uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL CALCULUS
SELF-EVALUATION II - SOLUTIONS
Degree in Applied Mathematics and Computation

Time: 90 minutes

Problem 1 (**2 points**)
Plot the graph of the function: $f(x) = x\sqrt{|x^2-1|}$.

SOLUTION:
$\mathrm{Dom}(f) = \mathbb{R}$ and it is odd. The function is:

$$f(x) = \begin{cases} x\sqrt{x^2-1}, & |x| \geq 1, \\ x\sqrt{1-x^2}, & |x| < 1. \end{cases}$$

We study the asymptotes:

$$\lim_{x\to\infty} x\sqrt{x^2-1} = \infty, \qquad \lim_{x\to\infty} \frac{x\sqrt{x^2-1}}{x} = \infty,$$
$$\lim_{x\to-\infty} x\sqrt{x^2-1} = -\infty, \qquad \lim_{x\to-\infty} \frac{x\sqrt{x^2-1}}{x} = \infty,$$

so, there are no horizontal nor oblique asymptotes. The derivative is:

$$f'(x) = \begin{cases} \sqrt{x^2-1} + \dfrac{x^2}{\sqrt{x^2-1}}, & |x| > 1, \\ \sqrt{1-x^2} - \dfrac{x^2}{\sqrt{1-x^2}}, & |x| < 1, \end{cases}$$

and

$$\lim_{x\to1^+} \sqrt{x^2-1} + \frac{x^2}{\sqrt{x^2-1}} = \infty, \qquad \lim_{x\to1^-} \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} = -\infty,$$
$$\lim_{x\to-1^+} \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} = -\infty, \qquad \lim_{x\to-1^-} \sqrt{x^2-1} + \frac{x^2}{\sqrt{x^2-1}} = \infty,$$

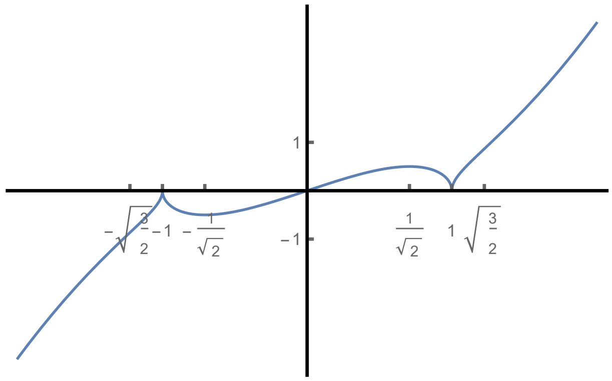
so, the points 1 and -1 have no derivative. Also, for $|x| < 1$:

$$f'(x) = \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} = 0 \Longrightarrow x = \pm\frac{1}{\sqrt{2}}.$$

These are the critical points. On $|x| > 1$ there are no critical points. Besides: $f' > 0$ on $|x| > 1$ and on $|x| < \frac{1}{\sqrt{2}}$, so there f is increasing, and $f' < 0$ on $\frac{1}{\sqrt{2}} < |x| < 1$, so there f is decreasing. The points -1 and $\frac{1}{\sqrt{2}}$ are a local maxima and $-\frac{1}{\sqrt{2}}$ and 1 are local minima. Besides,

$$f''(x) = \begin{cases} \dfrac{2x^3-3x}{(x^2-1)^{3/2}} & |x| > 1, \\ \dfrac{2x^3-3x}{(1-x^2)^{3/2}}, & |x| < 1, \end{cases} = 0 \quad \Longrightarrow \quad \begin{cases} x = 0, \\ x = \pm\sqrt{\frac{3}{2}}. \end{cases}$$

With this we obtain: $f'' > 0$ on $\left(-\sqrt{\frac{3}{2}}, -1\right) \cup (-1, 0) \cup \left(\sqrt{\frac{3}{2}}, \infty\right)$, here f is convex, and $f'' < 0$ on $\left(-\infty, -\sqrt{\frac{3}{2}}\right) \cup (0, 1) \cup \left(1, \sqrt{\frac{3}{2}}\right)$, where f is concave. The points 0, $-\sqrt{\frac{3}{2}}$ and $\sqrt{\frac{3}{2}}$ are inflection points.



Problem 2 (**3 points**)

a) Prove that the function $g(x) = \max\{\log(1 + x^2), |x| + \alpha\}$ verifies the hypothesis of the mean value theorem in any interval $[a, b] \in \mathbb{R}$ if and only if $\alpha = \log 2 - 1$.

b) For the previous value of α, obtain the point or points whose existence is guaranteed by the aforementioned theorem applied to the function g in the interval $[-1, 2]$.

c) Obtain the Taylor polynomial of $f(x) = \sin(x/2) + x^2 e^x$ of order 3 at the origin and estimate the error using that polynomial to approximate the function on $[-1/4, 1/4]$.

SOLUTION:

a) The function must be differentiable on $\mathbb{R}$. The logarithm is differentiable on $\mathbb{R}$, and $|x| + \alpha$ is differentiable outside the origin. If the functions $f(x) = \log(1 + x^2)$ and $h(x) = |x| + \alpha$ intersect at $x = \pm A$ (by symmetry), then we have $f(A) = h(A)$, $f'(A) = h'(A)$, which implies $A = 1$ and the value $\alpha = \log 2 - 1$. Observe that in this case:

$$g(x) = \begin{cases} \log(1 + x^2), & |x| \le 1, \\ |x| + \log 2 - 1, & |x| > 1. \end{cases}$$

b) By the mean value theorem, there exists some c for which:

$$g'(c) = \frac{g(2) - g(-1)}{2 + 1} = \frac{1}{3}.$$

On the interval $[1, 2]$ the derivative is 1, so we try for $c \in [-1, 1]$:

$$g'(c) = \frac{2c}{1 + c^2} \quad \Longrightarrow \quad c = 3 - 2\sqrt{2} \in [-1, 1].$$

c)

$$P_{3,0}f(x) = \left(\frac{x}{2} - \frac{x^3}{3!8}\right) + x^2(1 + x) = 2x + x^2 - \frac{2x^3}{3!}.$$

Now, we use the Lagrange formula for the Taylor remainder:

$$R_{3,0}f(x) = \frac{f^{IV)}(t)}{4!}x^4 \quad \Longrightarrow \quad |R_{3,0}f(x)| \le \frac{|f^{IV)}(t)|}{4!}\frac{1}{2^4}, \qquad t \in [-1/2, 1/2].$$

Now we bound the fourth derivative on $[-1/2, 1/2]$:

$$|f^{IV)}(t)| = \left|\frac{\sin(\frac{t}{2})}{2^4} + e^t(11 + 7t + t^2)\right| < 2^{-4} + e^{1/2}(11 + \frac{7}{2} + \frac{1}{4}) = 2^{-4} + \frac{59\sqrt{e}}{4} < 2^{-4} + \frac{59}{2} < 30$$

Then, $|R_{3,0}f(x)| \le \dfrac{30}{4!2^4} = \dfrac{5}{2^6}$.

Problem 3 (**3 points**)

a) Obtain the limit:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n, \qquad a,b>0,$$

b) If we have $\lim_{n\to\infty} a_n = \ell$, find: $\displaystyle\lim_{n\to\infty}\frac{a_1+\frac{a_2}{2}+\cdots+\frac{a_n}{n}}{\log(n+1)}$.

c) Study the convergence of the following recurring sequence and find its limit if it exists:

$$a_0=\frac{1}{2}, \qquad a_{n+1}=\sqrt{1+3a_n}-1$$

Solution:

a) Since $\lim_{n\to\infty}\sqrt[n]{a}=\lim_{n\to\infty}\sqrt[n]{b}=1$:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n = \mathrm{e}^{\lim_{n\to\infty} n\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}-1\right)} = L.$$

The exponent is, changing the variable and using L'Hôpital:

$$\lim_{x\to\infty}\frac{\mathrm{e}^{(\log a)/x}+\mathrm{e}^{(\log b)/x}-2}{2/x} = \lim_{t\to 0^+}\frac{\mathrm{e}^{t\log a}+\mathrm{e}^{t\log b}-2}{2t}$$

$$=\lim_{t\to 0^+}\frac{(\log a)\mathrm{e}^{t\log a}+(\log b)\mathrm{e}^{t\log b}}{2} = \frac{\log a+\log b}{2} = \log(ab)^{1/2}.$$

So, $L=\mathrm{e}^{\log(ab)^{1/2}}=\sqrt{ab}$.

b) We can use Stolz's criterion, since $\log(n+1)$ is increasing with limit ∞:

$$\lim_{n\to\infty}\frac{a_1+\frac{a_2}{2}+\cdots+\frac{a_n}{n}}{\log(n+1)} = \lim_{n\to\infty}\frac{\frac{a_n}{n}}{\log(n+1)-\log n} = \lim_{n\to\infty}\frac{a_n}{\log(\frac{n+1}{n})^n} = \lim_{n\to\infty} a_n = \ell.$$

c) $a_1=\sqrt{1+\frac{3}{2}}-1=\sqrt{\frac{5}{2}}-1>\frac{1}{2}$. We suppose now that $a_n>a_{n-1}$, then:

$$a_{n+1}=\sqrt{1+3a_n}-1>\sqrt{1+3a_{n-1}}-1=a_n.$$

so, by induction, the sequence is increasing. The possible limit satisfies:

$$\ell=\sqrt{1+3\ell}-1 \implies (\ell+1)^2=1+3\ell \implies \ell^2-\ell=\ell(\ell-1)=0 \implies \ell=0,\ \ell=1.$$

We have $a_0<1$, suppose now that $a_n<1$, then

$$a_{n+1}=\sqrt{1+3a_n}-1<\sqrt{4}-1=1,$$

thus, the sequence is bounded above by 1 and the sequence converges. The limit is $\ell=1$, since all the terms are positive and the sequence is increasing.

Problem 4 (**1 + 1 = 2 points**)

Study the convergence of the series:

$$a)\sum_{n=2}^{\infty}\frac{2}{(\log n)^{\log n}}, \qquad b)\sum_{n=2}^{\infty}\frac{(-1)^n n!\mathrm{e}^n}{n^{n+1}}.$$

Solution:

a) We can use the comparison test:

$$(\log n)^{\log n} = n^{\log(\log n)},$$

and $\log(\log n) > 2$ for $n > \mathrm{e}^{\mathrm{e}^2}$, so we compare with the 2-harmonic series, that converges:

$$\sum_{n=2}^{\infty} \frac{2}{(\log n)^{\log n}} < \sum_{n=2}^{\infty} \frac{2}{n^2}.$$

Then, our series converges.

b) This is an alternating series. For the absolute convergence we study: $\displaystyle\sum_{n=2}^{\infty} \frac{n!\mathrm{e}^n}{n^{n+1}}$. Using Stirling's formula we find that:

$$\lim_{n\to\infty} \frac{n!\mathrm{e}^n}{n^{n+1}} = \lim_{n\to\infty} \frac{n^n \mathrm{e}^{-n}\sqrt{2\pi n}\,\mathrm{e}^n}{n^{n+1}} = \lim_{n\to\infty} \sqrt{\frac{2\pi}{n}}.$$

Thus, our series can be compared with the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$, that is divergent. We obtain:

$$\lim_{n\to\infty} \frac{n!\mathrm{e}^n}{n^{n+1}} : \frac{1}{n^{1/2}} = \lim_{n\to\infty} \sqrt{2\pi},$$

so both series diverge and our original series does not converge absolutely. For the conditional convergence we use Leibniz's criterion: The general term without the sign, is like $\sqrt{\frac{2\pi}{n}}$, that decreases to zero, so the series converges conditionally.
