uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL CALCULUS. Problems

Degree in Applied Mathematics and Computation

Chapter 5

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5 Sequences and series of real numbers

5.1 Sequences of numbers

Problem 5.1.1

a) Let $\{x_n\}$ be a convergent sequence and $\{y_n\}$ a divergent one, what can we say about the product sequence $\{x_n y_n\}$, sum sequence $\{x_n + y_n\}$ and quotient sequence $\{y_n/x_n\}$ (supposing that $x_n \neq 0$ for all $n \in \mathbb{N}$)?

b) Prove that if $\{x_n\}$ is convergent, then the sequence $\{|x_n|\}$ is also convergent. Is the reciprocal true?

c) What can we say about a sequence of integer numbers that is convergent?

d) Show that every convergent sequence is bounded.

Problem 5.1.2 Consider a sequence α_n that verifies the recurrence relation

$$\alpha_{n+1} = \alpha_n + \alpha_{n-1}, \qquad n \geq 1.$$

a) Prove that if both a_n and b_n verify this relation, then $\alpha_n = Ca_n + Db_n$ also satisfies the relation for all $C, D \in \mathbb{R}$.

b) Look for solutions in the form $\alpha_n = r^n$, $r \in \mathbb{R}$.

c) Find a sequence with the following two first terms: $\alpha_0 = 0$, $\alpha_1 = 1$. (This is the famous *Fibonacci sequence*).

Problem 5.1.3 Obtain the limit (if it exists) of the sequence defined by the following recurrence relation:

$$u_n = \frac{u_{n-1} + u_{n-2}}{2}, \qquad u_0 = a, \quad u_1 = b.$$

(*Hint:* use the technique of the previous problem.)

Problem 5.1.4 Find the general term of the following sequences defined by recurrence and obtain the limit if it exists.

$$i)\ a_0 = 0, \quad a_{n+1} = \frac{a_n + 1}{2}; \qquad ii)\ b_0 = 1, \quad b_{n+1} = \sqrt{2b_n}\,.$$

Problem 5.1.5 Consider the two sequences:

$$i)\ a_n = \frac{2n+3}{3n+5}, \qquad ii)\ b_n = \sum_{j=1}^{n} \frac{1}{j},$$

prove that the first one is a Cauchy sequence and the second is not, using the definition.

Problem 5.1.6 Compute the following limits:

$$i) \quad \lim_{n\to\infty} \sqrt[n]{a}, \quad (a>0), \qquad ii) \quad \lim_{n\to\infty} n^{-3/n},$$

$$iii) \quad \lim_{n\to\infty} \sqrt[n]{a^n+b^n}, \quad (a,b>0), \qquad iv) \quad \lim_{n\to\infty} \left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n, \quad (a,b>0),$$

$$v) \quad \lim_{n\to\infty} n\left(\sqrt{n^2+1}-n\right), \qquad vi) \quad \lim_{n\to\infty} (\sqrt[4]{n^2+1}-\sqrt{n+1}),$$

$$vii) \quad \lim_{n\to\infty} \frac{2^{n+1}+3^{n+1}}{2^n+3^n}, \qquad viii) \quad \lim_{n\to\infty} \left(\frac{n^2+1}{n^2-3n}\right)^{\frac{n^2-1}{2n}}.$$

Problem 5.1.7 Calculate the following limits:

$$i) \quad \lim_{n\to\infty} \frac{n}{\pi}\sin n\pi, \qquad ii) \quad \lim_{n\to\infty} \frac{n(\mathrm{e}^{1/n}-\mathrm{e}^{\sin 1/n})}{1-n\sin 1/n},$$

$$iii) \quad \lim_{n\to\infty} \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{\log n}, \qquad iv) \quad \lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}},$$

$$v) \quad \lim_{n\to\infty} \frac{2^n}{n!}, \qquad vi) \quad \lim_{n\to\infty} \frac{n^2}{2^n},$$

$$vii) \quad \lim_{n\to\infty} \frac{n^{n-1}}{(n-1)^n}, \qquad viii) \quad \lim_{n\to\infty} \frac{1+2\sqrt{2}+3\sqrt[3]{3}+\cdots+n\sqrt[n]{n}}{n^2}.$$

Problem 5.1.8 Obtain the following limits:

$$i) \lim_{n\to\infty} \left(\cos\frac{b}{n}+a\sin\frac{b}{n}\right)^n; \qquad ii) \lim_{n\to\infty} \sqrt[u_n]{\frac{a-bu_n}{a+u_n}}, \quad \text{if} \quad \lim_{n\to\infty} u_n = 0,\ a>0.$$

Problem 5.1.9 Find the limits:

$$i) \lim_{n\to\infty} \frac{\displaystyle\sum_{k=1}^n \sin\frac{\pi}{k}}{\log n}, \qquad ii) \lim_{n\to\infty} \prod_{k=1}^n (2k-1)^{1/n^2}, \qquad iii) \lim_{n\to\infty} \sum_{k=1}^n \frac{k^2}{n^2}\sin\frac{1}{k}.$$

Problem 5.1.10 If $\lim_{n\to\infty} a_n = \ell$, find

$$\lim_{n\to\infty} \frac{a_1+\dfrac{a_2}{2}+\cdots+\dfrac{a_n}{n}}{\log(n+1)}.$$

Problem 5.1.11 Let $\{a_n\}$ be sequence of positive terms that satisfies $\lim_{n\to\infty}(a_n-n)=L$.

$a)$ Show that $\lim_{n\to\infty} \dfrac{a_n}{n} = 1$.

$b)$ Show that $\lim_{n\to\infty} n\log(a_n/n) = L$.

Problem 5.1.12 Consider a sequence of positive numbers, $\{a_n\}$, that verifies $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell$. Compute, using the Stolz criterion, the limit:

$$\lim_{n\to\infty} \sqrt[n^2]{\frac{a_n^n}{a_1 \cdot a_2 \cdots a_n}}.$$

Problem 5.1.13 Prove that the following sequences are monotonic, analyze if they are bounded and compute the limits if they exist.

$i)$ $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$ $\qquad$ $ii)$ $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$

$iii)$ $u_{n+1} = 3 + \frac{u_n}{2}, \quad u_0 = 0.$ $\qquad$ $iv)$ $u_{n+1} = 3 + 2u_n, \quad u_0 = 0.$

$v)$ $u_{n+1} = \frac{u_n^3 + 6}{7},$ $\qquad$ $a)$ $u_0 = 1/2,$ $\quad b)$ $u_0 = 3/2,$ $\quad c)$ $u_0 = 3.$

Problem 5.1.14 Consider the sequence defined by $a_{n+1} = \sqrt{1 + 3a_n} - 1,\ a_0 = 1/2.$

$a)$ Prove that it is convergent and compute its limit.

$b)$ Compute $\lim_{n\to\infty} \frac{a_{n+1} - 1}{a_n - 1}.$

Problem 5.1.15 We define a sequence by $b_{n+1} = 1 - \frac{b_n}{2}$, with $b_0 = 0.$

$a)$ Check that it is an oscillating sequence, that is: $\operatorname{sign}(b_{n+1} - b_n) = -\operatorname{sign}(b_n - b_{n-1}).$

$b)$ Calculate the possible limit ℓ.

$c)$ Show that $|b_{n+1} - \ell| = \frac{1}{2}|b_n - \ell|.$

$d)$ Show that indeed $\lim_{n\to\infty} b_n = \ell.$

Hint: $c)$ $|b_n - \ell| = (\frac{1}{2})^n\, \ell.$

Problem 5.1.16 Consider a sequence defined by $c_{n+1} = f(c_n)$, where $f(x) = \frac{1}{1+x}$, $c_0 = 0.$ Prove that it is convergent with the following steps:

$a)$ Calculate the possible limit ℓ.

$b)$ Show that if $x \in [1/2, 1]$ then $f(x) \in [1/2, 1].$

$c)$ Check that $|f'(x)| \le k < 1$ for every $x \in [1/2, 1].$

$d)$ Prove that $c_n \in [1/2, 1]$ for every $n \ge 1.$

$e)$ Prove the estimate $|c_{n+1} - \ell| \le k^n |c_1 - \ell|$ for every $n \ge 1.$

Problem 5.1.17

a) Use the technique of the previous problem with the sequence

$$d_0 = \frac{1}{2}, \quad d_{n+1} = 2 + \frac{4}{d_n},\ n \geq 0,$$

and the interval $[3, 10/3]$.

b) Compute $\lim_{n\to\infty} \dfrac{d_{n+1} - \ell}{d_n - \ell}$.

Problem 5.1.18 We consider a sequence of real numbers defined recursively by

$$x_1 = 1, \qquad x_n = \frac{x_{n-1}(1 + x_{n-1})}{1 + 2x_{n-1}}\,.$$

Prove that it is convergent and obtain its limit.

Problem 5.1.19 Describe the behaviour of the sequences defined recursively in the previous problems using a representation of each pair of consecutive terms in a cartesian system (*cobweb diagram*).

Problem 5.1.20 Let $\{x_n\}$ be a bounded sequence (*not* necessarily convergent) of positive terms. For $\alpha > 0$ we define the sequence

$$y_n = \frac{(x_1 + x_2 + \cdots + x_n)^\alpha}{n}.$$

a) If $0 < \alpha < 1$, show that $\lim_{n\to\infty} y_n = 0$.

b) Consider now $\alpha = 1$. If $\lim_{n\to\infty} x_n = \ell$, show that $\lim_{n\to\infty} y_n = \ell$. Give an example of a non-convergent sequence $\{x_n\}$ such that the sequence $\{y_n\}$ is convergent.

Problem 5.1.21 Given a bounded sequence $\{x_n\}$ (*not* necessarily convergent) we consider a new sequence defined by

$$y_n = \sup\{x_n, x_{n+1}, \ldots\}.$$

a) Prove that $\{y_n\}$ is a bounded monotonic sequence and so it is convergent.

b) Compute $\lim_{n\to\infty} y_n$ (known as *limit superior* of x_n) for the sequences:

$$i)\ x_n = \frac{1 + (-1)^n}{2}, \qquad ii)\ x_n = (-1)^n \left(3 + \frac{1}{n}\right).$$

5.2 Series of numbers

Problem 5.2.1
Study the convergence of the following series of positive terms:

i) $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n-1}\right)^n$,	*ii*) $\sum_{n=1}^{\infty} \frac{1}{(3n-1)^2}$,	*iii*) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n^4+1}}$,
iv) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$,	*v*) $\sum_{n=1}^{\infty} \frac{\lvert \sin n \rvert}{n^2+n}$,	*vi*) $\sum_{n=1}^{\infty} \sin(\frac{1}{n^2})$,
vii) $\sum_{n=1}^{\infty} \arcsin(\frac{1}{\sqrt{n}})$,	*viii*) $\sum_{n=1}^{\infty} \frac{3n-1}{(\sqrt{2})^n}$,	*ix*) $\sum_{n=1}^{\infty} \frac{n^n}{3^n n!}$,
x) $\sum_{n=1}^{\infty} (\sqrt[n]{n}-1)^n$,	*xi*) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} 3^{-n}$,	*xii*) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} \mathrm{e}^{-n}$,
xiii) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$,	*xiv*) $\sum_{n=2}^{\infty} \frac{n^2}{(\log n)^n}$,	*xv*) $\sum_{n=2}^{\infty} [\sqrt{n^2+1}-n]$,
xvi) $\sum_{n=2}^{\infty} \log(\frac{n+1}{n})$,	*xvii*) $\sum_{n=1}^{\infty} \frac{1}{n^{\log n}}$,	*xviii*) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}$.

Hints: (in general, we can apply more than one test to decide); *i*), *viii*), *x*), *xi*), *xiii*), *xiv*), root test; *ix*), ratio (quotient) test; *ii*), *iii*), *iv*), *v*), *vi*), *vii*), *xv*), *xvi*), *xvii*), *xviii*), comparison test; *xii*), compute the limit of the general term (see 3.1.22 *ii*)).

Problem 5.2.2 Prove that the series

$$\sum_{n=1}^{\infty} \left(\frac{a}{2n-1} - \frac{b}{2n+1}\right)$$

is convergent if and only if $a = b$.

Problem 5.2.3

a) Study the convergence of the series $\sum_{n=1}^{\infty} n(1+a)^n \mathrm{e}^{-an}$, for different values of $a > -1$.

b) Do the same with the series $\sum_{n=1}^{\infty} \frac{n^n}{a^n n!}$, for different values of $a > 0$.

c) Again the same question for the series $\sum_{n=1}^{\infty} \frac{n! \mathrm{e}^n}{n^{n+a}}$, for different values of $a \in \mathbb{R}$.

Hints: in *b*) and *c*) use the Stirling formula.

Problem 5.2.4 Analyze the absolute and conditional convergence of the following alternating series:

$$i)\ \sum_{n=2}^{\infty} \frac{(-1)^n}{\log n}, \qquad ii)\ \sum_{n=2}^{\infty} \sin(\pi n + 1/n),$$

$$iii)\ \sum_{n=1}^{\infty} (-1)^n (\operatorname{arctg} 1/n)^2, \qquad iv)\ \sum_{n=1}^{\infty} (-1)^n (\operatorname{arctg} n)^2,$$

$$v)\ \sum_{n=1}^{\infty} (-1)^n [\sqrt{n^2-1} - n], \qquad vi)\ \sum_{n=1}^{\infty} (-1)^n \log(\frac{n}{n+1}),$$

$$vii)\ \sum_{n=1}^{\infty} (-1)^n (1 - \cos(1/n)), \qquad viii)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{\log(\mathrm{e}^n + \mathrm{e}^{-n})}.$$

Problem 5.2.5 Use the Taylor expansion of the function $\operatorname{arctg} x$ to study the convergence of the series

$$\sum_{n=1}^{\infty} \Big(\operatorname{arctg} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n}} \Big).$$

Problem 5.2.6 Find how many terms are necessary to approximate the following sums with an error smaller than 10^{-3}:

$$i)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}, \qquad ii)\ \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Problem 5.2.7 Compute the sum of the following series:

$$i)\ \sum_{n=0}^{\infty} \frac{3^{n+1} - 2^{n-3}}{4^n}, \qquad ii)\ \sum_{n=1}^{\infty} \frac{n}{2^n}, \qquad iii)\ \sum_{n=0}^{\infty} \frac{4n+1}{3^n},$$

$$iv)\ \sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n(n+1)}}, \qquad v)\ \sum_{n=1}^{\infty} \log \left[\frac{n(n+2)}{(n+1)^2} \right].$$

Problem 5.2.8 Obtain the sum of the following series:

$$i)\ \sum_{n=0}^{\infty} a^{[n/2]} b^{[(n+1)/2]}, \quad (|ab| < 1), \qquad ii)\ \sum_{n=1}^{\infty} \frac{1}{2^n} \cos \frac{2n\pi}{3}.$$

(*Hint*: decompose the sums in two and three parts respectively)

Problem 5.2.9

a) If $a_n > -1$ for all n and $\lim_{n\to\infty} a_n = 0$, study the convergence of the series $\sum_n \log(1 + a_n)$ in terms of the convergence of the series $\sum_n a_n$.

b) If both series of positive terms $\sum_n a_n$ and $\sum_n b_n$ are convergent, prove that it is also convergent the series $\sum_n \sqrt{a_n b_n}$.

c) Prove that the two series $\sum_n \sqrt{a_n a_{n+1}}$ and $\sum_n \frac{\sqrt{a_n}}{n}$ are convergent if it is the series $\sum_n a_n$.

Problem 5.2.10

a) Show that the series $\sum_{n=0}^{\infty} b_n 10^{-n}$, where $b_n \in \{0,1,\dots,9\}$ for $n \geq 1$ and $b_0 \in \mathbb{Z}$, converges. What does this series represent and why is it important?

b) Compute the previous sum in the cases:

$$i)\quad b_n = 9, \quad n \geq 0; \qquad ii)\quad b_n = \begin{cases} 1 & n = 2k \\ 2 & n = 2k+1 \end{cases}, \quad k \geq 0.$$

Problem 5.2.11

a) Show that the equation $\operatorname{tg} x = x$ has a unique solution λ_n on each interval

$$\left(\frac{(2n-1)\pi}{2}, \frac{(2n+1)\pi}{2}\right), \qquad n = 1, 2, 3, \cdots$$

b) Prove that the series $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2}$ is convergent.

Problem 5.2.12 Consider the sequence defined by $x_{n+1} = \sqrt{1 + 2x_n} - 1$, $x_0 = 1$.

a) Show that it is convergent and compute the limit.

b) Find the limits

$$i)\quad \lim_{n\to\infty} \frac{x_{n+1}}{x_n}, \qquad ii)\quad \lim_{n\to\infty} n x_n.$$

c) Study the convergence of the series

$$i)\quad \sum_{n=0}^{\infty} x_n, \qquad ii)\quad \sum_{n=0}^{\infty} x_n^2.$$