uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL CALCULUS. Problems

Degree in Applied Mathematics and Computation

Chapter 6

Elena Romera
with the collaboration of Arturo de Pablo

Open Course Ware, UC3M

6 Sequences and series of functions

6.1 Sequences and series of functions

Problem 6.1.1

a) Consider the sequence of functions given by:

$$f_n(x) = \begin{cases} 1 & \text{if } |x| \leq n, \\ 0 & \text{if } |x| > n. \end{cases}$$

Obtain $\lim_{n\to\infty} f_n$ and study if the convergence is uniform on $\mathbb{R}$.

b) The same question for the sequence $g_n(x) = \dfrac{\sin nx}{n}$.

c) In this second sequence, is the limit of the derivatives equal to the derivative of the limit?

Problem 6.1.2

a) Prove that the series $\sum_{n=0}^{\infty} x(1-x)^n$ converges on $[0,1]$ to a function $f(x)$. Is this a uniform convergence?

b) For what values of x does the series $\sum_{n=0}^{\infty} 2^n \sin^n x$ converge?

Problem 6.1.3 Find the values of $x \in \mathbb{R}$ for which the following series converge:

$$i) \quad \sum_{n=1}^{\infty} \frac{(x-1)^n 2^n}{n}, \qquad ii) \quad \sum_{n=1}^{\infty} \frac{(2\sin x)^n}{n^2}.$$

Problem 6.1.4

a) Find $\lim_{n\to\infty} f_n(x)$, where $f_n(x) = \mathrm{e}^{-nx}$, $0 \leq x \leq 1$. Is this a uniform convergence?

b) The same question for the sequence

$$g_n(x) = \begin{cases} 2n^2 x & \text{if } 0 \leq x \leq 1/2n, \\ 2n - 2n^2 x & \text{if } 1/2n \leq x \leq 1/n, \\ 0 & \text{if } 1/n \leq x \leq 1. \end{cases}$$

Problem 6.1.5 Let $\{a_n\}_{n\in\mathbb{N}}$ be a decreasing monotonic sequence convergent to zero and define the new sequences

$$b_n = a_n - a_{n+1}, \qquad c_n = b_n - b_{n+1} = a_n - 2a_{n+1} + a_{n+2}.$$

a) Find $\sum_{n=1}^{\infty} b_n$.

b) Once we know that the previous series converges, find the limit

$$\lim_{n\to\infty} n(a_n - a_{n+1}),$$

if it exists

c) Find $\sum_{n=1}^{N} nc_n$, and use the previous limit to compute $\sum_{n=1}^{\infty} nc_n$.

d) Apply the previous sum to obtain, for $x \in (0,1)$,

$$\sum_{n=1}^{\infty} n(x-1)^2 x^n.$$

e) Obtain the sum directly and study the uniform convergence on $[0,1]$.

6.2 Taylor series

Problem 6.2.1 Obtain the convergence interval of the following series:

$$i)\ \sum_{n=1}^{\infty} \frac{x^n}{2^n n^2}, \qquad ii)\ \sum_{n=1}^{\infty} \frac{n!x^n}{n^n}, \qquad iii)\ \sum_{n=1}^{\infty} \frac{x^n}{n10^{n-1}},$$

$$iv)\ \sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}, \qquad v)\ \sum_{n=1}^{\infty} (3-2x)^n, \qquad vi)\ \sum_{n=1}^{\infty} \frac{(x-2)^n}{\sqrt{2n}}.$$

Problem 6.2.2 Find the interval of convergence of the series

$$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} x^n.$$

Problem 6.2.3 Obtain the Taylor series at 0 of the functions $f_k(x) = \dfrac{1}{(1-x)^k}$, for $k = 1, 2, 3$.

Problem 6.2.4 Compute the interval of convergence and the sum of the following Taylor series:

$$i)\ \sum_{n=1}^{\infty} \frac{x^n}{n}, \qquad ii)\ \sum_{n=0}^{\infty} (n+1)2^{-n}x^n.$$

Problem 6.2.5 Obtain the Taylor series and the interval of convergence of the following functions at 0.

$$i)\ f(x) = \sin^2 x, \qquad ii)\ f(x) = \frac{x}{a+bx}, \quad \text{with } a, b > 0,$$

$$iii)\ f(x) = \log\sqrt{\frac{1+x}{1-x}}, \qquad iv)\ f(x) = \frac{1}{2-x^2}, \qquad v)\ f(x) = \mathrm{e}^{x^2}.$$

Problem 6.2.6 Find the sum of the following series

$$i)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!}, \qquad ii)\ \sum_{n=1}^{\infty} \frac{n}{2^n},$$

$$iii)\ \sum_{n=1}^{\infty} \frac{1}{n2^n}, \qquad iv)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}.$$

Problem 6.2.7 Let C_0 be a circle of radius r.

a) Obtain a rectangle Q_0, inscribed in C_0, with maximum area.

b) Let now C_1 be the maximum inner circle for that rectangle, concentric with C_0, and inscribe in it a rectangle Q_1 of maximum area in C_1. Compute the sum of the areas of the sequence of circles $\{C_n\}_{n=0}^{\infty}$ that we have obtained with the process.

Problem 6.2.8 A function is defined by $f(x) = \sum_{n=1}^{\infty} \frac{n^x}{n!}$, compute the values of $f(0)$, $f(1)$ and $f(2)$.

Problem 6.2.9 Find a function $f(x)$, with power series expansion, that verifies

$$f'(x) = f(x) + x, \qquad f(0) = 2.$$

Problem 6.2.10 We know that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, use this to find the sum of the series

$$i) \quad \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}, \qquad ii) \quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2}.$$

Hints: *i*) decompose the sum we know in the even and odd terms; *ii*) decompose the general term as a sum of simple fractions.

Problem 6.2.11

a) Obtain the Taylor series of the function

$$f(x) = \frac{1 + x - (1-x)\mathrm{e}^{2x}}{\mathrm{e}^x},$$

and compute the convergence interval.

b) As an application, obtain the sum of the series

$$\sum_{n=1}^{\infty} \frac{n}{(2n+1)!}.$$

– $\mathcal{ERC}$–
– $\mathcal{A}_\delta$P–