uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL CALCULUS
CONTROL I - SOLUTIONS
Degree in Applied Mathematics and Computation

Time: 90 minutes

Problem 1 **(1 + 1 = 2 points)**
Compute the following limits (if they exist):

$$a)\ \lim_{x\to 0}(1+x^2)^{3/(2\arcsin x)} \qquad b)\ \lim_{x\to 0}\frac{\sec x - 1}{2x\sin x},$$

SOLUTION:
a) Since $\lim_{x\to 0}\frac{3}{2\arcsin x} = \infty$ our limit is, with the help of L'Hôpital:

$$\lim_{x\to 0}(1+x^2)^{3/(2\arcsin x)} = \mathrm{e}^{\lim_{x\to 0} 3x^2/(2\arcsin x)} = \mathrm{e}^{\lim_{x\to 0} 3x\sqrt{1-x^2}} = \mathrm{e}^0 = 1.$$

b) We use the conjugate and a famous known limit:

$$\lim_{x\to 0}\frac{\sec x - 1}{2x\sin x} = \lim_{x\to 0}\frac{1-\cos x}{2x\cos x\sin x} = \lim_{x\to 0}\frac{1-\cos^2 x}{2x\cos x(1+\cos x)\sin x} = \lim_{x\to 0}\frac{\sin x}{2x\cos x(1+\cos x)} = \frac{1}{4}.$$

Problem 2 **(2 points)**
Find the values of λ for which the function: $f(x) = \dfrac{1}{\lambda x^2 - 4\lambda x + 4}$ is continuous on $\mathbb{R}$.

SOLUTION:
It is continuous on $\mathbb{R}$ if $\lambda = 0$ because it is a constant. For $\lambda \neq 0$ we find the roots of the denominator:

$$x = \frac{4\lambda \pm \sqrt{16\lambda^2 - 16\lambda}}{2\lambda} = \frac{2\lambda \pm 2|\lambda|\sqrt{1-\frac{1}{\lambda}}}{\lambda} = 2 \pm 2\sqrt{1-\frac{1}{\lambda}}$$

The function is continuous if there are no real roots, so $\lambda > 0$ and we need also:

$$1 - \frac{1}{\lambda} < 0 \quad\Longrightarrow\quad 1 < \frac{1}{\lambda} \quad\Longrightarrow\quad \lambda < 1.$$

Then, $\lambda \in [0, 1)$.

Problem 3 **(2 points)**
Prove that a polynomial of even degree:

$$f(x) = a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1 x + a_0, \qquad a_{2n} \neq 0,$$

is bounded below if $a_{2n} > 0$ and it is bounded above if $a_{2n} < 0$.

Solution:
A polynomial is a continuous function on the real line. If it is of even degree and $a_{2n} > 0$ then we have the limits:

$$\lim_{x\to\pm\infty} (a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1 x + a_0) = \infty,$$

and this means that:

$$\forall M > 0,\ \exists N_1 > 0 \text{ such that } x > N_1 \Longrightarrow f(x) > M,$$

and also

$$\forall M > 0,\ \exists N_2 < 0 \text{ such that } x < N_2 \Longrightarrow f(x) > M.$$

Hence, f is bounded on $(-\infty, N_2)\bigcup(N_1,\infty)$, and the interval in the middle is closed: $[N_2, N_1]$, so f is bounded above and below on $[N_2, N_1]$. Joining the two things we have proved that the function f is bounded below on $\mathbb{R}$.
If $a_{2n} < 0$ then $-f(x)$ is of the previous kind and it is bounded below, so $f(x)$ is bounded above.

Problem 4 (**2 points**)
The equation

$$\begin{cases} e^{-f}f' = 2 - \log(x+1), \\ f(0) = 2, \end{cases}$$

defines a differentiable one-to-one (bijective) function f on the interval $(-1, 1)$. We define the function $g(x) = f^{-1}(x+2)$. Obtain the limit

$$\lim_{x\to 0} \frac{e^x - e^{-\sin x}}{g(x)}.$$

Solution:
Observe first that if f is continuous since f' exists, besides $f' = e^f(2-\log(x+1))$ is continuous which means that also the inverse g is continuous. We have

$$f(0) = 2, \quad \Longrightarrow \quad g(0) = f^{-1}(2) = 0, \quad f'(0) = \lim_{x\to 0} f'(x) = 2e^{f(0)} = 2e^2.$$

Even more, since $f'(0) \neq 0$, the following limit exists:

$$\lim_{x\to 0} g'(x) = \lim_{x\to 0} \frac{1}{f'(g(x))} = \frac{1}{2e^2}.$$

Now, we use L'Hôpital:

$$L = \lim_{x\to 0} \frac{e^x - e^{-\sin x}}{g(x)} = \lim_{x\to 0} \frac{e^x + \cos x\, e^{-\sin x}}{g'(x)} = 4e^2.$$

Problem 5 (**2 points**)
Find the absolute maxima and minima of the function $f(x) = 2x^{5/3} + 5x^{2/3}$ on the interval $[-2, 1]$.

Solution:
The function is continuous on $\mathbb{R}$, so it is on $[-2,1]$. Then the absolute maximum and minimum of f on that closed interval exist. To find them we compute $f'(x) = \dfrac{10}{3}(x+1)x^{-1/3}$. Thus, $f'(-1) = 0$, so $x = -1$ is a critical point, while $\nexists f'(0)$. Comparing the values of f at the critical point, at the point without derivative and at the endpoints of the interval we obtain:

$$f(0) = 0, \quad f(-1) = 3, \quad f(-2) = 2^{2/3}, \quad f(1) = 7,$$

we find that the maximum point is $x = 1$ and the minimum is $x = 0$.
