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Departamento de Matemáticas

DIFFERENTIAL CALCULUS. Solutions

Degree in Applied Mathematics and Computation

Chapter 6

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6 Sequences and series of functions

6.1 Sequences and series of functions

Problem 6.1.1 *a)* $\lim_{n\to\infty} f_n(x) = 1$, the convergence is not uniform since $\sup_{x\in\mathbb{R}} |f_n(x) - 1| = 1$. *b)* $\lim_{n\to\infty} g_n(x) = 0$, uniform convergence because $\sup_{x\in\mathbb{R}} |g_n(x) - 0| = 1/n \to 0$. *c)* The limit of the derivatives does not exist: $g_n'(x) = \cos nx$.

Problem 6.1.2 *a)* $f(x) = \sum_{n=0}^{\infty} x(1-x)^n = \dfrac{x}{1-(1-x)} = 1$, if $0 < x < 1$, $f(0) = 0$, $f(1) = 1$; the convergence cannot be uniform in $[0,1]$ because all the addends are continuous and f is not. *b)* This is a geometric series with ratio $2\sin x$; it converges if and only if $|\sin x| < 1/2$, that is $x \in (\pi/6 + k\pi, -\pi/6 + k\pi),\ k \in \mathbb{Z}$.

Problem 6.1.3 *i)* $\dfrac{1}{\rho} = \lim_{n\to\infty} \dfrac{2}{\sqrt[n]{n}} = 2$; at the endpoints: $x = \dfrac{3}{2} \Rightarrow a_n = \dfrac{(-1)^n}{n} \Rightarrow CC$, $x = -\dfrac{1}{2} \Rightarrow a_n = \dfrac{1}{n} \Rightarrow D$; finally, it converges for $x \in [1/2, 3/2)$. *ii)* converges for $x \in [\pi/6 + k\pi, -\pi/6 + k\pi],\ k \in \mathbb{Z}$.

Problem 6.1.4

$$a) \quad \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } 0 < x \le 1 \\ 1 & \text{if } x = 0. \end{cases}$$

The convergence cannot be uniform in $[0,1]$ since the limit is not a continuous function while all the f_n are continuous. *b)* $\lim_{n\to\infty} g_n(x) = 0$ for all $x \in [0,1]$; the convergence is not uniform since

$$\sup_{0\le x\le 1} |g_n(x) - 0| = n \to \infty.$$

Problem 6.1.5 *a)* $\sum_{n=1}^{\infty} b_n = a_1 - \lim_{n\to\infty} a_n = a_1$. *b)* $\lim_{n\to\infty} n(a_n - a_{n+1}) = \lim_{n\to\infty} nb_n = 0$, because in other case the series $\sum b_n$ cannot be convergent. *c)* $\sum_{n=1}^{N} nc_n = \sum_{n=1}^{N} b_n - Nb_{N+1}$ and so $\sum_{n=1}^{\infty} nc_n = a_1$. *d)* If $0 \le x < 1$, then

$$S(x) = \sum_{n=1}^{\infty} n(x-1)^2 x^n = \sum_{n=1}^{\infty} n(x^{n+2} - 2x^{n+1} + x^n) = \sum_{n=1}^{\infty} nc_n,$$

where $a_n = x^n$; thus $S(x) = a_1 = x$; if $x = 1$ the series converges obviously: $S(1) = 0$. *e)* $S(x) = (x-1)^2 \sum_{n=1}^{\infty} nx^n = (x-1)^2 \dfrac{x}{(x-1)^2} = x$, if $0 \le x < 1$, $S(1) = 0$; the convergence is not uniform in $[0,1]$ since all the addends are continuous and $S(x)$ is not.

6.2 Taylor series

Problem 6.2.1 *i)* $\frac{1}{\rho} = \lim_{n\to\infty} \frac{1}{2n^{2/n}} = \frac{1}{2}$; at the endpoints it is AC; $I = [-2, 2]$;
ii) Use Stirling $\frac{1}{\rho} = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{\mathrm{e}}$; at the endpoints $|a_n| \approx \sqrt{2\pi n} \to \infty \Rightarrow D$; $I = (-\mathrm{e}, \mathrm{e})$;
iii) $\rho = 10$; at the endpoints: $x = -10 \Rightarrow CC$, $x = 10 \Rightarrow D$; $I = [-10, 10)$; *iv)* $\rho = 1$; at the endpoints: $x = -1 \Rightarrow CC$, $x = 1 \Rightarrow D$; $I = [-1, 1)$; *v)* $a_n = 2^n(3/2 - x)^n$; $\rho = 1/2$; at the endpoints D; $I = (3/2 - 1/2, 3/2 + 1/2) = (1, 2)$; *vi)* $\rho = 1$; at the endpoints $x = 1 \Rightarrow CC$, $x = 3 \Rightarrow D$; $I = [1, 3)$.

Problem 6.2.2 $\rho = 1/\mathrm{e}$; at the endpoints it diverges by problem 5.2.1 *xii)*; $I = (-1/\mathrm{e}, 1/\mathrm{e})$.

Problem 6.2.3

$$\begin{aligned}
f_1(x) &= \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \\
f_2(x) &= \frac{1}{(1-x)^2} = f_1'(x) = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n, \\
f_3(x) &= \frac{1}{(1-x)^3} = \frac{1}{2} f_2'(x) = \frac{1}{2} \sum_{n=1}^{\infty} (n+1) n x^{n-1} = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n.
\end{aligned}$$

Problem 6.2.4

$$\begin{aligned}
&i) \quad \sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x), && x \in [-1, 1); \\
&ii) \quad \sum_{n=0}^{\infty} (n+1) 2^{-n} x^n = \frac{4}{(2-x)^2}, && x \in (-2, 2)
\end{aligned}$$

(using f_2 from the previous problem).

Problem 6.2.5

$$\begin{aligned}
&i) && \sin^2 x = \frac{1 - \cos 2x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1}}{(2n)!} x^{2n}, && x \in \mathbb{R}; \\
&ii) && \frac{x}{a + bx} = \frac{x}{a} \frac{1}{1 + bx/a} = \frac{x}{a} \sum_{n=0}^{\infty} \Big(\frac{-bx}{a}\Big)^n = \sum_{n=1}^{\infty} (-1)^{n-1} b^{n-1} a^{-n} x^n, && x \in (-a/b, a/b); \\
&iii) && \log \sqrt{\frac{1+x}{1-x}} = \frac{1}{2} \Big[\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} + \sum_{n=1}^{\infty} \frac{x^n}{n} \Big] = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}, && x \in (-1, 1)
\end{aligned}$$

(descompose the sum for n even and n odd);

$$\begin{aligned}
&iv) && \frac{1}{2 - x^2} = \frac{1}{2} \sum_{n=0}^{\infty} \Big(\frac{x^2}{2}\Big)^n = \sum_{n=0}^{\infty} 2^{-n-1} x^{2n}, && x \in (-\sqrt{2}, \sqrt{2}); \\
&v) && \mathrm{e}^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}, && x \in \mathbb{R}.
\end{aligned}$$

Problem 6.2.6 *i)* $S = \sum_{n=1}^{\infty} \frac{(-1/2)^n}{n!} = e^{-1/2} - 1$; *ii)* $S = \frac{1/2}{(1-1/2)^2} = 2$; *iii)* $S = \sum_{n=1}^{\infty} \frac{(1/2)^n}{n} = \log 2$; *iv)* $S = \operatorname{arctg} 1 - 1 = \frac{\pi}{4} - 1$.

Problem 6.2.7 *a)* It is a square with side $r\sqrt{2}$. *b)* Each radius is $r_{n+1} = \frac{r_n}{\sqrt{2}}$, so each area is $A_{n+1} = \frac{1}{2}A_n$; starting with $A_0 = \pi r^2$, we have $A_n = \frac{\pi r^2}{2^n}$, and so $\sum_{n=0}^{\infty} A_n = 2\pi r^2$.

Problem 6.2.8

$$\begin{aligned}
f(0) &= \sum_{n=1}^{\infty} \frac{1}{n!} = e - 1;\\
f(1) &= \sum_{n=1}^{\infty} \frac{n}{n!} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = \sum_{n=0}^{\infty} \frac{1}{n!} = e;\\
f(2) &= \sum_{n=1}^{\infty} \frac{n^2}{n!} = \sum_{n=1}^{\infty} \frac{n}{(n-1)!} = \sum_{n=1}^{\infty} \frac{n-1+1}{(n-1)!} = \sum_{n=2}^{\infty} \frac{1}{(n-2)!} + \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = 2e.
\end{aligned}$$

Problem 6.2.9 Differentiating the equation satisfied by f we arrive to

$$\begin{aligned}
&f'(x) = f(x) + x, && f''(x) = f'(x) + 1,\\
&f'''(x) = f''(x), && f^{n)}(x) = f''(x), \quad \forall\, n \geq 2.
\end{aligned}$$

Now we compute the derivatives at $x = 0$ to obtain the Taylor series of f at $x = 0$:

$$\begin{aligned}
&f(0) = 2, && f'(0) = f(0) + 0 = 2,\\
&f''(0) = f'(0) + 1 = 3, && f^{n)}(0) = 3, \quad \forall\, n \geq 2.
\end{aligned}$$

Then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n)}(0)x^n}{n!} = 2 + 2x + 3\sum_{n=2}^{\infty} \frac{x^n}{n!} = 2 + 2x + 3(e^x - x - 1) = 3e^x - x - 1.$$

Problem 6.2.10

$$\begin{aligned}
i)\quad & \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \left(1 - \frac{1}{4}\right)\frac{\pi^2}{6} = \frac{\pi^2}{8};\\
ii)\quad & \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) - \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 - \left(\frac{\pi^2}{6} - 1\right) = 2 - \frac{\pi^2}{6}.
\end{aligned}$$

Problem 6.2.11

$$a)\quad f(x) = (x+1)\mathrm{e}^{-x} + (x-1)\mathrm{e}^{x} = (x+1)\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!} + (x-1)\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n - 1}{n!}x^n + \sum_{n=0}^{\infty}\frac{(-1)^n + 1}{n!}x^{n+1} = \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n!} + \frac{(-1)^{n-1} + 1}{(n-1)!}\right]x^n$$

$$= \sum_{n=1}^{\infty}\frac{1-(-1)^n}{n(n-2)!}x^n = \sum_{k=0}^{\infty}\frac{4k}{(2k+1)!}x^{2k+1},$$

for all $x \in \mathbb{R}$.

$$b)\quad \sum_{n=1}^{\infty}\frac{n}{(2n+1)!} = \frac{1}{4}f(1) = \frac{1}{2\mathrm{e}}\,.$$

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