uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL CALCULUS
CONTROL II - SOLUTIONS
Degree in Applied Mathematics and Computation

Time: 90 minutes

Problem 1 (**2,5 points**)

Obtain the graphic representation of: $y = \dfrac{e^{-x}}{x^2-1}$. Deduce the convexity and concavity without the second derivative.

SOLUTION:
Dom $(f) = \mathbb{R} \setminus \{-1, 1\}$ and there are no symmetries. Also:

$$\lim_{x\to\infty} \frac{e^{-x}}{x^2-1} = 0^+, \quad \Longrightarrow \quad y = 0 \text{ horizontal asymptote for } x \to \infty.$$

With respect to $x \to -\infty$ there aren't asymptotes, nor horizontal nor oblique, since:

$$\lim_{x\to-\infty} \frac{e^{-x}}{x^2-1} = \infty, \qquad \lim_{x\to-\infty} \frac{e^{-x}}{x(x^2-1)} = -\infty.$$

At $x = -1$ there is a vertical asymptote from the two sides, because:

$$\lim_{x\to-1^-} \frac{e^{-x}}{x^2-1} = \infty, \qquad \lim_{x\to-1^+} \frac{e^{-x}}{x^2-1} = -\infty.$$

Also, at $x = 1$ there is a vertical asymptote from the two sides:

$$\lim_{x\to1^-} \frac{e^{-x}}{x^2-1} = -\infty, \qquad \lim_{x\to1^+} \frac{e^{-x}}{x^2-1} = +\infty.$$

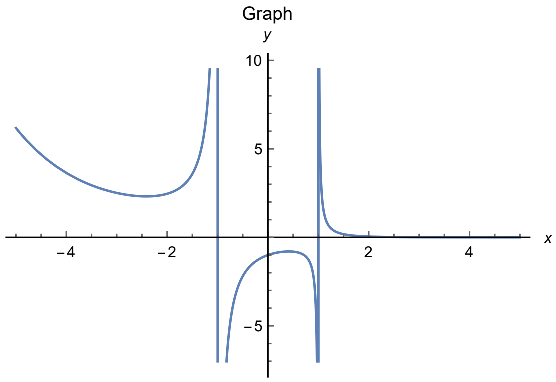
The derivative is:

$$f'(x) = \frac{-e^{-x}(x^2+2x-1)}{(x^2-1)^2} = 0 \quad \Longrightarrow \quad x = \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2} \text{ critical points.}$$

Besides, $f'(x) > 0$ if $x \in (-1-\sqrt{2}, -1) \cup (-1, -1+\sqrt{2})$, there f is increasing; and $f'(x) < 0$ on $(-\infty, -1-\sqrt{2}) \cup (-1+\sqrt{2}, 1) \cup (1, \infty)$, there f decreases, so $x = -1-\sqrt{2}$ is a local minimum and $x = -1+\sqrt{2}$ is a local maximum. The second derivative is long to study:

$$\begin{aligned} f''(x) &= \frac{e^{-x}(x^2+2x-1-2x-2)(x^2-1)^2 + 2(x^2-1)2xe^{-x}(x^2+2x-1)}{(x^2-1)^4} \\ &= \frac{e^{-x}[(x^2-3)(x^2-1)+4x(x^2+2x-1)]}{(x^2-1)^3}. \end{aligned}$$

But we can deduce from the asymptotes and the local extrema that f is convex on $(-\infty, -1) \cup (1, -\infty)$ and that f is concave on $(-1, 1)$. Also, there are no inflection points.



The vertical lines of the plot are not in the graph, they are produced by the computer.

Problem 2 (**1 + 1 = 2 points**)

a) Use Taylor's theorem to compute: (using other method it is worth 0.8 points)

$$\lim_{x\to 0}\frac{1}{x}\left(\frac{1}{x}-\cot x\right).$$

b) Obtain the Taylor polynomial (in its general form) of the functions $f(x)=\log(1-x)$ and $g(x)=\log(1-x^2)$ at $x_0=0$.

SOLUTION:

a) We use the Taylor approximation of $\cos x$ and $\sin x$:

$$\lim_{x\to 0}\frac{1}{x}\left(\frac{1}{x}-\cot x\right) = \quad = \lim_{x\to 0}\frac{\sin x - x\cos x}{x^2\sin x} =$$

$$= \lim_{x\to 0}\frac{x - x^3/6 + o(x^3) - x + x^3/2 + o(x^3)}{x^3+o(x^3)} = \frac{1}{3}.$$

b) The derivatives are:

$$f'(x)=\frac{-1}{1-x},\quad f''(x)=\frac{-1}{(1-x)^2},\quad f'''(x)=\frac{-2}{(1-x)^3},\quad \ldots,\quad f^{n)}(x)=\frac{-(n-1)!}{(1-x)^n},$$

Then $f(0)=0$, $f'(0)=-1$, $f''(0)=-1$, $f'''(0)=-2,\ldots,f^{n)}(0)=-(n-1)!$, and the polynomial of degree n is:

$$P_{n,0}f(x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots-\frac{x^n}{n}.$$

The polynomial for $g(x)$ is then:

$$P_{2n,0}g(x) = -x^2-\frac{x^4}{2}-\frac{x^6}{3}-\cdots-\frac{x^{2n}}{n}.$$

Problem 3 (**1 + 1,5 = 2.5 points**)

a) Compute the limit: $\lim_{n\to\infty}(\sqrt[4]{n^2+1}-\sqrt{n+1})$.

b) Study the convergence of the sequence defined by: $a_n=\sqrt{3+2a_{n-1}}$, $\qquad a_0=0$.

SOLUTION:

a) Multiply and divide by the conjugate twice:

$$\lim_{n\to\infty}(\sqrt[4]{n^2+1}-\sqrt{n+1}) = \lim_{n\to\infty}\frac{(\sqrt[4]{n^2+1}-\sqrt{n+1})(\sqrt[4]{n^2+1}+\sqrt{n+1}}{(\sqrt[4]{n^2+1}+\sqrt{n+1})}$$
$$= \lim_{n\to\infty}\frac{(\sqrt{n^2+1}-(n+1))}{(\sqrt[4]{n^2+1}+\sqrt{n+1})} = \lim_{n\to\infty}\frac{(n^2+1-(n+1)^2)}{(\sqrt[4]{n^2+1}+\sqrt{n+1})(\sqrt{n^2+1}+n+1)}$$
$$= \lim_{n\to\infty}\frac{-2n}{(\sqrt[4]{n^2+1}+\sqrt{n+1})(\sqrt{n^2+1}+n+1)} = 0.$$

b) It is increasing, since $a_1 = \sqrt{3} > 0 = a_0$ and if $a_n > a_{n-1}$ then:

$$a_{n+1} = \sqrt{3+2a_n} > \sqrt{3+2a_{n-1}} = a_n.$$

Also, we can obtain this using that the function that defines the recurrence is $f(x) = \sqrt{3+2x}$ and we check that

$$f'(x) = \frac{1}{\sqrt{3+2x}} > 0, \quad |f'(x)| < \frac{1}{\sqrt{3}} < 1.$$

This means that the sequence is monotonous and also convergent. Since $a_1 = \sqrt{3} > 0 = a_0$, the sequence is increasing (and also it is bounded below by zero). We compute the possible limits:

$$\ell = \sqrt{3+2\ell} \quad \Rightarrow \quad \ell^2 = 3+2\ell \quad \Rightarrow \quad \ell^2 - 2\ell - 3 = (\ell-3)(\ell+1) = 0.$$

Thus, $\ell = -1$ or $\ell = 3$. Let us prove if it is bounded above by 3: Using induction again: $a_0 < 3$ and if $a_{n-1} < 3$ then

$$a_n = \sqrt{3+2a_{n-1}} \leq \sqrt{3+6} = \sqrt{9} = 3.$$

Hence, the sequence is increasing and bounded above, so it has a limit, that is 3, which is the only possibility.

Problem 4 (**1 + 1 + 1 = 3 points**)

a) Study the convergence (conditional and absolute) of the series: $\displaystyle\sum_{n=0}^{\infty}\frac{(-4)^n}{e^n n!}$.

b) Study the convergence interval and the sum of the series:

$$\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n)!}.$$

c) Obtain the interval of convergence of the series: $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{n!}(x-2)^n$.

SOLUTION:

a) We study the series of the absolute values, using the quotient criterion:

$$\lim_{n\to\infty}\frac{4^{n+1}}{e^{n+1}(n+1)!}\frac{e^n n!}{4^n} = \lim_{n\to\infty}\frac{4}{e(n+1)} = 0 < 1,$$

So, the series converges absolutely and then it also converges conditionally.

b) The series of $\cos x$ is quite similar to this one, in fact:

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!} = x \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = x(\cos x - 1).$$

The series is convergent on $\mathbb{R}$, the same as the series of $\cos x$.

c) The powers are centered at $x = 2$, this is the center of the interval of convergence. Now consider $a_n = n^n/n!$, and compute the radius of convergence of the series

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1} n!}{n^n (n+1)!} = \lim_{n\to\infty} \frac{(n+1)^n \cdot (n+1) n!}{n^n (n+1) \cdot n!} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n}$$
$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \mathrm{e}.$$

This means that $R = \mathrm{e}^{-1}$ and the series converges absolutely when $|x - 2| < \mathrm{e}^{-1}$, that is, on $\left(2 - \mathrm{e}^{-1}, 2 + \mathrm{e}^{-1}\right)$.

Let us study now the end-points. We substitute the values: $x - 2 = \pm \mathrm{e}^{-1}$ and obtain the two series:

$$S_{\pm} = \sum_{n=1}^{\infty} (\pm 1)^n \frac{n^n}{n! \mathrm{e}^n}.$$

Since, when $n \to \infty$ we have $n! \sim \sqrt{2\pi n} n^n e^{-n}$,

$$\frac{n^n}{n! e^n} \sim \frac{1}{\sqrt{2\pi n}} = \frac{1}{\sqrt{2\pi}} \frac{1}{n^{1/2}}.$$

and so the convergence of S_+ is equivalent to the one of $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$, which diverges since the exponent is less than 1. The series S_- is alternating, with general term decreasing to zero, so, by Leibniz criterion, it converges (conditionally).

Joining everything, the interval of convergence is

$$\left[2 - \mathrm{e}^{-1}, 2 + \mathrm{e}^{-1}\right).$$