uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL CALCULUS
SELF-EVALUATION I - SOLUTIONS
Degree in Applied Mathematics and Computation

Time: 1 hour

Problem 1 (**0.5 + 1 + 1 + 0.5 = 3 points**)
Compute the following limits (if they exist):

$$a)\quad \lim_{x\to\pi/2}(\sin x)^{3/(\cos^2 x)},\qquad b)\quad \lim_{t\to 0}\frac{3-5\mathrm{e}^{2/t}}{2+\mathrm{e}^{2/t}},$$

$$c)\quad \lim_{x\to 1}\log x\cdot\log(x-1),\qquad d)\quad \lim_{x\to 0}\frac{1+\sin x-\mathrm{e}^x}{\operatorname{arctg} x}\,.$$

SOLUTION:
a) This is of the kind 1^∞:

$$\lim_{x\to\pi/2}(\sin x)^{3/(\cos^2 x)} = \exp(\lim_{x\to\pi/2}\frac{3(\sin x-1)}{\cos^2 x}) = \exp(\lim_{x\to\pi/2}\frac{3(-\cos^2 x)}{\cos^2 x(\sin x+1)}) = \exp(\frac{-3}{2}).$$

b) When $t\to 0^-$ we have $2/t\to-\infty$, so $\mathrm{e}^{2/t}\to 0$, and then:

$$\lim_{t\to 0}\frac{3-5\mathrm{e}^{2/t}}{2+\mathrm{e}^{2/t}} = \frac{3}{2}.$$

If $t\to 0^+$, then $2/t\to\infty$ and $\mathrm{e}^{2/t}\to\infty$, this means:

$$\lim_{t\to 0}\frac{3-5\mathrm{e}^{2/t}}{2+\mathrm{e}^{2/t}} = -5\,.$$

Then the limit does not exist.
c) We build a quotient where numerator and denominator tend to ∞ to use L'Hôpital's rule:

$$\lim_{x\to 1}\log x\cdot\log(x-1) = \lim_{x\to 1}\frac{\log(x-1)}{1/\log x} = \lim_{x\to 1}\frac{1/(x-1)}{-1/(x\log^2 x)} = \lim_{x\to 1}\frac{x\log^2 x}{1-x} = \lim_{x\to 1}\frac{\log^2 x+2\log x}{-1} = 0.$$

d) Numerator and denominator tend to zero, we use L'Hôpital's rule:

$$\lim_{x\to 0}\frac{1+\sin x-\mathrm{e}^x}{\operatorname{arctg} x} = \lim_{x\to 0}\frac{\cos x-\mathrm{e}^x}{1/(1+x^2)} = 0.$$

Problem 2 (**0.5 + 0.5 = 1 point**)
Study the continuity of the functions:

$$a)\ f(x) = \frac{\sqrt{1-\sqrt{9-x^2}}}{x},\qquad b)\ g(x) = \arcsin(\log|x-1|).$$

Solution:
The functions are continuous in their domains, since they are composition of continuous functions.
a) First of all $x \neq 0$. For the inner root we need:

$$9 - x^2 \geq 0 \Longrightarrow 9 \geq x^2 \Longrightarrow x \in [-3, 3],$$

and also

$$1 - \sqrt{9 - x^2} \geq 0 \Longrightarrow 1 \geq \sqrt{9 - x^2} \Longrightarrow 1 \geq 9 - x^2 \Longrightarrow x^2 \geq 8 \Longrightarrow |x| \geq 2\sqrt{2},$$

joining the three conditions, $\text{Dom}(f) = \{x \in \mathbb{R} : \ 2\sqrt{2} \leq |x| \leq 3\}$.
b) On one side, we need

$$x - 1 \neq 0 \Longrightarrow x \neq 1.$$

Since the domain of arcsin is $[-1, 1]$, we must have $\log |x - 1| \in [-1, 1]$, so

$$|x-1| \in [\mathrm{e}^{-1}, \mathrm{e}] \iff x - 1 \in [\mathrm{e}^{-1}, \mathrm{e}] \cup [-\mathrm{e}, -\mathrm{e}^{-1}] \iff x \in [1 + \mathrm{e}^{-1}, 1 + \mathrm{e}] \cup [1 - \mathrm{e}, 1 - \mathrm{e}^{-1}].$$

Hence, $\text{Dom}(g) = [1 + \mathrm{e}^{-1}, 1 + \mathrm{e}] \cup [1 - \mathrm{e}, 1 - \mathrm{e}^{-1}]$.

Problem 3 (2 points)
Prove that the equation

$$2x + \sin \frac{\pi x}{2} = \frac{10}{1 + \sqrt{x}}$$

has exactly one root in $[0, \infty)$ and find an interval $[n, n+1)$, with $n \in \mathbb{N}$, where this root is found.

Solution:
The roots are the zeroes of the function

$$f(x) = 2x + \sin \frac{\pi x}{2} - \frac{10}{1 + \sqrt{x}}$$

with domain $[0, \infty)$, there it is continuous and it is differentiable in $(0, \infty)$. The derivative is:

$$f'(x) = 2 + \frac{\pi}{2} \cos \frac{\pi x}{2} + \frac{5}{\sqrt{x}(1 + \sqrt{x})^2} > 0, \qquad x > 0,$$

since $|\cos \frac{\pi x}{2}| < 1 \Longrightarrow 2 + \frac{\pi}{2} \cos \frac{\pi x}{2} > 0$ and $\frac{5}{\sqrt{x}(1 + \sqrt{x})^2} > 0$, then f is increasing in $[0, \infty)$, so at most it has one root in that interval. We seek now a change of sign to use Bolzano's theorem:

$$f(0) = -5 < 0, \qquad f(1) = 3 - 5 = -2 < 0, \qquad f(2) = 4 - \frac{10}{1 + \sqrt{2}} < 0, \qquad f(3) = 5 - \frac{10}{1 + \sqrt{3}} > 0.$$

Then the root is in $[2, 3)$.

Problem 4 (2 points)
Consider a function f such that $f(1/2) = -3$ and $f'(x) = \sqrt{x^2 + 2}$. If

$$g(x) = x^2 f(\frac{x-1}{x}),$$

obtain the tangent line to the graph of g at the point $x = 2$.

Solution:
We need $g(2) = 4f(\frac{1}{2}) = -12$ and also the derivative:

$$g'(x) = 2xf(\frac{x-1}{x}) + x^2 f'(\frac{x-1}{x})\frac{1}{x^2} = 2xf(\frac{x-1}{x}) + x^2 \sqrt{(\frac{x-1}{x})^2 + 2} \cdot \frac{1}{x^2}$$

$$g'(2) = 4f(\frac{1}{2}) + \sqrt{\frac{1}{4} + 2} = -12 + \frac{3}{2} = \frac{-21}{2}$$

The tangent line is then:

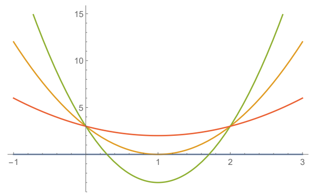
$$y = g(2) + g'(2)(x-2) = -12 - \frac{-21}{2}(x-2) \Longleftrightarrow y = 9 - \frac{21x}{2}.$$

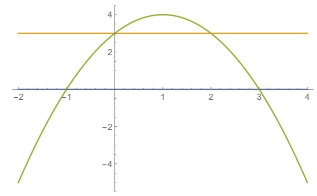
Problem 5 (**2 points**)
Obtain the minimum value of α for which the function $f(x) = |\alpha x^2 - 2\alpha x + 3|$ is differentiable on the whole real line.

SOLUTION:
For $\alpha = 0$ the function is constant, so it is differentiable on the real line. If $\alpha \neq 0$, in order to have a differentiable function, the polynomial cannot have two different roots. This means:

$$4\alpha^2 - 12\alpha \leq 0 \Longleftrightarrow \alpha(\alpha - 3) \leq 0 \Longleftrightarrow \alpha \in [0, 3]$$

The minimum value of α is then 0.





In the graph appear the cases $\alpha > 3$, $\alpha = 3$ and $0 < \alpha < 3$ at the left and $\alpha = 0$ and $\alpha < 0$ at the right.

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