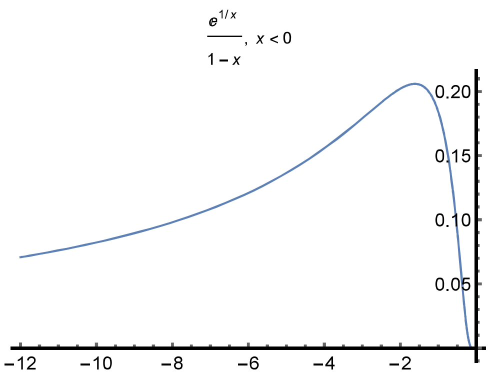

(x − 2) x^{2/3}
4
2
−2
−4
−6
−2
−1
1
2
3
4


(x² − 1) log ((1 + x) / 1 − x))
0.5
−0.5
−1.0
−0.5
0.5
1.0

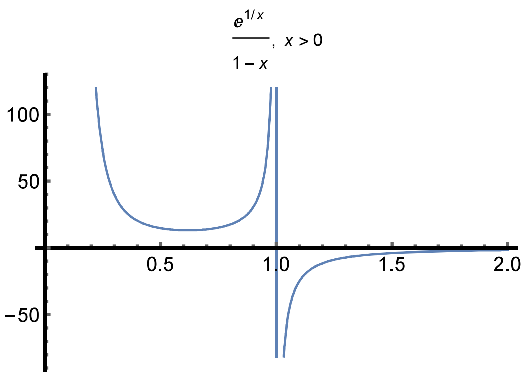

x / log x
8
6
4
2
−2
−4
1
2
3
4
5



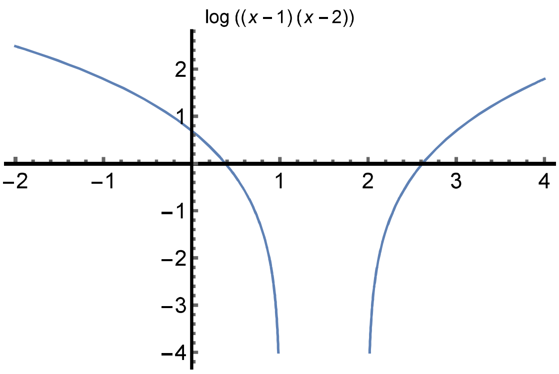
x^2 − 1 / x^2 + 1
1
−4
−2
2
4
−1



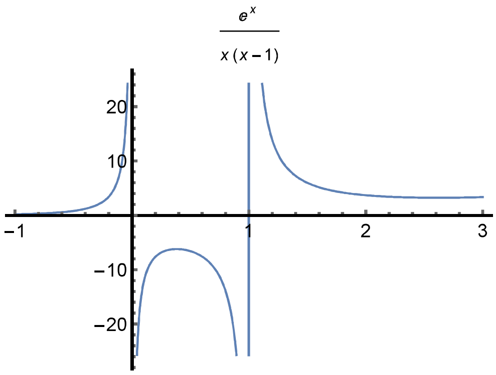
e^{1/x} / 1 − x, x < 0
0.20
0.15
0.10
0.05
−12
−10
−8
−6
−4
−2



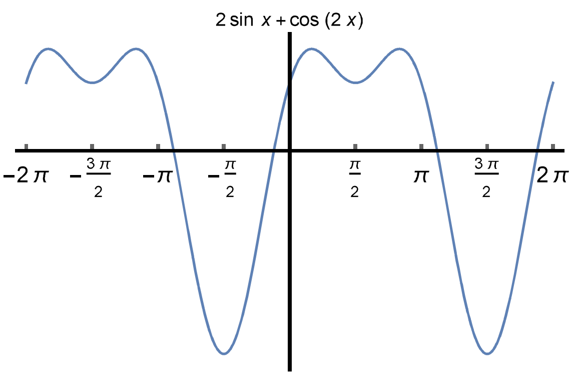
e^{1/x} / 1 − x, x > 0
100
50
−50
0.5
1.0
1.5
2.0



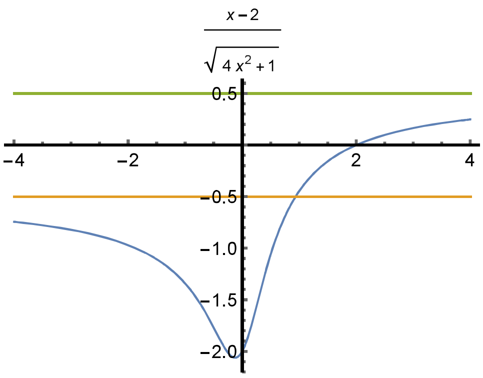
log ((x − 1) (x − 2))
2
1
−1
−2
−3
−4
−2
−1
1
2
3
4



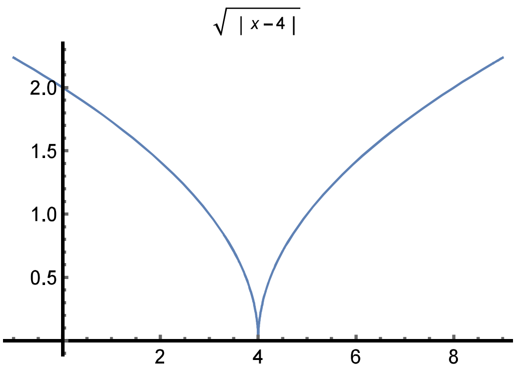
e^x / x (x − 1)
20
10
−10
−20
−1
1
2
3

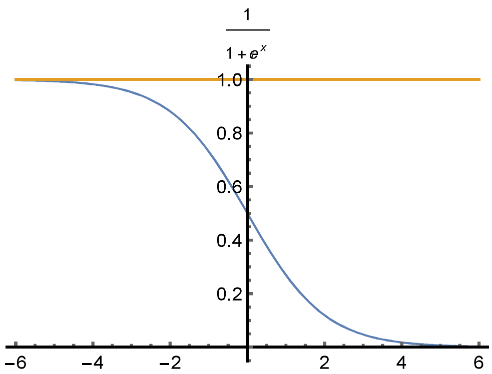


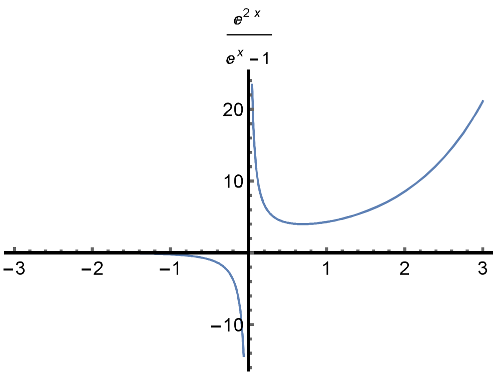
2 sin x + cos (2 x)
−2 π
− 3π/2
−π
− π/2
π/2
π
3π/2
2 π

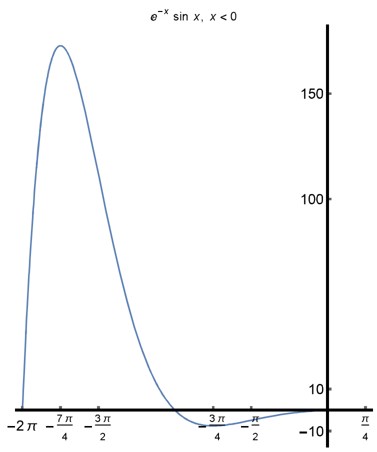


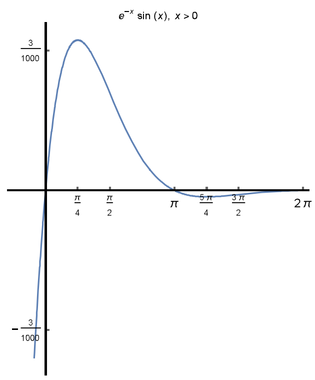
x − 2 / √(4 x^2 + 1)
0.5
−0.5
−1.0
−1.5
−2.0
−4
−2
2
4

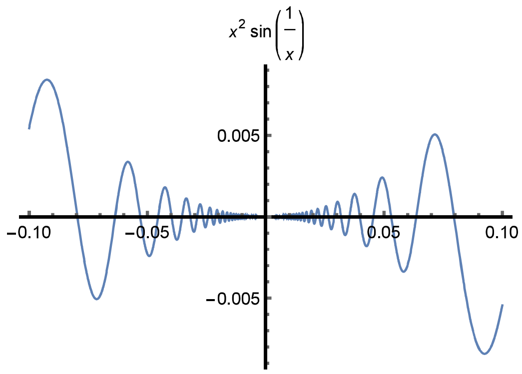

√|x − 4|
2.0
1.5
1.0
0.5
2
4
6
8

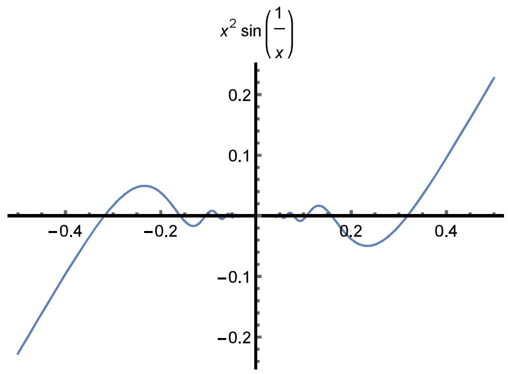

1 / (1 + e^x)
1.0
0.8
0.6
0.4
0.2
−6
−4
−2
2
4
6


e^{2x} / (e^x − 1)
20
10
−10
−3
−2
−1
1
2
3


e^{−x} sin x, x < 0
150
100
10
−10
−2π
−7π/4
−3π/2
−3π/4
−π/2
π/4

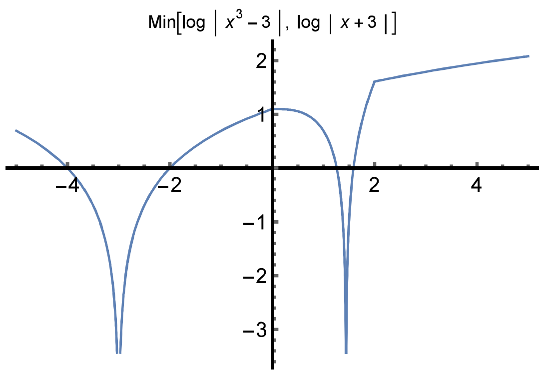

e^{−x} sin (x), x > 0
3/1000
−3/1000
π/4
π/2
π
5π/4
3π/2
2π

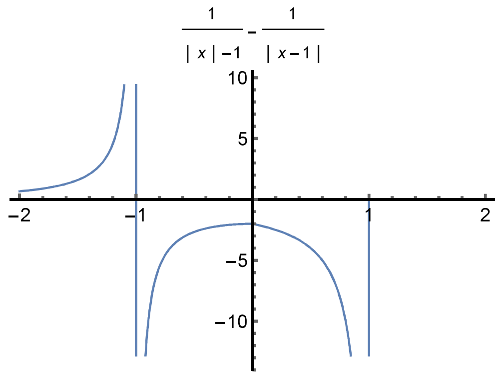


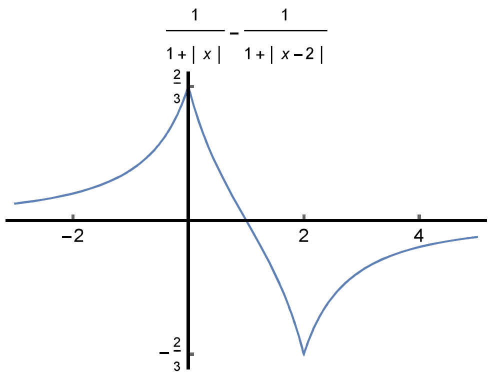


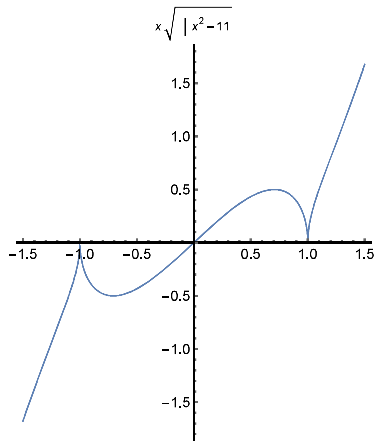
Some of the mappings appear twice, using different scales to observe some special behaviour.

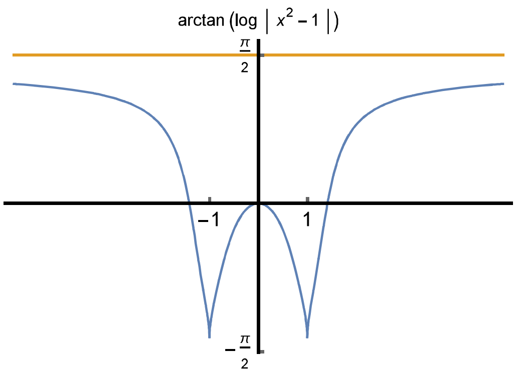
Problem 4.1.5

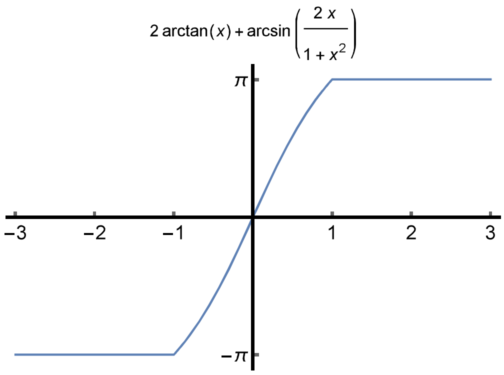






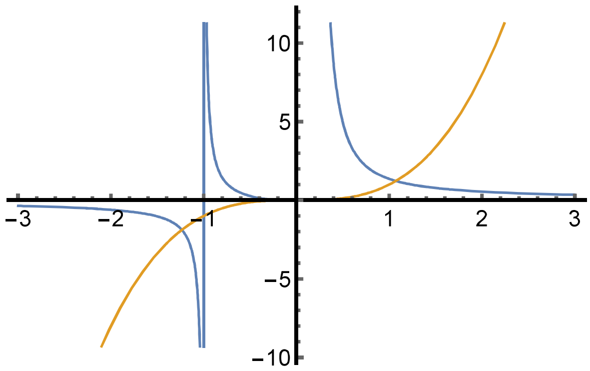




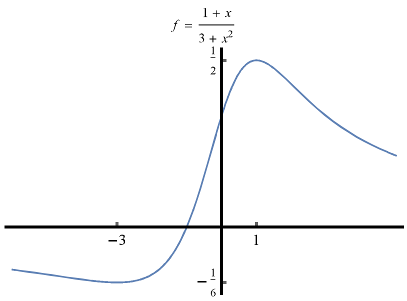


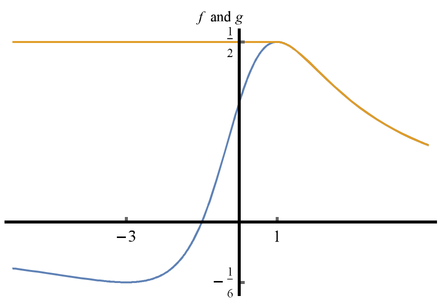
Problem 4.1.6

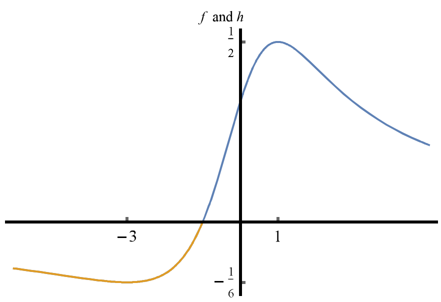
There are three solutions of the equation $\dfrac{e^{1/x}}{1+x} = x^3$, one is $x = 0$, another is in $(0, \infty)$ and the third is in $(-\infty, -1)$. It is enough to study the monotonicity of the function $g(x) = \dfrac{e^{1/x}}{1+x} - x^3$ in each one of the intervals.



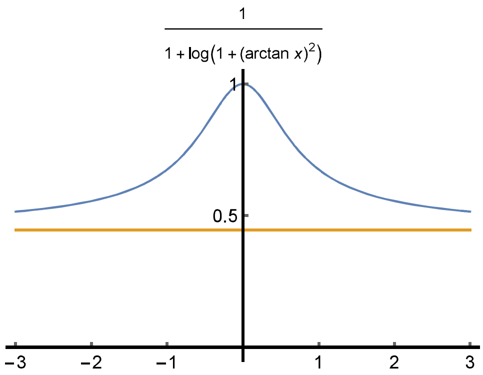
Problem 4.1.7





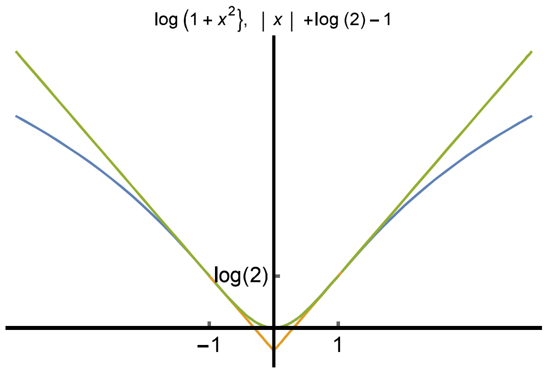


Problem 4.1.8



a) $Img(f) = [1, 1 + \pi^2/4)$.
b) $A > 0$ or $A \leq -\log(1 + \pi^2/4)$.
c) $\sup(g) = \max(g) = 1$, $\inf(g) = \dfrac{1}{1 + \log(1 + \pi^2/4)}$.

Problem 4.1.9



a) $x = \pm 1$, $y = \pm x + \log 2 - 1$. *b)* If the functions $f(x) = \log(1 + x^2)$ and $h(x) = |x| + \alpha$ intersect at $x = \pm A$ (by symmetry), then we have $f(A) = h(A)$, $f'(A) = h'(A)$, which implies $A = 1$, $\alpha = \log 2 - 1$.
c) $\dfrac{g(2) - g(-1)}{2 + 1} = \dfrac{1}{3} = g'(c) = \dfrac{2c}{1 + c^2}$ implies $c = 3 - 2\sqrt{2}$.

Problem 4.1.10 *a)* , $f'(x) = x^{p-1} - b = 0 \Longrightarrow x = b^{1/(p-1)}$, f' is positive for $x > b^{1/(p-1)}$ and negative for $0 < x < b^{1/(p-1)}$, so $x = b^{1/(p-1)}$ is an absolute minimum in $(0, \infty)$, so for any $0 < a$:

$$f(a) = \frac{a^p}{p} - ba \geq \frac{b^{p/(p-1)}}{p} - bb^{1/(p-1)} = -\frac{1}{q}b^q \Longrightarrow \frac{a^p}{p} + \frac{b^q}{q} \geq ab.$$

b) $\lambda = \frac{1}{p} \Longrightarrow 1 - \lambda = \frac{1}{q}$. By the concavity of $\log x$:

$$\log\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \geq \frac{1}{p}\log(a^p) + \frac{1}{q}\log(b^q) = \log(ab) \Longrightarrow \frac{a^p}{p} + \frac{b^q}{q} \geq ab.$$

4.2 Taylor polynomial

Problem 4.2.1 *i)* $e^x \sin x = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots\right)\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots\right)$, so $P_{5,0}(x) = x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5$; *ii)* $P_{5,0}(x) = 1 - 3x^2 + \frac{19}{6}x^4$; *iii)* $P_{5,0}(x) = x - \frac{13}{6}x^3 + \frac{121}{120}x^5$; *iv)* $P_{5,0}(x) = -x - \frac{3}{2}x^2 - \frac{4}{3}x^3 - x^4 - \frac{89}{120}x^5$; *v)* $P_{5,0}(x) = x^2 - \frac{1}{6}x^4$; *vi)* $P_{5,0}(x) = 1 + x^3$.

Problem 4.2.2 $P_{4,4}(x) = -56 + 21(x-4) + 37(x-4)^2 + 11(x-4)^3 + (x-4)^4$.

Problem 4.2.3 *a)* $P_{n,-1}(x) = -\sum_{k=0}^{n}(x+1)^k$; *b)* $P_{n,0}(x) = x\sum_{k=0}^{n-1}\frac{(-2x)^k}{k!} = \sum_{k=1}^{n}\frac{(-1)^{k-1}x^k}{(k-1)!}$;
c) $P_{n,0}(x) = 1 + 2\sum_{k=0}^{n}\frac{x^k}{k!} + \sum_{k=0}^{n}\frac{(2x)^k}{k!} = 4 + \sum_{k=1}^{n}\frac{2+2^k}{k!}x^k$.

Problem 4.2.4 *i)* $\lim_{x\to 0}\frac{\sin x}{x^\alpha} = \lim_{x\to 0}\frac{\sin x}{x}\,x^{1-\alpha} = 0$.
ii) $\lim_{x\to 0}\frac{\log(1+x^2)}{x} = 0$. *iii)* $\lim_{x\to\infty}\frac{\log x}{x} = 0$. *iv)* $\lim_{x\to 0}\frac{\operatorname{tg} x - \sin x}{x^2} = 0$.

Problem 4.2.5 *i)* $L = \lim_{x\to 0}\frac{1 + x + x^2/2 + o(x^2) - x + o(x^2) - 1}{x^2} = \frac{1}{2}$.
ii) $L = \lim_{x\to 0}\frac{x - x^3/6 + x^5/120 + o(x^5) - x + x^3/6}{x^5} = \frac{1}{120}$.
iii) $L = \lim_{x\to 0}\frac{1 + o(x) - 1 + x/2 + o(x)}{x + o(x^2)} = \frac{1}{2}$.
iv) $L = \lim_{x\to 0}\frac{x + x^3/3 + o(x^3) - x + x^3/6 + o(x^3)}{x^3} = \frac{1}{2}$.
v) $L = \lim_{x\to 0}\frac{x - x + x^3/6 + o(x^3)}{x(1 - 1 + 9x^2/2 + o(x^3))} = \frac{1}{27}$.
vi) $L = \lim_{x\to 0}\frac{1 - x^2/2 + o(x^3) + 1 + x + x^2/2 + x^3/6 - x - 2}{x^3} = \frac{1}{6}$.
vii) $L = \lim_{x\to 0}\frac{\sin x - x}{x\sin x} = \lim_{x\to 0}\frac{x + o(x^2) - x}{x(x + o(x^2))} = 0$.
viii) $L = \lim_{x\to 0}\frac{\sin x - x\cos x}{x^2\sin x} = \lim_{x\to 0}\frac{x - x^3/6 + o(x^3) - x + x^3/2 + o(x^3)}{x^3 + o(x^3)} = \frac{1}{3}$.
ix) $L = \lim_{t\to 0}\frac{\sqrt{1+t} + \sqrt{1-t} - 2}{t^2} =$
$= \lim_{t\to 0}\frac{1 + t/2 - t^2/8 + o(t^2) + 1 - t/2 - t^2/8 + o(t^2) - 2}{t^2} = -\frac{1}{4}$.
x) $L = \lim_{t\to 0}\left[\frac{1}{t} - \frac{1}{t^2}\left(t - \frac{t^2}{2} + o(t^2)\right)\right] = \frac{1}{2}$.

Problem 4.2.6 $P_{4,0}(x) = 1 + x^4$; so we have $f'(0) = f''(0) = f'''(0) = 0$, $f^{iv)}(0) > 0$, and then the origin is a minimum.

Problem 4.2.7

a) $\dfrac{1}{\sqrt{1.1}} = 1 - \dfrac{0.1}{2} + \dfrac{0.03}{8} - \dfrac{0.005}{16} + \varepsilon = 0.95343 + \varepsilon$, with $|\varepsilon| < \dfrac{35 \cdot (0.1)^4}{2^7} < 0.00003$.

b) $\sqrt[3]{28} = 3 + \dfrac{1}{27} - \dfrac{1}{2187} + \varepsilon = 3.03658 + \varepsilon$, with $|\varepsilon| < \dfrac{10}{177147 \cdot 3!} < 0.00001$.

Problem 4.2.8 *a)* $P_{3,0}(x) = 2 + x + \dfrac{x^3}{6}$. *b)* $|\varepsilon| < \dfrac{4 \cdot (1/4)^4}{4!} < 0.0007$.

Problem 4.2.9 $|\varepsilon| < \dfrac{3}{(n+1)!} < 10^{-3} \Rightarrow n \geq 6$.

– $\mathcal{ERC}$–

– $\mathcal{A}_\delta$P–

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