uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

# DIFFERENTIAL CALCULUS. Problems

**Degree in Applied Mathematics and Computation**

**Chapter 4**

Elena Romera
with the collaboration of Arturo de Pablo

Open Course Ware, UC3M

# 4 Local study of a function

## 4.1 Graphic representation

**Problem 4.1.1** Study the concavity and the convexity of the following functions:

$$i)\quad f(x) = (x-2)x^{2/3}, \qquad ii)\quad f(x) = x(x-2)^{3/2},$$
$$iii)\quad f(x) = |x|e^{|x|}, \qquad iv)\quad f(x) = \log(x^2 - 6x + 8).$$

**Problem 4.1.2**

*a*) Prove that if $f$ and $g$ are convex functions twice differentiable and $f$ is an increasing function, then $h = f \circ g$ is convex.

*b*) Prove the same but now without the differentiability hypothesis.

**Problem 4.1.3**

*a*) Sketch the graph of the function $f(x) = x + \log|x^2 - 1|$.

*b*) From it, draw the plot of the functions

$$i)\quad g(x) = |x| + \log|x^2 - 1|, \qquad ii)\quad h(x) = \Big|x + \log|x^2 - 1|\Big|.$$

**Problem 4.1.4** Represent graphically the following functions:

$$i)\quad y = e^x \sin x, \qquad ii)\quad y = \sqrt{x^2-1} - 1, \qquad iii)\quad y = xe^{1/x},$$
$$iv)\quad y = x^2 e^x, \qquad v)\quad y = (x-2)x^{2/3}, \qquad vi)\quad y = (x^2-1)\log\frac{1+x}{1-x},$$
$$vii)\quad y = \frac{x}{\log x}, \qquad viii)\quad y = \frac{x^2-1}{x^2+1}, \qquad ix)\quad y = \frac{e^{1/x}}{1-x},$$
$$x)\quad y = \log[(x-1)(x-2)], \qquad xi)\quad y = \frac{e^x}{x(x-1)}, \qquad xii)\quad y = 2\sin x + \cos 2x,$$
$$xiii)\quad y = \frac{x-2}{\sqrt{4x^2+1}}, \qquad xiv)\quad y = \sqrt{|x-4|}, \qquad xv)\quad y = \frac{1}{1+e^x},$$
$$xvi)\quad y = \frac{e^{2x}}{e^x - 1}, \qquad xvii)\quad y = e^{-x}\sin x, \qquad xviii)\quad y = x^2\sin(1/x).$$

**Problem 4.1.5** Represent the graph of the following functions:

$$i)\quad f(x) = \min\big\{\log|x^3-3|, \log|x+3|\big\}, \qquad ii)\quad g(x) = \frac{1}{|x|-1} - \frac{1}{|x-1|},$$
$$iii)\quad h(x) = \frac{1}{1+|x|} - \frac{1}{1+|x-a|}, \quad a > 0, \qquad iv)\quad k(x) = x\sqrt{|x^2-1|},$$
$$v)\quad p(x) = \mathrm{arctg}(\log(|x^2-1|)), \qquad vi)\quad w(x) = 2\,\mathrm{arctg}\, x + \arcsin\Big(\frac{2x}{1+x^2}\Big).$$

**Problem 4.1.6** Plot the graph of the function

$$f(x) = \frac{e^{1/x}}{1+x}, \qquad x \neq 0; \qquad f(0) = 0,$$

and study in a reasoned way how many solutions the equation $\dfrac{e^{1/x}}{1+x} = x^3$ has on $\mathbb{R}$.

**Problem 4.1.7** Given the function $f(x) = \dfrac{1+x}{3+x^2}$, represent the graph of the functions

$$i) \quad g(x) = \sup_{y>x} f(y), \qquad ii) \quad h(x) = \inf_{y>x} f(y).$$

**Problem 4.1.8**

*a*) Calculate the image of the function $f(x) = 1 + (\operatorname{arctg} x)^2$.

*b*) Calculate the values of $A \in \mathbb{R}$ such that the function

$$g(x) = \frac{1}{A + \log f(x)}$$

is continuous on $\mathbb{R}$.

*c*) Find the supremum and the infimum of $g$ if $A = 1$.

*d*) Sketch the graph of $g$ in this last case.

**Problem 4.1.9** We consider the function $f(x) = \log(1 + x^2)$.

*a*) Calculate the tangent lines at its inflection points and sketch the graph of $f$ and those lines.

*b*) Prove that the function $g(x) = \max\{\, f(x), \, |x| + \alpha \,\}$ verifies the hypotheses of the mean value theorem on any interval $[a, b] \subset \mathbb{R}$ if and only if $\alpha = \log 2 - 1$.

*c*) For the previous value of $\alpha$, obtain the point or points whose existence is guaranteed by the aforementioned theorem applied to the function $g$ on the interval $[-1, 2]$.

**Problem 4.1.10** Prove in two different ways the following inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \qquad a, b > 0, \quad p, q > 1, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

*a*) minimizing the function $f(x) = \dfrac{x^p}{p} - bx$;

*b*) using the concavity of the function $g(x) = \log x$, and applying the definition to $x = a^p$, $y = b^p$, $\lambda = \dfrac{1}{p}$.

## 4.2 Taylor polynomial

**Problem 4.2.1** Write the Taylor polynomial of order 5 at the origin of the following functions:

$$i)\quad e^x \sin x, \qquad ii)\quad e^{-x^2}\cos 2x, \qquad iii)\quad \sin x \cos 2x,$$

$$iv)\quad e^x \log(1-x), \qquad v)\quad (\sin x)^2, \qquad vi)\quad \frac{1}{1-x^3}.$$

**Problem 4.2.2** Write the polynomial $x^4 - 5x^3 + x^2 - 3x + 4$ in powers of $x - 4$.

**Problem 4.2.3** Obtain the Taylor polynomial of order $n$ at the given points of the following functions:

*a)* $f(x) = 1/x$ at $a = -1$;

*b)* $f(x) = xe^{-2x}$ at $a = 0$;

*c)* $f(x) = (1 + e^x)^2$ at $a = 0$.

**Problem 4.2.4** Prove the formulas:

$$i)\quad \sin x = o(x^\alpha), \quad \forall\, \alpha < 1, \qquad \text{when } x \to 0;$$

$$ii)\quad \log(1+x^2) = o(x), \qquad \text{when } x \to 0;$$

$$iii)\quad \log x = o(x), \qquad \text{when } x \to \infty;$$

$$iv)\quad \operatorname{tg} x - \sin x = o(x^2), \qquad \text{when } x \to 0.$$

**Problem 4.2.5** Find the following limits using Taylor's theorem:

$$i)\quad \lim_{x\to 0} \frac{e^x - \sin x - 1}{x^2}, \qquad ii)\quad \lim_{x\to 0} \frac{\sin x - x + x^3/6}{x^5},$$

$$iii)\quad \lim_{x\to 0} \frac{\cos x - \sqrt{1-x}}{\sin x}, \qquad iv)\quad \lim_{x\to 0} \frac{\operatorname{tg} x - \sin x}{x^3},$$

$$v)\quad \lim_{x\to 0} \frac{x - \sin x}{x(1-\cos 3x)}, \qquad vi)\quad \lim_{x\to 0} \frac{\cos x + e^x - x - 2}{x^3},$$

$$vii)\quad \lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right), \qquad viii)\quad \lim_{x\to 0} \frac{1}{x}\left(\frac{1}{x} - \cot x\right),$$

$$ix)\quad \lim_{x\to\infty} x^{3/2}(\sqrt{x+1} + \sqrt{x-1} - 2\sqrt{x}), \qquad x)\quad \lim_{x\to\infty} \left[x - x^2 \log(1 + 1/x)\right].$$

**Problem 4.2.6** Obtain the Taylor polynomial of order 4 at the origin for the function $f(x) = 1 + x^3 \sin x$ and decide if $f$ has at that point a maximum, a minimum or an inflection point.

**Problem 4.2.7**

*a*) Calculate approximately the value of $\dfrac{1}{\sqrt{1.1}}$ using a Taylor polynomial of degree 3. How much is the error?

*b*) Approximate $\sqrt[3]{28}$ using a Taylor polynomial of degree 2. Evaluate the error.

**Problem 4.2.8**

*a*) Approximate the function $f(x) = \cos x + e^x$ using a Taylor polynomial of third order at the origin.

*b*) Estimate the error when we use the previous approximation for $x \in [-1/4, 1/4]$.

**Problem 4.2.9** How many terms are necessary in the Taylor expansion at the origin of the function $f(x) = e^x$ in order to obtain a polynomial which approximates it on $[-1, 1]$ with three exact figures?

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