uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL CALCULUS. Solutions

Degree in Applied Mathematics and Computation

Chapter 5

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5 Sequences and series of real numbers

5.1 Sequences of numbers

Problem 5.1.1 *a*) The product sequence can be anything; for example with $x_n = 1/n$, $y_n = n$, $x_n y_n = 1$ converges; if $x_n = 1/n$ and $y_n = n^2$, $x_n y_n = n$ diverges. The sum sequence is divergent because in other case the difference $(x_n + y_n) - x_n$ would be convergent. The quotient sequence is divergent because in other case the product $(y_n/x_n) \cdot x_n$ would be convergent.
b) $||x_n| - |\ell|| \le |x_n - \ell|$, by the problem 1.1.1 *c*). The reciprocal is false, for example $x_n = (-1)^n$.
c) Take $\varepsilon = 1/2$, the only way to obtain $|x_n - \ell| < 1/2$ with $x_n \in \mathbb{Z}$ is that $x_n = \ell \in \mathbb{Z}$ from some n onwards.
d) If $|x_n - \ell| < \varepsilon$ for $n \ge N$, then

$$|x_n| \le \max\{|\ell + \varepsilon|, |\ell - \varepsilon|, |x_1|, |x_2|, \dots, |x_N|\}.$$

Problem 5.1.2 *a*) $\alpha_{n+1} = C(a_n + a_{n-1}) + D(b_n + b_{n-1}) = \alpha_n + \alpha_{n+1}$. *b*) $r^{n+1} = r^n + r^{n-1} \Longrightarrow r^2 - r - 1 = 0 \Longrightarrow r_1 = \dfrac{1+\sqrt{5}}{2}$, $\quad r_2 = \dfrac{1-\sqrt{5}}{2}$. The value of r_1 is known as the *golden ratio.*
c) We use the two previous parts and look for a sequence of the form: $\alpha_n = Cr_1^n + Dr_2^n$, with the values $\alpha_0 = 0$ and $\alpha_1 = 1$. We obtain:

$$\alpha_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Problem 5.1.3 We try sequences of the form $u_n = r^n$ and obtain $2r^n - r - 1 = 0$, so $r_1 = 2$ and $r_2 = -1$, hence $u_n = C2^n + D(-1)^n$. Now, $u_0 = a$ and $u_1 = b$ imply that $C = \frac{a+b}{3}$ and $D = \frac{2a-b}{3}$. The sequence is then

$$u_n = \frac{a+b}{3}2^n + \frac{2a-b}{3}(-1)^n \quad \Longrightarrow \quad \lim_{n\to\infty} u_n = \begin{cases} -\infty, & \text{if } \ a+b<0, \\ \text{does not exist} & \text{if } \ a+b=0, \\ \infty, & \text{if } \ a+b>0. \end{cases}$$

Problem 5.1.4 *i*) $a_n = 1 - 2^{-n}$; $\lim\limits_{n\to\infty} a_n = 1$.
ii) $b_n = 2^{a_n}$, with a_n from the previous part; $\lim\limits_{n\to\infty} b_n = 2$.

Problem 5.1.5 *i*) Compute for $n < m$:

$$a_n - a_m = \frac{2n+3}{3n+5} - \frac{2m+3}{3m+5} = \frac{(2n+3)(3m+5) - (2m+3)(3n+5)}{(3n+5)(3m+5)} = \frac{n-m}{9nm + 15n + 15m + 25}$$

and this tends to zero as $n, m \to \infty$, so it is a Cauchy sequence. *ii*) We consider now the specific case of $n = 2^k$ and $m = 2^{k+1}$ for $k \in \mathbb{N}$:

$$\sum_{j=2^k}^{j=2^{k+1}} \frac{1}{j} = \frac{1}{2^k} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}} > 2^k \cdot \frac{1}{2^{k+1}} = \frac{1}{2}$$

and this does not tend to zero as $n = 2^k$ grows to infinity, so this is not a Cauchy sequence.

Problem 5.1.6 *i)* $L = \exp(\lim_{n\to\infty} \frac{\log a}{n}) = 1$; *ii)* $L = \exp(\lim_{n\to\infty} \frac{-3\log n}{n}) = 1$; *iii)* If $a \geq b$, $L = \lim_{n\to\infty} a\sqrt[n]{1+(b/a)^n} = a$, so $L = \max\{a, b\}$; *iv)* $L = \lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{1/x} = \sqrt{ab}$ by the problem 3.1.22 *iv)*; *v)* multiplying by the conjugate $L = 1/2$; *vi)* multiplying twice by the conjugate $L = 0$; *vii)* divide by 3^n, $L = 3$; *viii)* $L = \exp[\lim_{n\to\infty} \frac{(n^2-1)(3n+1)}{2n(n^2-3n)}] = \mathrm{e}^{3/2}$;

Problem 5.1.7 *i)* $L = \lim_{n\to\infty} 0 = 0$; *ii)* $L = \lim_{n\to\infty} \mathrm{e}^{\sin 1/n}\frac{\mathrm{e}^{1/n-\sin 1/n}}{1/n - \sin 1/n} = 1$; *iii)* use Stolz $L = \lim_{n\to\infty} \frac{1/n}{\log(n/(n-1))} = 1$; *iv)* use Stirling $L = \lim_{n\to\infty} \frac{n}{(n/\mathrm{e})(2\pi n)^{1/2n}} = \mathrm{e}$; (it can be solved also using Stolz, but it is longer); *v)* by comparison $0 \leq L \leq \lim_{n\to\infty} \frac{2}{n} = 0$ (or use Stirling); *vi)* use Stolz twice $L = \lim_{n\to\infty} \frac{2}{2^n} = 0$; *vii)* $L = \lim_{n\to\infty} \frac{1}{n}\left(\frac{n}{n-1}\right)^n = 0$; *viii)* use Stolz $L = \lim_{n\to\infty} \frac{n^{1+1/n}}{n^2-(n-1)^2} = 1/2$.

Problem 5.1.8 *i)* $L = \exp(\lim_{x\to 0} \frac{\cos bx + a\sin bx - 1}{x}) = \mathrm{e}^{ab}$; *ii)* $L = \exp(\lim_{x\to 0} \frac{(a-bx)/(a+x)-1}{x}) = \mathrm{e}^{-(b+1)/a}$.

Problem 5.1.9 *i)* $L = \lim_{n\to\infty} \frac{\sin(\pi/n)}{\log(n/(n-1))} = \pi$. *ii)* $L = \exp(\lim_{n\to\infty} \frac{\sum_{k=1}^{\infty}\log(2k-1)}{n^2}) = \exp(\lim_{n\to\infty} \frac{\log(2n-1)}{n^2-(n-1)^2}) = 1$. *iii)* $L = \lim_{n\to\infty} \frac{n^2\sin(1/n)}{n^2-(n-1)^2} = \frac{1}{2}$.

Problem 5.1.10 $L = \lim_{n\to\infty} \frac{a_n/n}{\log((n+1)n)} = \mathrm{e}$.

Problem 5.1.11 *a)* $\lim_{n\to\infty} \frac{a_n - n - L}{n} = 0 \Rightarrow \lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \frac{n+L}{n} = 1$; *b)* $\lim_{n\to\infty} n\log\left(\frac{a_n}{n}\right) = \lim_{n\to\infty} n\left(\frac{a_n}{n} - 1\right) = L$.

Problem 5.1.12 $L = \exp\left(\lim_{n\to\infty} \frac{n\log a_n - \sum_{k=1}^{n}\log a_k}{n^2}\right) = \mathrm{e}^a$; and now we obtain a using Stolz,

$$a = \lim_{n\to\infty} \frac{n\log a_n - (n-1)\log a_{n-1} - \log a_n}{n^2-(n-1)^2} = \lim_{n\to\infty} \frac{(n-1)\log(a_n/a_{n-1})}{2n-1} = \frac{\log \ell}{2}.$$

So $L = \sqrt{\ell}$.

Problem 5.1.13 *i)* $a_{n+1} = f(a_n)$, where $f(x) = \sqrt{2x}$. It is monotonous since f is increasing; it is increasing monotonous since $a_2 = \sqrt{2\sqrt{2}} > \sqrt{2} = a_1$. It is bounded from above, $a_n < 2$, this is proved by induction: $a_1 < 2$ and $a_n < 2 \Rightarrow a_{n+1} < \sqrt{4} = 2$. Then, there exists $\ell = \lim_{n\to\infty} a_n$, that verifies $\ell = \sqrt{2\ell}$, which implies $\ell = 2$, since $\ell \geq a_1 = \sqrt{2} > 0$. In fact, by problem 5.1.4 *ii)*, the sequence is explicit, $a_n = 2^{1-2^{-n}} \to 2$.
ii) Now $f(x) = \sqrt{2+x}$, that is also increasing, besides $a_2 = \sqrt{2+\sqrt{2}} > \sqrt{2} = a_1$, so the sequence is increasing monotonous; on the other hand, we have $a_n < 2$, so there exists $\ell = \lim_{n\to\infty} a_n$, that satisfies $\ell = \sqrt{2+\ell}$, this means $\ell = 2$, because $\ell \geq \sqrt{2} > -1$.
iii) $f(x) = 3 + x/2$ is increasing, $u_1 = 3 > 0 = u_0$ and $u_n < 6$: then there exists $\ell = \lim_{n\to\infty} u_n$, that satisfies $\ell = 3 + \frac{\ell}{2}$, this implies $\ell = 6$. This sequence is also explicit (it is a geometric progression) $u_n = 6(1 - 2^{-n}) \to 6$.
iv) $f(x) = 3 + 2x$ is increasing, $u_1 = 3 > 0 = u_0$, but the sequence is a non-bounded monotonic one, so $\lim_{n\to\infty} u_n = \infty$. Again it is explicit: $u_n = 3(2^n - 1) \to \infty$.
v) $f(x) = \frac{x^3+6}{7}$ is increasing, so the sequence is monotonous. It is increasing or decreasing monotonic according with the first two terms. *a)* $u_0 = 1/2$, $u_1 = \frac{49}{56} > u_0$, increasing, since $u_n < 1$ the limit exists $\ell = \lim_{n\to\infty} u_n$, that verifies $\ell = \frac{\ell^3+6}{7}$, this means $\ell = 1$, 2 or -3; finally $\ell = 1$ since $1/2 \leq \ell \leq 1$.
b) $u_0 = 3/2$, $u_1 = \frac{75}{56} < u_0$, decreasing; since $u_n > 1$ the limit exists $\ell = \lim_{n\to\infty} u_n = 1$, since $1 \leq \ell \leq 3/2$.
c) $u_0 = 3$, $u_1 = \frac{33}{7} > 3$, increasing and non-bounded sequence, so $\lim_{n\to\infty} u_n = \infty$.

Problem 5.1.14 *a)* $a_{n+1} = f(a_n)$ with $f(x) = \sqrt{1+3x} - 1$, $f'(x) = \frac{3}{2\sqrt{1+3x}} > 0$. We have $a_n < 1$ and $a_{n+1} > a_n$. Then there exists $\ell = \lim_{n\to\infty} a_n$, that verifies $\ell = \sqrt{1+3\ell} - 1$, thus $\ell = 1$, since $1/2 \leq \ell \leq 1$.

$$b) \qquad \lim_{n\to\infty} \frac{\sqrt{1+3a_n} - 2}{a_n - 1} = \lim_{x\to 1} \frac{\sqrt{1+3x} - 2}{x - 1} = \frac{3}{4}.$$

This limit can also be obtained multiplying by the conjugate:

$$\lim_{n\to\infty} \frac{3a_n - 3}{(a_n - 1)(\sqrt{1+3a_n} + 2)} = \lim_{n\to\infty} \frac{3}{\sqrt{1+3a_n} + 2} = \frac{3}{4}.$$

Problem 5.1.15 *a)* $b_{n+1} - b_n = -\frac{1}{2}(b_n - b_{n-1})$ (in fact $f(x) = 1 - x/2$ is decreasing).
b) $\ell = 1 - \frac{\ell}{2} \Rightarrow \ell = \frac{2}{3}$.
c) $|b_{n+1} - \frac{2}{3}| = |\frac{1}{3} - \frac{b_n}{2}| = \frac{1}{2}|b_n - \frac{2}{3}|$.

d) $|b_n - \frac{2}{3}| = \frac{1}{2}|b_{n-1} - \frac{2}{3}| = \frac{1}{4}|b_{n-2} - \frac{2}{3}| = \cdots = \frac{1}{2^n}(b_0 - \frac{2}{3}) \to 0.$

This is also an explicit sequence, $b_n = \frac{2}{3}\left[1 - \left(\frac{-1}{2}\right)^n\right] \to \frac{2}{3}$.

Problem 5.1.16 *a)* $\ell = \frac{1}{1+\ell} \Rightarrow \ell = \frac{-1+\sqrt{5}}{2}$, since directly $\ell \geq 0$. *b)* $x \in [1/2, 1] \Rightarrow \frac{1}{1+x} \in [1/2, 2/3] \subset [1/2, 1]$. *c)* $|f'(x)| = \frac{1}{(1+x)^2} \leq \frac{4}{9} < 1$ if $x \geq \frac{1}{2}$. *d)* Use induction starting with $c_1 = 1$ and the previous part. *e)*

$$|c_{n+1} - l| = \left|\frac{1}{1+c_n} - l\right| = \left|\frac{l(l-c_n)}{1+c_n}\right| < l|c_n - l| < \cdots < l^n|c_1 - l|.$$

Problem 5.1.17

a)

- $\ell = 2 + \frac{4}{\ell} \Rightarrow \ell = 1 + \sqrt{5}$ (since $\ell > 0$);
- $x \in [3, \frac{10}{3}] \Rightarrow 2 + \frac{4}{x} \in [\frac{16}{5}, \frac{10}{3}] \subset [3, \frac{10}{3}]$;
- $d_4 = \frac{34}{11} \in [3, \frac{10}{3}] \Rightarrow d_n \in [3, \frac{10}{3}] \quad \forall n \geq 4$;
- $|f'(x)| = \frac{4}{x^2} \leq \frac{4}{9} < 1$ if $x \geq 3$.

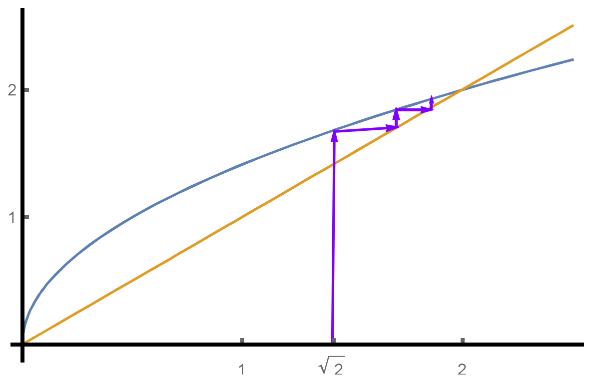
b)

$$\lim_{n\to\infty} \frac{2 + 4/d_n - \ell}{d_n - \ell} = \lim_{x\to\ell} \frac{2 + 4/x - \ell}{x - \ell} = -\frac{4}{\ell^2}.$$

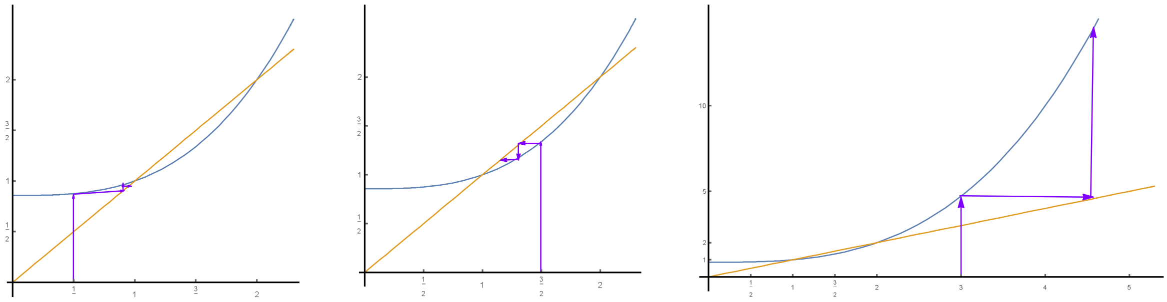
Problem 5.1.18 $x_n = f(x_{n-1})$ with $f(t) = \frac{t(1+t)}{1+2t}$, that is an increasing function. Since $x_2 = \frac{2}{3} < 1 = x_1$ it is a decreasing monotonous sequence. Besides $x_n > 0$, so there exists $\lim_{n\to\infty} x_n = 0$.

Problem 5.1.19 For example:

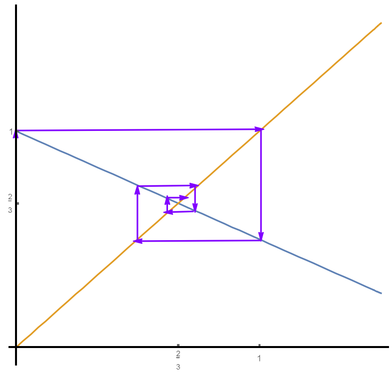
5.1.13 *i)* $u_{n+1} = \sqrt{2u_n}, \quad u_0 = 1.$



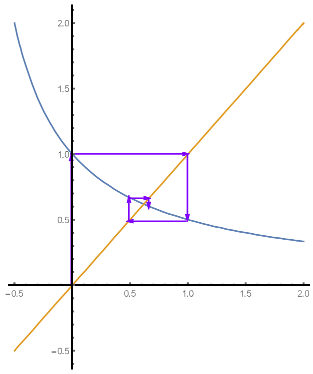
5.1.13: *v*) $u_{n+1} = \dfrac{u_n^3 + 6}{7}$, *a*) $x_0 = 1/2$, *b*) $u_0 = 3/2$, *c*) $u_0 = 3$.



5.1.15: $b_{n+1} = 1 - \dfrac{b_n}{2}$, $b_0 = 0$.



5.1.16: $c_{n+1} = \dfrac{1}{1 + c_n}$, $c_0 = 0$.



Problem 5.1.20 *a*) Observe that $y_n \geq 0$ $\forall n \in \mathbb{N}$. Consider now $K > 0$ an upper bound for $\{x_n\}$, then

$$0 \leq y_n \leq \frac{(nK)^\alpha}{n} = K^\alpha n^{\alpha - 1},$$

that tends to zero as $n \to \infty$ since $-1 < \alpha - 1 < 0$. Using the pinching lemma we obtain that $\lim\limits_{n\to\infty} y_n = 0$. *b*) For $\alpha = 1$ we use Stolz:

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{x_{n+1}}{n + 1 - n} = l.$$

An example of what they ask is $x_n = (-1)^n$, that is not convergent. The corresponding sequence y_n is $-1/n$ for n odd and zero for n even, so $\lim\limits_{n\to\infty} y_n = 0$.

Problem 5.1.21 *a*) By definition, $y_n \geq y_{n+1}$, so it is decreasing, and it is bounded because $\{x_n\}$ is bounded, so $\{y_n\}$ is convergent *b*) *i*) 1. *ii*) 3.

5.2 Series of numbers

Problem 5.2.1 Notation: C = convergent and D = divergent.
i) $\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{2} < 1 \Rightarrow C$.
ii) $\lim_{n\to\infty} \frac{a_n}{1/n^2} = \frac{1}{9} \Rightarrow C$. *iii)* $\lim_{n\to\infty} \frac{a_n}{1/n^2} = \frac{1}{\sqrt{2}} \Rightarrow C$. *iv)* $\lim_{n\to\infty} \frac{a_n}{1/n} = 1 \Rightarrow D$.
v) $a_n \leq \frac{1}{n^2} \Rightarrow C$. *vi)* $\lim_{n\to\infty} \frac{a_n}{1/n^2} = 1 \Rightarrow C$. *vii)* $\lim_{n\to\infty} \frac{a_n}{1/\sqrt{n}} = 1 \Rightarrow D$.
viii) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{\sqrt{2}} < 1 \Rightarrow C$. *ix)* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\mathrm{e}}{3} < 1 \Rightarrow C$. *x)* $\lim_{n\to\infty} \sqrt[n]{a_n} = 0 < 1 \Rightarrow C$.
xi) $\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{\mathrm{e}}{3} < 1 \Rightarrow C$. *xii)* $\lim_{n\to\infty} a_n = \mathrm{e}^{-1/2} \neq 0 \Rightarrow D$. *xiii)* $\lim_{n\to\infty} \sqrt[n]{a_n} = 0 < 1 \Rightarrow C$.
xiv) $\lim_{n\to\infty} \sqrt[n]{a_n} = 0 < 1 \Rightarrow C$. *xv)* $\lim_{n\to\infty} \frac{a_n}{1/n} = \frac{1}{2} \Rightarrow D$. *xvi)* $\lim_{n\to\infty} \frac{a_n}{1/n} = 1 \Rightarrow D$.
xvii) $a_n \leq \frac{1}{n^2}$ si $n > \mathrm{e}^2 \Rightarrow C$. *xviii)* $(\log n)^{\log n} = n^{\log(\log n)} \geq n^2$ si $n > \mathrm{e}^{\mathrm{e}^2} \Rightarrow C$.

Problem 5.2.2 $a_n = \frac{2(a-b)n + a + b}{4n^2 - 1}$. Then $\lim_{n\to\infty} \frac{a_n}{1/n^2} = \frac{a}{2}$ if $a = b$ and the series converges, while $\lim_{n\to\infty} \frac{a_n}{1/n} = \frac{a-b}{2}$ if $a \neq b$ and the series diverges.

Problem 5.2.3 *a)* $\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1+a}{\mathrm{e}^a} < 1$ for all $a > -1$, $a \neq 0 \Rightarrow C$; if $a = 0$ the series is $\sum_{n=1}^{\infty} n = \infty$. *b)* Use the Stirling formula, $\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{\mathrm{e}}{a}$, so $a > \mathrm{e} \Rightarrow C$, $a < \mathrm{e} \Rightarrow D$; if $a = \mathrm{e}$, then $a_n \approx \frac{1}{\sqrt{2\pi n}} \Rightarrow D$. *c)* Use again, $a_n \approx \frac{\sqrt{2\pi n}}{n^a}$, so $C \Leftrightarrow a > 3/2$.

Problem 5.2.4 Notation: AC = absolutely convergent, CC = conditionally convergent.
i) $\lim_{n\to\infty} |a_n| = 0$, but $\lim_{n\to\infty} \frac{|a_n|}{1/n} = \infty \Rightarrow CC$; *ii)* $a_n = \frac{(-1)^n}{n} + o\Big(\frac{1}{n}\Big) \Rightarrow CC$;
iii) $|a_n| = \frac{1}{n^2} + o\Big(\frac{1}{n^2}\Big) \Rightarrow AC$; *iv)* $\lim_{n\to\infty} |a_n| = \frac{\pi^2}{4} \Rightarrow D$; *v)* $a_n = \frac{(-1)^n}{2n} + o\Big(\frac{1}{n}\Big) \Rightarrow CC$;
vi) $a_n = \frac{(-1)^{n+1}}{n} + o\Big(\frac{1}{n}\Big) \Rightarrow CC$; *vii)* $|a_n| = \frac{1}{2n^2} + o\Big(\frac{1}{n^2}\Big) \Rightarrow AC$;
viii) $a_n = \frac{(-1)^n}{n} + o\Big(\frac{1}{n}\Big) \Rightarrow CC$.

Problem 5.2.5 $\operatorname{arctg} x = x - \frac{x^3}{3} + o(x^3)$ for $x \to 0$, so we have for $n \to \infty$
$|a_n| = \frac{1}{3n^{3/2}} + o(\frac{1}{n^{3/2}}) \Rightarrow AC$.

Problem 5.2.6 *i)* $|\varepsilon| \leq \dfrac{1}{(N+1)!} < \dfrac{1}{10^3}$ if $N > 6$; *ii)* $|\varepsilon| \leq \displaystyle\int_N^\infty \frac{dx}{x^4} = \frac{1}{3N^3} < \frac{1}{10^3}$ if $N > 7$.

Problem 5.2.7 *i)* $S = 3\displaystyle\sum_{n=0}^\infty \left(\frac{3}{4}\right)^n - \frac{1}{2^3}\sum_{n=0}^\infty \left(\frac{1}{2}\right)^n = 3 \cdot \frac{1}{1-3/4} - \frac{1}{8} \cdot \frac{1}{1-1/2} = \frac{47}{4}$;

ii) $S = \dfrac{1/2}{(1-1/2)^2} = 2$;

iii) $S = 4 \cdot \dfrac{1/3}{(1-1/3)^2} + \dfrac{1}{1-1/3} = \dfrac{9}{2}$;

iv) $S = \displaystyle\sum_{n=1}^\infty \left[\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right] = 1 - \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 1$;

v) $S = \displaystyle\sum_{n=1}^\infty \left[\log\frac{n+2}{n+1} - \log\frac{n+1}{n}\right] = -\log 2 + \lim_{n\to\infty} \log\frac{n+2}{n+1} = -\log 2.$

Problem 5.2.8 *i)* For $n = 2k$, $k \in \mathbb{N}$: $a^{[n/2]}b^{[(n+1)/2]} = a^k b^k = (ab)^k$. If now $n = 2k+1$, $k \in \mathbb{N}$: $a^{[n/2]}b^{[(n+1)/2]} = a^k b^{k+1} = b(ab)^k$. Hence we have two geometric series, the original series converges absolutely, so we can reorder to sum:

$$\sum_{n=0}^\infty a^{[n/2]}b^{[(n+1)/2]} = \sum_{k=0}^\infty (ab)^k + b\sum_{k=0}^\infty (ab)^k = \frac{1+b}{1-ab}\,.$$

ii) For $n = 3k$, $k \in \mathbb{N}$: $\cos\frac{2\pi n}{3} = \cos(2\pi k) = 1$. If $n = 3k+1$, $k \in \mathbb{N}$: $\cos\frac{2\pi n}{3} = \cos(2\pi k + \frac{2\pi}{3}) = \cos\frac{2\pi}{3} = -\frac{1}{2}$. Finally, if $n = 3k+2$, $k \in \mathbb{N}$: $\cos\frac{2\pi n}{3} = \cos(2\pi k + \frac{4\pi}{3}) = \cos\frac{4\pi}{3} = -\frac{1}{2}$. Since our series converges absolutely we can change the order in the sum:

$$\sum_{n=1}^\infty \frac{1}{2^n}\cos\frac{2\pi n}{3} = \sum_{k=1}^\infty \frac{1}{2^{3k}} - \frac{1}{2}\sum_{k=0}^\infty \frac{1}{2^{3k+1}} - \frac{1}{2}\sum_{k=0}^\infty \frac{1}{2^{3k+2}}$$

$$= \left(1 - \frac{1}{4} - \frac{1}{8}\right)\sum_{k=0}^\infty \frac{1}{2^{3k}} - 1 = \frac{5}{8}\frac{1}{1-1/8} - 1 = -\frac{2}{7}.$$

Problem 5.2.9 *a)* We compute the limit:

$$\lim_{n\to\infty} \frac{\log(1+a_n)}{a_n} = \lim_{n\to\infty} \log\left(1+a_n\right)^{1/a_n} = \lim_{n\to\infty} \frac{1+a_n-1}{a_n} = 1.$$

Therefore, both series have the same character. *b)* For every $n \in \mathbb{N}$ we have that $\sqrt{a_n b_n} \leq a_n + b_n$, so:

$$\sum_n \sqrt{a_n b_n} \leq \sum_n a_n + \sum_n b_n,$$

since both series are convergent, the new series is convergent. *c)* As in the previous part: $\displaystyle\sum_n \sqrt{a_n a_{n+1}} \leq 2\sum_n a_n$. so it is convergent. For the second series: the general terms a_n and $1/n^2$ give convergent series, so our series converges.

Problem 5.2.10 *a*) $b_0 \le S \le b_0 + \sum_{n=1}^{\infty} 9 \cdot 10^{-n} \le b_0 + 9 \cdot \frac{1/10}{1-1/10} = b_0 + 1$. It represents the decimal development of a real number with integer part b_0 (or b_0+1 in some particular cases, as in the following example).

$$b) \qquad i) \quad 9.999999\cdots = 9\sum_{n=0}^{\infty} 10^{-n} = 10;$$
$$ii) \quad 1.212121\cdots = \sum_{k=0}^{\infty} 10^{-2k} + 2\sum_{k=0}^{\infty} 10^{-(2k+1)} = \frac{40}{33}.$$

Problem 5.2.11 *a*) Consider $f(x) = \operatorname{tg} x - x$. We have $f'(x) = \operatorname{tg}^2 x \ge 0$, and also

$$\lim_{n\to[(2n-1)\pi/2]^+} f(x) = -\infty, \qquad \lim_{n\to[(2n+1)\pi/2]^-} f(x) = +\infty,$$

for every $n \in \mathbb{N}$; so there is one and only one root in that interval.
b) Since $\lambda_n \in ((2n-1)\pi/2, (2n+1)\pi/2)$, then $\lim_{n\to\infty} \frac{1/\lambda_n^2}{1/n^2} = \frac{1}{\pi^2}$, and the series converges.

Problem 5.2.12 *a*) It is a decreasing monotonous sequence of positive terms, so it converges; the limit verifies $\ell = \sqrt{1+2\ell} - 1$, that is $\ell = 0$.

b) *i*) $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{t\to 0} \frac{\sqrt{1+2t}-1}{t} = 1.$

ii) use Stolz, $\lim_{n\to\infty} \frac{n}{1/x_n} = \lim_{n\to\infty} \frac{x_n x_{n+1}}{x_n - x_{n+1}} = \lim_{t\to 0} \frac{t(\sqrt{1+2t}-1)}{t-\sqrt{1+2t}+1} = 2.$

c) *i*) diverges and *ii*) converges, since by the previous part $x_n = \frac{2}{n} + o\left(\frac{1}{n}\right)$. (Observe that the quotient (ratio) criterion does not work here).

– $\mathcal{ERC}$–
– $\mathcal{A}_\delta$P–